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# ADMISSIBLE STRUCTURES

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# 1. PRELIMINARIES

Let $\sigma \supseteq \{\mathrm{U}, \in, \varnothing\}$ be a fixing signature where $\mathrm{U}, \in$ , $\varnothing$ are interpreted in models as urelements, membership relation and empty set respectively. As usual,

$\exists y \in x\varphi$ and $\forall y \in x\varphi$ are shortenings of $\exists y((y \in x) \wedge \varphi)$ and $\forall y((y \in x) \rightarrow \varphi)$ respectively, where $\varphi$ is a formula in signature $\sigma$.

*Axioms of* $\mathrm{KPU}_\sigma$ :

**1. existence of empty set:**
$(\neg \mathrm{U}(\varnothing) \wedge \forall y \in \varnothing \neg(y = y))$;

**2. extensionality:**
$\forall x \forall y((\neg \mathrm{U}(x) \wedge \neg \mathrm{U}(y)) \rightarrow (\forall z((z \in x) \leftrightarrow (z \in y)) \rightarrow (x = y)))$;

**3. pair:**
$\forall x \forall y \exists z((x \in z) \wedge (y \in z))$;

**4. union:**
$\forall x \exists y \forall z \in x \forall t \in z(t \in y)$;

**5. foundation** (for every formula $\varphi(x, z_1, \ldots, z_n)$ in $\sigma$)**:**
$\forall z_1 \ldots \forall z_n(\exists x \varphi(x, z_1, \ldots, z_n) \rightarrow (\exists x \varphi(x, z_1, \ldots, z_n) \wedge \forall y \in x \neg \varphi(y, z_1, \ldots, z_n)))$;

**6. set structure:**
$\forall x(\exists y(y \in x) \to \neg \mathrm{U}(x))$;

A class of $\Delta_0$ *formulas* is the least class containing atomic formulas and closed under $\wedge$, $\vee$, $\to$, $\neg$ and $\forall y \in x$, $\exists y \in x$.

**7. $\Delta_0$ separation** (for every $\Delta_0$ formula $\varphi(x, z_1, \ldots, z_n)$ in $\sigma$ which doesn't contain $y$ free)**:**
$\forall z_1 \ldots \forall z_n \forall x \exists y((z \in y) \leftrightarrow ((z \in x) \wedge \varphi))$;

**8. $\Delta_0$ collection** (for every $\Delta_0$ formula $\varphi(x, z_1, \ldots, z_n)$ in $\sigma$ which doesn't contain $y$ free):
$\forall z_1 \ldots \forall z_n(\forall z \in x \exists u \varphi \to \exists y \forall z \in x \exists u \in y \varphi)$;

Further, we will consider models in some finite signature only.

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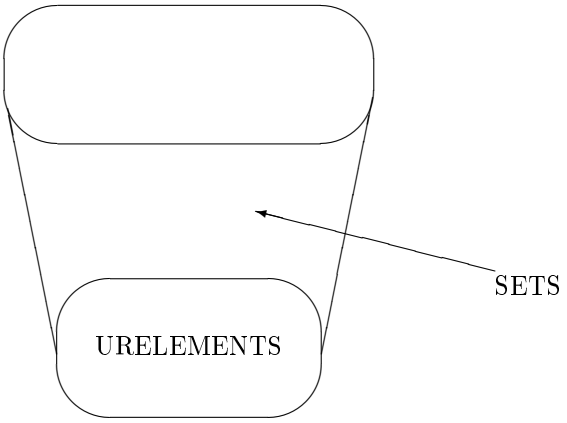


Ordinals in models of KPU are defined as in the set theory. A set $a$ is called *transitive* iff $\forall y \in a \forall z \in y(z \in a)$. A transitive set $a$ is called an *ordinal* if it consists of transitive sets.

An *ordinal* or a *height* of a model $\mathfrak{A}_{\mathfrak{M}}$ of KPU is collection of all ordinals of this model (we denote this collection as $\mathrm{Ord}(\mathfrak{A}_{\mathfrak{M}})$). A model $\mathfrak{A}_{\mathfrak{M}}$ of KPU is called an *admissible set* if $\mathrm{Ord}(\mathfrak{A}_{\mathfrak{M}})$ is standard i.e. a well ordered set.

**Examples.**

1. Every model of the theory ZF can be considered as a model of KPU.

2. Let $\mathbb{HYP}(\mathfrak{N})$ be the least by inclusion admissible set which contains a structure $\mathfrak{N}$ as an element. E.g if $\mathfrak{N}$ is the standard model of arithmetics then $\mathrm{Ord}(\mathbb{HYP}(\mathfrak{N})) = \omega_1^{CK}$. Moreover any countable admissible ordinal greater than $\omega$ is represented as
$\mathrm{Ord}(\mathbb{HYP}(\mathfrak{N}, P))$ for a proper $P \subseteq \omega$ where $\mathfrak{N}$ is the standard model of arithmetics.

