Some considerations around the Weak Vopěnka Principle

Joan Bagaria

Accessible categories and their connections
School of Mathematics, University of Leeds
17-20 July 2018

Vopěnka's Principle (VP)

Definition (P. Vopěnka, 1960's)

Vopěnka's Principle asserts that there is no rigid proper class of graphs.

Equivalent forms of VP

The following are equivalent:

- VP
- (Adámek-Rosický [AR])[1] No locally presentable category has a large rigid class of objects.
- For every proper class of structures of the same type there exist distinct A and B in the class and an (elementary) embedding of A into B.
- For every $n > 0$ there exists a $C^{(n)}$-extendible cardinal.
- **Ord** cannot be fully embedded into **Gra**.

[1] J. Adámek and Jiří Rosický, *Locally Presentable and Accessible Categories*. London Math. Soc. Lecture Note Series, 189. Cambridge Univ. Press 1994.

Equivalent forms of VP

The following are equivalent:

- VP
- (Adámek-Rosický [AR])[1] No locally presentable category has a large rigid class of objects.
- For every proper class of structures of the same type there exist distinct A and B in the class and an (elementary) embedding of A into B.
- For every $n > 0$ there exists a $C^{(n)}$-extendible cardinal.
- **Ord** cannot be fully embedded into **Gra**.

[1] J. Adámek and Jiří Rosický, *Locally Presentable and Accessible Categories*. London Math. Soc. Lecture Note Series, 189. Cambridge Univ. Press 1994.

Equivalent forms of VP

The following are equivalent:

- VP
- (Adámek-Rosický [AR])[1] No locally presentable category has a large rigid class of objects.
- For every proper class of structures of the same type there exist distinct A and B in the class and an (elementary) embedding of A into B.
- For every $n > 0$ there exists a $C^{(n)}$-extendible cardinal.
- **Ord** cannot be fully embedded into **Gra**.

[1] J. Adámek and Jiří Rosický, *Locally Presentable and Accessible Categories*. London Math. Soc. Lecture Note Series, 189. Cambridge Univ. Press 1994.

Equivalent forms of VP

The following are equivalent:

- VP
- (Adámek-Rosický [AR])[1] No locally presentable category has a large rigid class of objects.
- For every proper class of structures of the same type there exist distinct A and B in the class and an (elementary) embedding of A into B.
- For every $n > 0$ there exists a $C^{(n)}$-extendible cardinal.
- **Ord** cannot be fully embedded into **Gra**.

[1] J. Adámek and Jiří Rosický, *Locally Presentable and Accessible Categories*. London Math. Soc. Lecture Note Series, 189. Cambridge Univ. Press 1994.

Equivalent forms of VP

The following are equivalent:

- VP
- (Adámek-Rosický [AR])[1] No locally presentable category has a large rigid class of objects.
- For every proper class of structures of the same type there exist distinct A and B in the class and an (elementary) embedding of A into B.
- For every $n > 0$ there exists a $C^{(n)}$-extendible cardinal.
- **Ord** cannot be fully embedded into **Gra**.

[1] J. Adámek and Jiří Rosický, *Locally Presentable and Accessible Categories*. London Math. Soc. Lecture Note Series, 189. Cambridge Univ. Press 1994.

Equivalent forms of VP

The following are equivalent:

- VP
- (Adámek-Rosický [AR])[1] No locally presentable category has a large rigid class of objects.
- For every proper class of structures of the same type there exist distinct A and B in the class and an (elementary) embedding of A into B.
- For every $n > 0$ there exists a $C^{(n)}$-extendible cardinal.
- **Ord** cannot be fully embedded into **Gra**.

[1] J. Adámek and Jiří Rosický, *Locally Presentable and Accessible Categories*. London Math. Soc. Lecture Note Series, 189. Cambridge Univ. Press 1994.

The Weak Vopěnka's Principle (WVP)

Definition (Adámek-Rosický)

The Weak Vopěnka's Principle asserts that $\mathbf{Ord}^{\mathbf{op}}$ cannot be fully embedded into $\mathbf{Gra}$.

The motivation for this principle, given in [AR], is the following:

- *VP* holds iff every full subcategory of a locally presentable category closed under colimits is coreflective.
- *WVP* holds iff every full subcategory of a locally presentable category closed under limits is reflective.

The Weak Vopěnka's Principle (WVP)

Definition (Adámek-Rosický)

The Weak Vopěnka's Principle asserts that $\mathbf{Ord}^{\mathbf{op}}$ cannot be fully embedded into $\mathbf{Gra}$.

The motivation for this principle, given in [AR], is the following:

- *VP* holds iff every full subcategory of a locally presentable category closed under colimits is coreflective.
- *WVP* holds iff every full subcategory of a locally presentable category closed under limits is reflective.

VP vs. WVP

Proposition

VP implies WVP.

Proof.

Suppose, for a contradiction, that $\langle A_i \rangle_{i \in OR}$ is a sequence of graphs such that for every $i < j$ there is a unique homo $h_{ji} : A_j \to A_i$. We can find a proper class $C \subseteq OR$ such that $|A_i| < |A_j|$, for every $i < j$ in C.

By VP there are $i < j$ in C and a homo $g : A_i \to A_j$. Then, $g \circ h_{ji} = Id_j$ and $h_{ji} \circ g = Id_i$, hence $|A_i| = |A_j|$. A contradiction. □

Question (Adámek-Rosický, 1994)

Does WVP imply VP?

VP vs. WVP

Proposition

VP implies WVP.

Proof.

Suppose, for a contradiction, that $\langle A_i \rangle_{i \in OR}$ is a sequence of graphs such that for every $i < j$ there is a unique homo $h_{ji} : A_j \to A_i$. We can find a proper class $C \subseteq OR$ such that $|A_i| < |A_j|$, for every $i < j$ in C.
By VP there are $i < j$ in C and a homo $g : A_i \to A_j$. Then, $g \circ h_{ji} = Id_j$ and $h_{ji} \circ g = Id_i$, hence $|A_i| = |A_j|$. A contradiction. □

Question (Adámek-Rosický, 1994)

Does WVP imply VP?

Consistency strength

One way to show that a statement φ does not imply a statement ψ is to show that ψ has greater consistency strength than φ.
Consistency strength is measured by large cardinals.

Thus, on the one hand, if one could show that, e.g., WVP follows from, say, the existence of (a proper class of) measurable, or supercompact, or extendible cardinals, then it could not possibly imply VP, because we know that VP is equivalent to the existence of even stronger large cardinals, i.e., $C^{(n)}$-extendible, for all $n > 0$.

On the other hand, to prove that WVP implies VP one may take an incremental approach, namely, by showing that WVP implies the existence of ever stronger large cardinals, up to $C^{(n)}$-extendible, for all $n > 0$.

Consistency strength

One way to show that a statement φ does not imply a statement ψ is to show that ψ has greater consistency strength than φ. Consistency strength is measured by large cardinals.

Thus, on the one hand, if one could show that, e.g., WVP follows from, say, the existence of (a proper class of) measurable, or supercompact, or extendible cardinals, then it could not possibly imply VP, because we know that VP is equivalent to the existence of even stronger large cardinals, i.e., $C^{(n)}$-extendible, for all $n > 0$.

On the other hand, to prove that WVP implies VP one may take an incremental approach, namely, by showing that WVP implies the existence of ever stronger large cardinals, up to $C^{(n)}$-extendible, for all $n > 0$.

Consistency strength

One way to show that a statement φ does not imply a statement ψ is to show that ψ has greater consistency strength than φ. Consistency strength is measured by large cardinals.

Thus, on the one hand, if one could show that, e.g., *WVP* follows from, say, the existence of (a proper class of) measurable, or supercompact, or extendible cardinals, then it could not possibly imply *VP*, because we know that *VP* is equivalent to the existence of even stronger large cardinals, i.e., $C^{(n)}$-extendible, for all $n > 0$.

On the other hand, to prove that *WVP* implies *VP* one may take an incremental approach, namely, by showing that *WVP* implies the existence of ever stronger large cardinals, up to $C^{(n)}$-extendible, for all $n > 0$.

Consistency strength

One way to show that a statement φ does not imply a statement ψ is to show that ψ has greater consistency strength than φ. Consistency strength is measured by large cardinals.

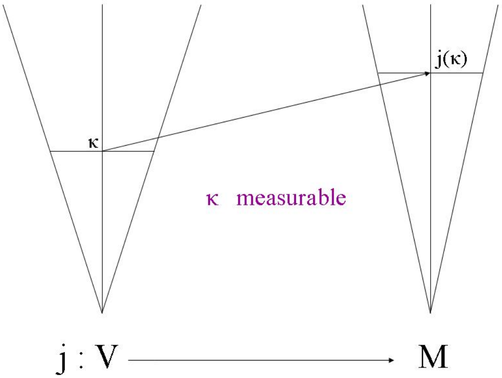
Thus, on the one hand, if one could show that, e.g., *WVP* follows from, say, the existence of (a proper class of) measurable, or supercompact, or extendible cardinals, then it could not possibly imply *VP*, because we know that *VP* is equivalent to the existence of even stronger large cardinals, i.e., $C^{(n)}$-extendible, for all $n > 0$.

On the other hand, to prove that *WVP* implies *VP* one may take an incremental approach, namely, by showing that *WVP* implies the existence of ever stronger large cardinals, up to $C^{(n)}$-extendible, for all $n > 0$.

Measurable cardinals

Definition (Ulam, Tarski-Ulam (1930), D. Scott (1961))

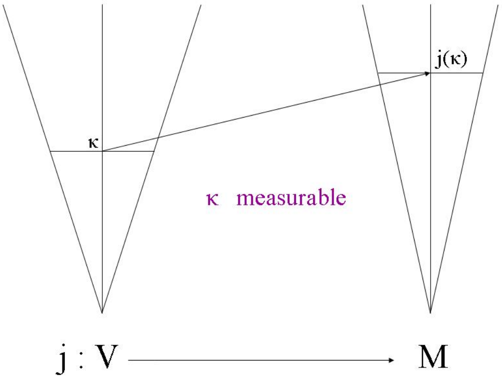
κ is measurable if there exists a κ-complete non-principal ultrafilter over κ. Equivalently: if there exists an elementary embedding $j : V \to M$, with M transitive and with critical point κ.



Measurable cardinals

Definition (Ulam, Tarski-Ulam (1930), D. Scott (1961))

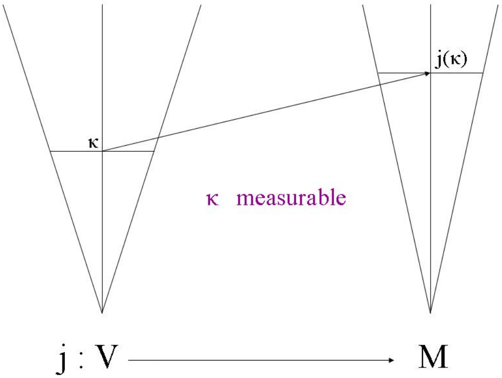
κ is measurable if there exists a κ-complete non-principal ultrafilter over κ. Equivalently: if there exists an elementary embedding $j : V \to M$, with M transitive and with critical point κ.