3. Let $M$ be an arbitrary collection of urelements with given structure $\mathfrak{M}$ on it and let $HF(M)$ be the least set which is closed under $\cup$, $x \mapsto \{x\}$ and contains $M \cup \{\varnothing\}$ as a subset. Then $\mathbb{HF}(\mathfrak{M})$ is the least admissible set over $\mathfrak{M}$. It is called a *hereditarily finite set* over $\mathfrak{M}$. Notice that $\mathrm{Ord}(\mathbb{HF}(\mathfrak{M})) = \omega$.

Constructive representation: $\mathrm{HF}_0(M) = M$; $\mathrm{HF}_{n+1}(M) = HF_n(M) \cup \mathcal{P}_{fin}(\mathrm{HF}_n(M))$; $\mathrm{HF}(M) = \cup_{n<\omega}\mathrm{HF}_n(M)$.

4. Let $D$ be a nonprincipal ultrafilter over $\omega$. Then ${}^{(\mathbb{HF}(\varnothing))^{\omega}}\!\big/_{D}$ is a model of KPU such that it has continuum many natural ordinals and hence it is not admissible set.

5. Let $D$ be a nonprincipal ultrafilter over $\omega$ and let $\mathbb{HF}(S_n)$ be hereditarily finite set over $S_n$ in the empty language having $n$ elements where $n \in \omega$. Then $\prod\limits_{D} \mathbb{HF}(S_n)$ is a model of KPU such that it has continuum many natural ordinals too. However $\mathrm{Th}(\prod\limits_{D} \mathbb{HF}(S_n))$ has no admissible set.

$\Sigma_n$ *formulas* have form $\exists x_1 \forall x_2 \ldots Q x_n \varphi$;
$\Pi_n$ *formulas* have form $\forall x_1 \exists x_2 \ldots Q x_n \varphi$,
where $\varphi$ is $\Delta_0$ in both cases.

Class of $\Sigma$ *formulas* is the least one containing $\Delta_0$ formulas and is closed under $\vee$, $\wedge$, $\forall x \in y$, $\exists x \in y$ and $\exists x$.

Let $\mathbb{A}$ be an admissible set and let $R$ be a relation on $\mathrm{dom}(\mathbb{A})$. We say that $R \in \Sigma_n^{\mathbb{A}}$ ($R \in \Pi_n^{\mathbb{A}}$) if $R$ is definable by some $\Sigma_n(\Pi_n)$ formula in $\mathbb{A}$ (possibly with parameters). We say that $R \in \Delta_n^{\mathbb{A}}$ if $R \in \Pi_n^{\mathbb{A}} \cap \Sigma_n^{\mathbb{A}}$.

$R$ is called $\mathbb{A}$*-computably enumerable* ($\mathbb{A}$-c.e.) if $R \in \Sigma_1^{\mathbb{A}}$. $R$ is called $\mathbb{A}$*-computable* ($\mathbb{A}$-c.) if $R \in \Delta_1^{\mathbb{A}}$. A partial function $F$ is called $\mathbb{A}$*-computable* if its graph $\Gamma_F$ is $\mathbb{A}$-computable.

$\omega \subseteq \mathrm{Ord}(\mathbb{A})$ and $\omega$ is $\mathbb{A}$-c., for any admissible set $\mathbb{A}$.

**$\Sigma$ Reflection Principle.** Class of all $\Sigma$ predicates coincide with class of all $\mathbb{A}$-computably enumerable sets, for any KPU-model $\mathbb{A}$.

**Proposition 1.** Let $\mathbb{A}$ be a KPU-model. Then $B$ is $\mathbb{A}$-c.e. iff $B = \varnothing$ or there exists a total $\mathbb{A}$-c. function $F$ with range $B$.

*Proof.* Let $B \neq \varnothing$ be definable by $\Sigma_1$ formula $\exists u \Phi_0(x, u)$ and let $b_0 \in B$.

$$F(x) = \begin{cases} b & \text{if } x = \langle b, u \rangle \text{ for some } u,\ \mathbb{A} \models \Phi_0(b, u); \\ b_0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{A}$ be KPU-model. Define two topologies on $\mathcal{P}(\mathrm{dom}(\mathbb{A}))$.

*Strong topology* $\tau_s$ is determined by open basis consisting of sets of kind $V_a = \{M \mid M \subseteq \mathrm{dom}(\mathbb{A}), a \subseteq M\}$, $a \in \mathrm{dom}(\mathbb{A}) \setminus U(\mathbb{A})$.

*Weak topology* $\tau_w$ is determined by open pre-basis consisting of sets of kind $V_{\{a\}}$, $a \in \mathrm{dom}(\mathbb{A})$.

A continuous map $F : \langle \mathcal{P}(\mathrm{dom}(\mathbb{A})), \tau_s \rangle \to \langle \mathcal{P}(\mathrm{dom}(\mathbb{A})), \tau_w \rangle$ is called a *weakly continuous operator*.

Every weakly continuous operator is monotonic so it has the least fixed point. This point can be found as follows: $\Gamma_0 = \varnothing$, $\Gamma_{\alpha+1} = F(\Gamma_\alpha)$, $\Gamma_\eta = \bigcup_{\beta<\alpha} \Gamma_\beta$ if $\eta$ is limit.

A weakly continuous operator $F$ is called $\Sigma$ *operator* if $\Gamma^*_F = \{\langle a, b\rangle \mid a \in \mathrm{dom}(\mathbb{A}) \setminus U(\mathbb{A}),\ b \in F(a)\}$ is $\mathbb{A}$-c.e.

**Gandy Theorem** Let $\mathbb{A}$ be an admissible set and let $F$ be $\Sigma$ operator. Then the least fixed point $\Gamma_*$ of $F$ is $\mathbb{A}$-c.e. Moreover, $\Gamma_* = \Gamma_{\mathrm{Ord}(\mathbb{A})}$.

Let $\Psi(x, P^+)$ be $\Sigma$ formula and let $F_\Psi(M) = \{b \mid \langle \mathbb{A}, M\rangle \models \Psi(b)\}$, for any subset $M$ of admissible set $\mathbb{A}$. Then $F_\Psi$ is $\Sigma$ operator.

**Proposition.** Let $\mathbb{A}$ be an admissible set over $\mathfrak{M}$. Then $\mathrm{HF}(\mathrm{dom}(\mathfrak{M}))$ is $\mathbb{A}$-c.e. subset.

*Proof.* Let $\Psi(x, P^+)$ be $U(x) \vee \exists y \exists z (P(y) \wedge P(z) \wedge ((x = \{y\}) \vee (x = y \cup z)))$.

**Proposition.** Let $\mathbb{A}$ be an admissible set and let $M$ be $\mathbb{A}$-c. Then $M^{<\omega}$ is also $\mathbb{A}$-c.

*Proof.* Let $\Psi_0(x, P^+)$ be $(x \in M) \vee \exists y \exists z ((x = \langle y, z \rangle) \wedge P(y) \wedge (z \in M))$ and let $\Psi_1(x, Q^+)$ be $(x \notin M) \wedge ((x \text{ is not ordered pair }) \vee \forall y \in \mathrm{TC}(x) \forall z \in \mathrm{TC}(x) ((x = \langle y, z \rangle) \rightarrow Q(y) \vee (z \notin M)))$.

**Proposition.** Let $\mathbb{A}$ be an admissible set. Then $\mathrm{HF}(\varnothing)$ is $\mathbb{A}$-c. subset, $\mathbb{HF}(\varnothing) \leqslant_{\mathrm{end}} \mathbb{A}$ and any $\mathbb{HF}(\varnothing)$-c. subset is $\mathbb{A}$-c.

Let $\mathbb{A}$ be KPU-model.

**Satisfaction Theorem.** There exists binary $\mathbb{A}$-c.e. predicate $\mathrm{Tr}_\Sigma$ such that for any $n \in \omega \subseteq \mathrm{Nat}(\mathbb{A})$ and $a \in \mathrm{dom}(\mathbb{A})^{<\omega}$,

$\langle n, a\rangle \in \mathrm{Tr}_\Sigma \Leftrightarrow n$ goedel number of $\Sigma$ formula $\Phi, \mathbb{A} \models \Phi(a)$

**Parametrization Theorem.** There exists $n+1$-ary $\mathbb{A}$-c.e. predicate $T(e, x_1, \ldots, x_n)$ which parametrizes the class of $n$-ary predicates, whith indices from $\mathrm{dom}(\mathbb{A})$.