Measurable cardinals

Definition (Ulam, Tarski-Ulam (1930), D. Scott (1961))

κ is measurable if there exists a κ-complete non-principal ultrafilter over κ. Equivalently: if there exists an elementary embedding $j : V \to M$, with M transitive and with critical point κ.



WVP implies many measurables

Theorem (Adámek-Rosický)

WVP implies that there exists a proper class of measurable cardinals.

The key step in the proof is the following:

Theorem (Adámek-Rosický)

$\mathbf{Set}^{\mathbf{op}}$ *is bounded iff there is no proper class of measurable cardinals.*

It is then shown that if $\mathbf{Set}^{\mathbf{op}}$ is bounded, then there is a full and faithful functor $E : \mathbf{Gra}^{\mathbf{op}} \to \mathbf{Gra}$.

Thus, assuming there is no proper class of measurable cardinals, by composing a full embedding of $\mathbf{Ord}^{\mathbf{op}}$ to $\mathbf{Gra}^{\mathbf{op}}$ with E we obtain a full embedding of $\mathbf{Ord}^{\mathbf{op}}$ into $\mathbf{Gra}$, thus contradicting *WVP*.

WVP implies many measurables

Theorem (Adámek-Rosický)

WVP implies that there exists a proper class of measurable cardinals.

The key step in the proof is the following:

Theorem (Adámek-Rosický)

Set$^{\mathbf{op}}$ *is bounded iff there is no proper class of measurable cardinals.*

It is then shown that if **Set**$^{\mathbf{op}}$ is bounded, then there is a full and faithful functor $E : \mathbf{Gra}^{\mathbf{op}} \to \mathbf{Gra}$.

Thus, assuming there is no proper class of measurable cardinals, by composing a full embedding of **Ord**$^{\mathbf{op}}$ to **Gra**$^{\mathbf{op}}$ with E we obtain a full embedding of **Ord**$^{\mathbf{op}}$ into **Gra**, thus contradicting *WVP*.

WVP implies many measurables

Theorem (Adámek-Rosický)

WVP implies that there exists a proper class of measurable cardinals.

The key step in the proof is the following:

Theorem (Adámek-Rosický)

$\mathbf{Set}^{\mathbf{op}}$ *is bounded iff there is no proper class of measurable cardinals.*

It is then shown that if $\mathbf{Set}^{\mathbf{op}}$ is bounded, then there is a full and faithful functor $E : \mathbf{Gra}^{\mathbf{op}} \to \mathbf{Gra}$.

Thus, assuming there is no proper class of measurable cardinals, by composing a full embedding of $\mathbf{Ord}^{\mathbf{op}}$ to $\mathbf{Gra}^{\mathbf{op}}$ with E we obtain a full embedding of $\mathbf{Ord}^{\mathbf{op}}$ into $\mathbf{Gra}$, thus contradicting *WVP*.

WVP implies many measurables

Theorem (Adámek-Rosický)

WVP implies that there exists a proper class of measurable cardinals.

The key step in the proof is the following:

Theorem (Adámek-Rosický)

$\mathbf{Set}^{\mathbf{op}}$ *is bounded iff there is no proper class of measurable cardinals.*

It is then shown that if $\mathbf{Set}^{\mathbf{op}}$ is bounded, then there is a full and faithful functor $E : \mathbf{Gra}^{\mathbf{op}} \to \mathbf{Gra}$.

Thus, assuming there is no proper class of measurable cardinals, by composing a full embedding of $\mathbf{Ord}^{\mathbf{op}}$ to $\mathbf{Gra}^{\mathbf{op}}$ with E we obtain a full embedding of $\mathbf{Ord}^{\mathbf{op}}$ into $\mathbf{Gra}$, thus contradicting *WVP*.

A translation exercise

Suppose **Set^{op}** is bounded, i.e., there is a cardinal κ such that every cardinal λ is the canonical colimit of the category

$$Card(\kappa) := \{\mu : \mu \text{ is a cardinal smaller than } \kappa\}.$$

Each arrow $\lambda \to \mu$ may be identified with a partition q of λ into $\leqslant \mu$ many classes (i.e., with a quotient $q : \lambda \to \lambda_q$, where $|\lambda_q| \leqslant \mu$).

The canonical colimit of the set $Q_\kappa(\lambda)$ of all quotients q is the set $\lim Q_\kappa(\lambda)$ of all the tuples $\langle t_q \rangle_q$ such that $t_q \in \lambda_q$ and if q' is a coarser partition than q, then $t_q \subseteq t_{q'}$ (i.e., for every $f : \lambda_q \to \lambda_{q'}$ with $f \circ q = q'$, we have $f(t_q) = t_{q'}$).

The limit cone is formed by the projection of $\lim Q_\kappa(\lambda)$ on each coordinate q, and the canonical map $h_\lambda : \lambda \to \lim Q_\kappa(\lambda)$ is given by

$$h_\lambda(\alpha) = \langle q(\alpha) \rangle_q$$

A translation exercise

Suppose $\mathbf{Set}^{\mathbf{op}}$ is bounded, i.e., there is a cardinal κ such that every cardinal λ is the canonical colimit of the category

$$Card(\kappa) := \{\mu : \mu \text{ is a cardinal smaller than } \kappa\}.$$

Each arrow $\lambda \to \mu$ may be identified with a partition q of λ into $\leqslant \mu$ many classes (i.e., with a quotient $q : \lambda \to \lambda_q$, where $|\lambda_q| \leqslant \mu$).

The canonical colimit of the set $Q_\kappa(\lambda)$ of all quotients q is the set $\lim Q_\kappa(\lambda)$ of all the tuples $\langle t_q \rangle_q$ such that $t_q \in \lambda_q$ and if q' is a coarser partition than q, then $t_q \subseteq t_{q'}$ (i.e., for every $f : \lambda_q \to \lambda_{q'}$ with $f \circ q = q'$, we have $f(t_q) = t_{q'}$).

The limit cone is formed by the projection of $\lim Q_\kappa(\lambda)$ on each coordinate q, and the canonical map $h_\lambda : \lambda \to \lim Q_\kappa(\lambda)$ is given by

$$h_\lambda(\alpha) = \langle q(\alpha) \rangle_q$$

A translation exercise

Suppose $\mathbf{Set}^{\mathbf{op}}$ is bounded, i.e., there is a cardinal κ such that every cardinal λ is the canonical colimit of the category

$$Card(\kappa) := \{\mu : \mu \text{ is a cardinal smaller than } \kappa\}.$$

Each arrow $\lambda \to \mu$ may be identified with a partition q of λ into $\leqslant \mu$ many classes (i.e., with a quotient $q : \lambda \to \lambda_q$, where $|\lambda_q| \leqslant \mu$).

The canonical colimit of the set $Q_\kappa(\lambda)$ of all quotients q is the set $\lim Q_\kappa(\lambda)$ of all the tuples $\langle t_q \rangle_q$ such that $t_q \in \lambda_q$ and if q' is a coarser partition than q, then $t_q \subseteq t_{q'}$ (i.e., for every $f : \lambda_q \to \lambda_{q'}$ with $f \circ q = q'$, we have $f(t_q) = t_{q'}$).

The limit cone is formed by the projection of $\lim Q_\kappa(\lambda)$ on each coordinate q, and the canonical map $h_\lambda : \lambda \to \lim Q_\kappa(\lambda)$ is given by

$$h_\lambda(\alpha) = \langle q(\alpha) \rangle_q$$

A translation exercise

Suppose $\mathbf{Set}^{\mathbf{op}}$ is bounded, i.e., there is a cardinal κ such that every cardinal λ is the canonical colimit of the category

$$Card(\kappa) := \{\mu : \mu \text{ is a cardinal smaller than } \kappa\}.$$

Each arrow $\lambda \to \mu$ may be identified with a partition q of λ into $\leqslant \mu$ many classes (i.e., with a quotient $q : \lambda \to \lambda_q$, where $|\lambda_q| \leqslant \mu$).

The canonical colimit of the set $Q_\kappa(\lambda)$ of all quotients q is the set $\lim Q_\kappa(\lambda)$ of all the tuples $\langle t_q \rangle_q$ such that $t_q \in \lambda_q$ and if q' is a coarser partition than q, then $t_q \subseteq t_{q'}$ (i.e., for every $f : \lambda_q \to \lambda_{q'}$ with $f \circ q = q'$, we have $f(t_q) = t_{q'}$).

The limit cone is formed by the projection of $\lim Q_\kappa(\lambda)$ on each coordinate q, and the canonical map $h_\lambda : \lambda \to \lim Q_\kappa(\lambda)$ is given by

$$h_\lambda(\alpha) = \langle q(\alpha) \rangle_q$$

A translation exercise

Thus, the following are equivalent:

(1) **Set**$^{\mathbf{op}}$ is bounded.

(2) There is a cardinal κ such that for every $\lambda \geqslant \kappa$ the following holds:

$(*)_{\kappa,\lambda}$: For all $\langle t_q \rangle_q \in \lim Q_\kappa(\lambda)$ there exists some $\alpha < \lambda$ such that $\langle t_q \rangle_q = \langle q(\alpha) \rangle_q$.

With this, the proof of

Theorem (Adámek-Rosický)

Set$^{\mathbf{op}}$ *is bounded iff there is no proper class of measurable cardinals.*

goes as follows:

A translation exercise

Thus, the following are equivalent:

(1) **Set**$^{\mathbf{op}}$ is bounded.

(2) There is a cardinal κ such that for every $\lambda \geqslant \kappa$ the following holds:

$(*)_{\kappa,\lambda}$: For all $\langle t_q \rangle_q \in \lim Q_\kappa(\lambda)$ there exists some $\alpha < \lambda$ such that $\langle t_q \rangle_q = \langle q(\alpha) \rangle_q$.

With this, the proof of

Theorem (Adámek-Rosický)

Set$^{\mathbf{op}}$ *is bounded iff there is no proper class of measurable cardinals.*

goes as follows:

A translation exercise

Thus, the following are equivalent:

(1) **Set**$^{\mathbf{op}}$ is bounded.

(2) There is a cardinal κ such that for every $\lambda \geqslant \kappa$ the following holds:

$(*)_{\kappa,\lambda}$: For all $\langle t_q \rangle_q \in \lim Q_\kappa(\lambda)$ there exists some $\alpha < \lambda$ such that $\langle t_q \rangle_q = \langle q(\alpha) \rangle_q$.

With this, the proof of

Theorem (Adámek-Rosický)

$\mathbf{Set}^{\mathbf{op}}$ is bounded iff there is no proper class of measurable cardinals.

goes as follows:

A translation exercise

Thus, the following are equivalent:

(1) $\mathbf{Set}^{\mathbf{op}}$ is bounded.

(2) There is a cardinal κ such that for every $\lambda \geqslant \kappa$ the following holds:

$(*)_{\kappa,\lambda}$: For all $\langle t_q \rangle_q \in \lim Q_\kappa(\lambda)$ there exists some $\alpha < \lambda$ such that $\langle t_q \rangle_q = \langle q(\alpha) \rangle_q$.