**Proposition.** Let $\mathbb{HF}(\mathfrak{M})$ is a hereditarily finite sets over $\mathfrak{M}$. Then there exists $\Sigma$ function Term with $\mathbb{A}$-c. domain $\subseteq \mathrm{Ord}(\mathbb{HF}(\mathfrak{M})) \times \mathrm{dom}(\mathfrak{M})^{<\omega}$ and with range $\mathrm{dom}(\mathbb{HF}(\mathfrak{M}))$ with the following properties:

1. if $x = \mathrm{Term}(n, \overline{a})$ then all coordinates in $\overline{a}$ are distinct and $\mathrm{sp}(x) = \mathrm{sp}(\overline{a})$;
2. it is single-valued under first coordinate;
3. pre-image on second coordinate is finite;
4. there exists a strongly computable sequence of finite groups $S_n$ such that $\mathrm{Term}(n, \overline{a}) = \mathrm{Term}(n, \overline{b}) \Leftrightarrow \exists \pi \in S_n[\pi(\overline{a}) = \overline{b}]$;
5. there exists a strongly computable sequence of finite sets $A_n$ such that $k \in A_n \Leftrightarrow \exists \overline{a} \exists \overline{b} \subseteq \overline{a}[\mathrm{Term}(k, \overline{b}) \in \mathrm{Term}(n, \overline{a})]$.

Moreover, $\Sigma$ formula defining Term is independent of $\mathbb{HF}(\mathfrak{M})$.

**Theorem.** Let $\mathbb{HF}(\mathfrak{M})$ be a hereditarily finite set over $\mathfrak{M}$ in some finite signature. Then $C$ is definable by $\Sigma$ formula $\Phi$ with parameters $\overline{s}$ iff $C = \{\mathrm{Term}(n, \overline{u}) \mid \mathfrak{M} \models \varphi(\overline{u}\,\overline{s}),$ for some $\varphi \in A_n^{\Phi}, n \in \omega, \overline{u} \in \mathrm{dom}(\mathfrak{M})^{<\omega}\}$ where $A_n^{\Phi}$ is a computable sequence of c.e. sets consisting of existentional formulas of signature of $\mathfrak{M}$. Moreover, such a sequence determines some $\mathbb{HF}(\mathfrak{M})$-c.e. subset.

**Reduction Principle.** If $\mathfrak{M}$ is a structure of decidable model complete theory in a finite signature then $\mathbb{HF}(\mathfrak{M})$ satisfies the reduction property.

*Proof.* Let $B_n$, $C_n$ determine $\mathbb{HF}(\mathfrak{M})$-c.e. subsets as in Theorem and let $B_{n,s}$, $C_{n,s}$ be their strongly computable approximations such that on every step at most one new element appeared. Then $B'_{n,0} = C'_{n,0} = \varnothing$; $B'_{n,s+1} = B'_{n,s} \cup \{\varphi_0 \wedge e(\bigwedge_{\psi \in B'_{n,s} \cup C'_{n,s}} \bigwedge_{\pi \in S_n} \neg \psi^{\pi}) \mid \varphi_0 \in B_{n,s+1} \setminus B_{n,s}\}$; $C'_{n,s+1} = C'_{n,s} \cup \{\varphi_0 \wedge e(\bigwedge_{\psi \in B'_{n,s+1} \cup C'_{n,s}} \bigwedge_{\pi \in S_n} \neg \psi^{\pi}) \mid \varphi_0 \in C_{n,s+1} \setminus C_{n,s}\}$.

We say that a structure $\mathfrak{M}$ of a signature $\sigma$ *has $\Sigma$-definable Skolem functions* if for any formula $\varphi(\overline{y}, x)$, there exists a $\mathbb{HF}(\mathfrak{M})$-c. function $F_\varphi$ such that $\mathbb{HF}(\mathfrak{M}) \models \exists x \varphi(\overline{y}, x) \to \varphi(\overline{y}, F_\varphi(\overline{y}))$ and it can be found effectively from $\varphi$.

**Uniformization Principle (A.I.Stukachev).** Let $\mathfrak{M}$ be a structure of decidable model complete theory. Then $\mathbb{HF}(\mathfrak{M})$ satisfies the uniformization property iff some inessential expansion of $\mathfrak{M}$ has $\Sigma$-definable Skolem functions.

*Proof.* (1) For any formula $\varphi(y_1, \ldots, y_m, x_1, \ldots, x_n)$, there exist $\mathbb{A}$-c. functions $F_1, \ldots, F_n$ such that $\mathbb{HF}(\mathfrak{M}) \models \exists \overline{x} \varphi(\overline{y}, \overline{x}) \to \varphi(\overline{y}, F_1(\overline{y}), \ldots, F_n(\overline{y}))$. Moreover, they can be found effectively from $\varphi$.