With this, the proof of

Theorem (Adámek-Rosický)

$\mathbf{Set}^{\mathbf{op}}$ *is bounded iff there is no proper class of measurable cardinals.*

goes as follows:

($\Rightarrow$): Assume $(*)_{\kappa,\lambda}$ holds for all $\lambda \geqslant \kappa$. Fix some cardinal $\lambda \geqslant \kappa$ and suppose $\mathcal{U}$ is a λ-complete ultrafilter on λ. We will show it is principal, hence λ is not measurable.

For each $q \in Q_\kappa(\lambda)$, there is a unique $t_q \in \lambda_q$ with $t_q \in \mathcal{U}$.

Since $\mathcal{U}$ is a filter, $\langle t_q \rangle_q \in \lim Q_\kappa(\lambda)$.

By $(*)_{\kappa,\lambda}$, there is some $\alpha < \lambda$ such that $\langle t_q \rangle_q = \langle q(\alpha) \rangle_q$.

Letting q be the partition $\{\{\alpha\}, \lambda \setminus \{\alpha\}\}$ we have that $q(\alpha) = \{\alpha\} \in \mathcal{U}$. So, $\mathcal{U}$ is principal.

($\Rightarrow$): Assume $(*)_{\kappa,\lambda}$ holds for all $\lambda \geqslant \kappa$. Fix some cardinal $\lambda \geqslant \kappa$ and suppose $\mathcal{U}$ is a λ-complete ultrafilter on λ. We will show it is principal, hence λ is not measurable.
For each $q \in Q_\kappa(\lambda)$, there is a unique $t_q \in \lambda_q$ with $t_q \in \mathcal{U}$.
Since $\mathcal{U}$ is a filter, $\langle t_q \rangle_q \in \lim Q_\kappa(\lambda)$.
By $(*)_{\kappa,\lambda}$, there is some $\alpha < \lambda$ such that $\langle t_q \rangle_q = \langle q(\alpha) \rangle_q$.
Letting q be the partition $\{\{\alpha\}, \lambda \setminus \{\alpha\}\}$ we have that $q(\alpha) = \{\alpha\} \in \mathcal{U}$. So, $\mathcal{U}$ is principal.

($\Rightarrow$): Assume $(*)_{\kappa,\lambda}$ holds for all $\lambda \geqslant \kappa$. Fix some cardinal $\lambda \geqslant \kappa$ and suppose $\mathcal{U}$ is a λ-complete ultrafilter on λ. We will show it is principal, hence λ is not measurable.

For each $q \in Q_\kappa(\lambda)$, there is a unique $t_q \in \lambda_q$ with $t_q \in \mathcal{U}$.

Since $\mathcal{U}$ is a filter, $\langle t_q \rangle_q \in \lim Q_\kappa(\lambda)$.

By $(*)_{\kappa,\lambda}$, there is some $\alpha < \lambda$ such that $\langle t_q \rangle_q = \langle q(\alpha) \rangle_q$.

Letting q be the partition $\{\{\alpha\}, \lambda \setminus \{\alpha\}\}$ we have that $q(\alpha) = \{\alpha\} \in \mathcal{U}$. So, $\mathcal{U}$ is principal.

($\Rightarrow$): Assume $(*)_{\kappa,\lambda}$ holds for all $\lambda \geqslant \kappa$. Fix some cardinal $\lambda \geqslant \kappa$ and suppose $\mathcal{U}$ is a λ-complete ultrafilter on λ. We will show it is principal, hence λ is not measurable.

For each $q \in Q_\kappa(\lambda)$, there is a unique $t_q \in \lambda_q$ with $t_q \in \mathcal{U}$.

Since $\mathcal{U}$ is a filter, $\langle t_q \rangle_q \in \lim Q_\kappa(\lambda)$.

By $(*)_{\kappa,\lambda}$, there is some $\alpha < \lambda$ such that $\langle t_q \rangle_q = \langle q(\alpha) \rangle_q$.

Letting q be the partition $\{\{\alpha\}, \lambda \setminus \{\alpha\}\}$ we have that $q(\alpha) = \{\alpha\} \in \mathcal{U}$. So, $\mathcal{U}$ is principal.

($\Rightarrow$): Assume $(*)_{\kappa,\lambda}$ holds for all $\lambda \geqslant \kappa$. Fix some cardinal $\lambda \geqslant \kappa$ and suppose $\mathcal{U}$ is a λ-complete ultrafilter on λ. We will show it is principal, hence λ is not measurable.
For each $q \in Q_\kappa(\lambda)$, there is a unique $t_q \in \lambda_q$ with $t_q \in \mathcal{U}$.
Since $\mathcal{U}$ is a filter, $\langle t_q \rangle_q \in \lim Q_\kappa(\lambda)$.
By $(*)_{\kappa,\lambda}$, there is some $\alpha < \lambda$ such that $\langle t_q \rangle_q = \langle q(\alpha) \rangle_q$.
Letting q be the partition $\{\{\alpha\}, \lambda \setminus \{\alpha\}\}$ we have that $q(\alpha) = \{\alpha\} \in \mathcal{U}$. So, $\mathcal{U}$ is principal.

($\Leftarrow$): There exists κ such that there is no κ-complete non-principal ultrafilter on any cardinal $\lambda \geqslant \kappa$.

Given $\lambda \geqslant \kappa$, and $\langle t_q \rangle_q \in \lim Q_\kappa(\lambda)$, define $\mathcal{U} \subseteq \mathcal{P}(\lambda)$ by:

$$A \in \mathcal{U} \quad \text{iff} \quad t_q \cap A \neq \emptyset, \text{all } q.$$

It is easily checked that $\mathcal{U}$ is a κ-complete filter. So, by our assumption, $\mathcal{U}$ is principal, i.e., for some $\alpha < \lambda$, $\alpha \in t_q$, all q. So, $(*)_{\kappa,\lambda}$ holds.

($\Leftarrow$): There exists κ such that there is no κ-complete non-principal ultrafilter on any cardinal $\lambda \geqslant \kappa$.
Given $\lambda \geqslant \kappa$, and $\langle t_q \rangle_q \in \lim Q_\kappa(\lambda)$, define $\mathcal{U} \subseteq \mathcal{P}(\lambda)$ by:

$$A \in \mathcal{U} \quad \text{iff} \quad t_q \cap A \neq \emptyset, \text{all } q.$$

It is easily checked that $\mathcal{U}$ is a κ-complete filter. So, by our assumption, $\mathcal{U}$ is principal, i.e., for some $\alpha < \lambda$, $\alpha \in t_q$, all q. So, $(*)_{\kappa,\lambda}$ holds.

($\Leftarrow$): There exists κ such that there is no κ-complete non-principal ultrafilter on any cardinal $\lambda \geqslant \kappa$.
Given $\lambda \geqslant \kappa$, and $\langle t_q \rangle_q \in \lim Q_\kappa(\lambda)$, define $\mathcal{U} \subseteq \mathcal{P}(\lambda)$ by:

$$A \in \mathcal{U} \quad \text{iff} \quad t_q \cap A \neq \emptyset, \text{all } q.$$

It is easily checked that $\mathcal{U}$ is a κ-complete filter. So, by our assumption, $\mathcal{U}$ is principal, i.e., for some $\alpha < \lambda$, $\alpha \in t_q$, all q. So, $(*)_{\kappa,\lambda}$ holds.

Some bonuses

$\neg(*)_{\kappa,\kappa}$ yields a normal κ-complete ultrafilter over κ:

We first pick any $\langle t_q \rangle_q$ witnessing $\neg(*)_{\kappa,\kappa}$ and obtain a κ-complete non-principal ultrafilter $\mathcal{U}$ on κ by defining:

$$A \in \mathcal{U} \quad \text{iff} \quad t_q \cap A \neq \emptyset, \text{all } q$$

Then we take the ultrapower embedding

$$j : V \to M \cong Ult(V, \mathcal{U})$$

and define $\mathcal{U}^* \subseteq \mathcal{P}(\kappa)$ by:

$$A \in \mathcal{U}^* \quad \text{iff} \quad \kappa \in j(A)$$

Then $\mathcal{U}^*$ is a κ-complete normal ultrafilter.

Some bonuses

$\neg(*)_{\kappa,\kappa}$ yields a normal κ-complete ultrafilter over κ:
We first pick any $\langle t_q \rangle_q$ witnessing $\neg(*)_{\kappa,\kappa}$ and obtain a κ-complete non-principal ultrafilter $\mathcal{U}$ on κ by defining:

$$A \in \mathcal{U} \quad \text{iff} \quad t_q \cap A \neq \emptyset, \text{all } q$$

Then we take the ultrapower embedding

$$j : V \to M \cong Ult(V, \mathcal{U})$$

and define $\mathcal{U}^* \subseteq \mathcal{P}(\kappa)$ by:

$$A \in \mathcal{U}^* \quad \text{iff} \quad \kappa \in j(A)$$

Then $\mathcal{U}^*$ is a κ-complete normal ultrafilter.

Some bonuses

$\neg(*)_{\kappa,\kappa}$ yields a normal κ-complete ultrafilter over κ:
We first pick any $\langle t_q \rangle_q$ witnessing $\neg(*)_{\kappa,\kappa}$ and obtain a κ-complete non-principal ultrafilter $\mathcal{U}$ on κ by defining:

$$A \in \mathcal{U} \quad \text{iff} \quad t_q \cap A \neq \emptyset, \text{all } q$$

Then we take the ultrapower embedding

$$j : V \to M \cong Ult(V, \mathcal{U})$$

and define $\mathcal{U}^* \subseteq \mathcal{P}(\kappa)$ by:

$$A \in \mathcal{U}^* \quad \text{iff} \quad \kappa \in j(A)$$

Then $\mathcal{U}^*$ is a κ-complete normal ultrafilter.

Some bonuses

For each q, let t^*_q be the unique class of q that belongs to $\mathcal{U}^*$.

Then $\langle t^*_q \rangle_q \in \lim Q_\kappa(\kappa)$ and, moreover, it witnesses $\neg(*)_{\kappa,\kappa}$.

Finally, define $\mathcal{V} \subseteq \mathcal{P}(\kappa)$ by:

$$A \in \mathcal{V} \quad \text{iff} \quad t^*_q \cap A \neq \emptyset, \text{ all } q.$$

Then, $\mathcal{V} = \mathcal{U}^*$ and is thus a κ-complete, non-principal, normal ultrafilter on κ.

Some bonuses

For each q, let t_q^* be the unique class of q that belongs to $\mathcal{U}^*$.

Then $\langle t_q^* \rangle_q \in \lim Q_\kappa(\kappa)$ and, moreover, it witnesses $\neg(*)_{\kappa,\kappa}$.

Finally, define $\mathcal{V} \subseteq \mathcal{P}(\kappa)$ by:

$$A \in \mathcal{V} \quad \text{iff} \quad t_q^* \cap A \neq \emptyset, \text{ all } q.$$

Then, $\mathcal{V} = \mathcal{U}^*$ and is thus a κ-complete, non-principal, normal ultrafilter on κ.

Some bonuses

For each q, let t^*_q be the unique class of q that belongs to $\mathcal{U}^*$.

Then $\langle t^*_q \rangle_q \in \lim Q_\kappa(\kappa)$ and, moreover, it witnesses $\neg(*)_{\kappa,\kappa}$.

Finally, define $\mathcal{V} \subseteq \mathcal{P}(\kappa)$ by:

$$A \in \mathcal{V} \quad \text{iff} \quad t^*_q \cap A \neq \emptyset, \text{ all } q.$$

Then, $\mathcal{V} = \mathcal{U}^*$ and is thus a κ-complete, non-principal, normal ultrafilter on κ.

Some bonuses

Thus we obtain a new characterisation of measurable cardinals.

Corollary

κ is measurable iff $\neg(*)_{\kappa,\kappa}$ holds, i.e., iff there exists $\langle t_q \rangle_q \in \lim Q_\kappa(\kappa)$ such that $\bigcap_q t_q = \emptyset$.

From measurable to supercompact

In order to draw more large-cardinal strength from *WVP*, e.g., a proper class of supercompact cardinals, we could try to obtain a similar characterization of supercompact cardinals.

Recall:

Definition

A cardinal κ is supercompact iff for every $\lambda \geqslant \kappa$ there exists a κ-complete, non-principal, normal, and fine ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa(\lambda)$.

$\mathcal{U}$ is fine if for every $\alpha < \lambda$ the set $\{x \in \mathcal{P}_\kappa(\lambda) : \alpha \in x\}$ belongs to $\mathcal{U}$.

From measurable to supercompact

In order to draw more large-cardinal strength from *WVP*, e.g., a proper class of supercompact cardinals, we could try to obtain a similar characterization of supercompact cardinals.

Recall:

Definition

A cardinal κ is supercompact iff for every $\lambda \geqslant \kappa$ there exists a κ-complete, non-principal, normal, and fine ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa(\lambda)$.

$\mathcal{U}$ is fine if for every $\alpha < \lambda$ the set $\{x \in \mathcal{P}_\kappa(\lambda) : \alpha \in x\}$ belongs to $\mathcal{U}$.

From measurable to supercompact

In order to draw more large-cardinal strength from *WVP*, e.g., a proper class of supercompact cardinals, we could try to obtain a similar characterization of supercompact cardinals.

Recall:

Definition

A cardinal κ is supercompact iff for every $\lambda \geqslant \kappa$ there exists a κ-complete, non-principal, normal, and fine ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa(\lambda)$.

$\mathcal{U}$ is fine if for every $\alpha < \lambda$ the set $\{x \in \mathcal{P}_\kappa(\lambda) : \alpha \in x\}$ belongs to $\mathcal{U}$.

Characterizing supercompactness

Now, for $\lambda \geqslant \kappa$, consider the following principle:

$(**)_{\kappa,\lambda}$: For all $\langle t_q \rangle_q \in \lim Q_\kappa(\mathcal{P}_\kappa(\lambda))$ either there exists some $x \in \mathcal{P}_\kappa(\lambda)$ such that $\langle t_q \rangle_q = \langle q(x) \rangle_q$, or there exist q such that $\bigcup t_q \neq \lambda$.

Proposition

κ is supercompact iff $\neg(**)_{\kappa,\lambda}$ holds for every $\lambda \geqslant \kappa$.
i.e., iff for all $\lambda \geqslant \kappa$, there exists $\langle t_q \rangle_q \in \lim Q_\kappa(\mathcal{P}_\kappa(\lambda))$ such that $\bigcup t_q = \lambda$ for all q, and $\bigcap_q t_q = \emptyset$.

Characterizing supercompactness

Now, for $\lambda \geqslant \kappa$, consider the following principle:

$(**)_{\kappa,\lambda}$: For all $\langle t_q \rangle_q \in \lim Q_\kappa(\mathcal{P}_\kappa(\lambda))$ either there exists some $x \in \mathcal{P}_\kappa(\lambda)$ such that $\langle t_q \rangle_q = \langle q(x) \rangle_q$, or there exist q such that $\bigcup t_q \neq \lambda$.

Proposition

κ is supercompact iff $\neg(**)_{\kappa,\lambda}$ holds for every $\lambda \geqslant \kappa$.

i.e., iff for all $\lambda \geqslant \kappa$, there exists $\langle t_q \rangle_q \in \lim Q_\kappa(\mathcal{P}_\kappa(\lambda))$ such that $\bigcup t_q = \lambda$ for all q, and $\bigcap_q t_q = \emptyset$.

Characterizing supercompactness

Now, for $\lambda \geqslant \kappa$, consider the following principle:

$(**)_{\kappa,\lambda}$: For all $\langle t_q \rangle_q \in \lim Q_\kappa(\mathcal{P}_\kappa(\lambda))$ either there exists some $x \in \mathcal{P}_\kappa(\lambda)$ such that $\langle t_q \rangle_q = \langle q(x) \rangle_q$, or there exist q such that $\bigcup t_q \neq \lambda$.

Proposition

κ is supercompact iff $\neg(**)_{\kappa,\lambda}$ holds for every $\lambda \geqslant \kappa$.
i.e., iff for all $\lambda \geqslant \kappa$, there exists $\langle t_q \rangle_q \in \lim Q_\kappa(\mathcal{P}_\kappa(\lambda))$ such that $\bigcup t_q = \lambda$ for all q, and $\bigcap_q t_q = \emptyset$.

Translating back

Recall:

Theorem (Adámek-Rosický)

$\mathbf{Set^{op}}$ *is bounded iff there is no proper class of measurable cardinals.*

which can now be rephrased as:

Theorem

$\mathbf{Set^{op}}$ *is bounded iff* $(*)_{\kappa,\kappa}$ *holds for a tail of cardinals* κ.

Translating back

Let us write $(**)_\kappa$ for "$(**)_{\kappa,\lambda}$ holds for all $\lambda \geqslant \kappa$".

Question

Is there some (accessible) category $\mathcal{K}$ such that $\mathcal{K}^{op}$ is bounded iff $(**)_\kappa$ holds for a tail of cardinals κ?

Translating back

Let us write $(**)_\kappa$ for "$(**)_{\kappa,\lambda}$ holds for all $\lambda \geqslant \kappa$".

Question

Is there some (accessible) category $\mathcal{K}$ such that $\mathcal{K}^{op}$ is bounded iff $(**)_\kappa$ holds for a tail of cardinals κ?

Characterizing strong compactness

Similar considerations apply to strong compactness. Namely, for $\lambda \leqslant \kappa$, consider the following principle:

$(***)_{\kappa,\lambda}$: For some X and some κ-closed $\pi \subseteq Q_\lambda(X)$, there exists some $\langle t_q \rangle_q \in \lim \pi$ that does not extend to any $\langle t_q \rangle_q \in \lim Q_\lambda(X)$.

Proposition

κ is λ-strongly compact iff $\neg(***)_{\kappa,\lambda}$ holds.
i.e., iff for all X and all κ-closed $\pi \subseteq Q_\lambda(X)$, every $\langle t_q \rangle_q \in \lim \pi$ extends to some $\langle t_q \rangle_q \in \lim Q_\lambda(X)$.

Characterizing strong compactness

Similar considerations apply to strong compactness. Namely, for $\lambda \leqslant \kappa$, consider the following principle:

$(***)_{\kappa,\lambda}$: For some X and some κ-closed $\pi \subseteq Q_\lambda(X)$, there exists some $\langle t_q \rangle_q \in \lim \pi$ that does not extend to any $\langle t_q \rangle_q \in \lim Q_\lambda(X)$.

Proposition

κ is λ-strongly compact iff $\neg(***)_{\kappa,\lambda}$ holds.

i.e., iff for all X and all κ-closed $\pi \subseteq Q_\lambda(X)$, every $\langle t_q \rangle_q \in \lim \pi$ extends to some $\langle t_q \rangle_q \in \lim Q_\lambda(X)$.

Characterizing strong compactness

Similar considerations apply to strong compactness. Namely, for $\lambda \leqslant \kappa$, consider the following principle:

$(***)_{\kappa,\lambda}$: For some X and some κ-closed $\pi \subseteq Q_\lambda(X)$, there exists some $\langle t_q \rangle_q \in \lim \pi$ that does not extend to any $\langle t_q \rangle_q \in \lim Q_\lambda(X)$.

Proposition

κ is λ-strongly compact iff $\neg(***)_{\kappa,\lambda}$ holds.
i.e., iff for all X and all κ-closed $\pi \subseteq Q_\lambda(X)$, every $\langle t_q \rangle_q \in \lim \pi$ extends to some $\langle t_q \rangle_q \in \lim Q_\lambda(X)$.

Strong compactness

So, we can ask:

Question

Is there some (accessible) category $\mathcal{K}$ such that $\mathcal{K}^{op}$ is bounded iff $(***)_{\kappa,\lambda}$ holds for a tail of cardinals κ?

If the answer to any of the two last questions is yes, then (there is a good chance, depending on the nature of $\mathcal{K}$ that) the *WVP* implies the existence of a proper class of supercompact cardinals, or a proper class of λ-strongly compact cardinals, respectively.

Strong compactness

So, we can ask:

Question

Is there some (accessible) category $\mathcal{K}$ such that $\mathcal{K}^{op}$ is bounded iff $(***)_{\kappa,\lambda}$ holds for a tail of cardinals κ?

If the answer to any of the two last questions is yes, then (there is a good chance, depending on the nature of $\mathcal{K}$ that) the *WVP* implies the existence of a proper class of supercompact cardinals, or a proper class of λ-strongly compact cardinals, respectively.

A (perhaps unreasonable) conjecture

Recall Jensen's Global $\square$ principle:

There is a class sequence $\langle C_\alpha : \alpha \in Sing \rangle$ such that:

1. $C_\alpha \subseteq \alpha$ is club.
2. $ot(C_\alpha) < \alpha$
3. If β is a limit point of C_α, then $C_\alpha \cap \beta = C_\beta$.

Conjecture

Global $\square$ implies the negation of WVP.

A (perhaps unreasonable) conjecture

Recall Jensen's Global $\square$ principle:

There is a class sequence $\langle C_\alpha : \alpha \in Sing \rangle$ such that:

1. $C_\alpha \subseteq \alpha$ is club.
2. $ot(C_\alpha) < \alpha$
3. If β is a limit point of C_α, then $C_\alpha \cap \beta = C_\beta$.

Conjecture

Global $\square$ implies the negation of WVP.

A (perhaps unreasonable) conjecture

Recall Jensen's Global $\square$ principle:

There is a class sequence $\langle C_\alpha : \alpha \in Sing \rangle$ such that:

1. $C_\alpha \subseteq \alpha$ is club.
2. $ot(C_\alpha) < \alpha$
3. If β is a limit point of C_α, then $C_\alpha \cap \beta = C_\beta$.

Conjecture

Global $\square$ implies the negation of WVP.