(2) Let $A_n$ determine $\mathbb{HF}(\mathfrak{M})$-c.e. predicate $Q(x,y)$ and let $A_{n,s}$ be its strongly computable approximation such that on every step at most one new element is appeared in it. We also propose that if $n > s$ then $A_{n,s} = \varnothing$. Let $f$ be computable function such that if $\langle x, y\rangle = \mathrm{Term}(n, \overline{u})$ for some $\overline{u}$ then $x = \mathrm{Term}(f(n), \overline{v})$ for some $\overline{v}$. Then $A'_{n,0} = \varnothing$, $A'_{n,s+1} = A'_{n,s} \cup \{\langle F_1(\overline{y}), \ldots, F_n(\overline{y})\rangle \mid F_i$ Skolem function for $\varphi_0 \wedge \bigwedge_{\psi \in A_{n',s}, f(n)=f(n')} \bigwedge_{\pi \in S_{f(n)}} \neg \exists \overline{x} \psi^{\pi}(\overline{y}, \overline{x}), \varphi_0 \in A_{n,s+1} \setminus A_{n,s}\}$.

(3) Conversely, it suffice to uniformize $\mathbb{HF}(\mathfrak{M})$-c.e. predicate $\{\langle\langle \varphi, \overline{y}\rangle, x\rangle \mid \varphi$ goedel number of some formula, $\mathfrak{M} \models \varphi(\overline{y}, x)\}$.

**Theorem** Let $\mathbb{A}$ be an admissible set. Then the following holds:

$$\begin{tikzcd}
 & \Sigma_1^{\mathbb{A}} \arrow[dr, hook] & & \Sigma_2^{\mathbb{A}} \arrow[dr, hook] & \cdots \\
\Delta_1^{\mathbb{A}} \arrow[ur, hook] \arrow[dr, hook] & & \Delta_2^{\mathbb{A}} \arrow[ur, hook] \arrow[dr, hook] & & \\
 & \Pi_1^{\mathbb{A}} \arrow[ur, hook] & & \Pi_2^{\mathbb{A}} \arrow[ur, hook] & \cdots
\end{tikzcd}$$

Moreover, all the inclusions are strong.

**Proposition** Let $\mathbb{A}$ be an admissible set. Then the following conditions hold for every $m \geqslant 1$:

1. given $n \geqslant 1$ there exists a $n+1$-ary $\Sigma_m^{\mathbb{A}}$ predicate which is universal for class of all $n$-ary $\Sigma_m^{\mathbb{A}}$ predicates;

2. given $n \geqslant 1$ there exists a $n+1$-ary $\Pi_m^{\mathbb{A}}$ predicate which is universal for class of all $n$-ary $\Pi_m^{\mathbb{A}}$ predicates;

3. given $n \geqslant 1$ there is no $n+1$-ary $\Delta_m^{\mathbb{A}}$ predicate which is universal for class of all $n$-ary $\Delta_m^{\mathbb{A}}$ predicates;

4. $R(x_1, \ldots, x_n) \in \Sigma_m^{\mathbb{A}}$ iff there exists

$$Q(y, y_1, \ldots, y_n) \in \Delta_m^{\mathbb{A}}$$

that $R = \{\langle a_1, \ldots, a_n \rangle \mid \exists y\, Q(y, a_1, \ldots, a_n)\}$, $n \geqslant 1$;

5. $R(x_1, \ldots, x_n) \in \Pi_m^{\mathbb{A}}$ iff there exists

$$Q(y, y_1, \ldots, y_n) \in \Delta_m^{\mathbb{A}}$$

that $R = \{\langle a_1, \ldots, a_n \rangle \mid \forall y\, Q(y, a_1, \ldots, a_n)\}$, $n \geqslant 1$.

6. $\Sigma_m^{\mathbb{A}}$ is closed under $\wedge$, $\vee$, $\exists x$, $\exists x \in a$ and is not closed under $\neg$;

7. $\Pi_m^{\mathbb{A}}$ is closed under $\wedge$, $\vee$, $\forall x$, $\forall x \in a$ and is not closed under $\neg$;

8. $\Delta_m^{\mathbb{A}}$ is closed under $\wedge$, $\vee$, $\neg$.

**Examples.**

1. Let $\mathbb{A}$ be an admissible set. Then $\Sigma_1^{\mathbb{A}}$ is closed under $\forall x \in a$, by $\Sigma$ reflection principle.

2. Let $\mathbb{A}$ be an arbitrary hereditarily finite set. Then $\Sigma_n^{\mathbb{A}}$ is closed under $\forall x \in a$.

3. Closure of $\Sigma_2^{\mathbb{HYP}(\mathfrak{N})}$ under $\forall x \in a$, $\vee$, $\wedge$, $\exists x$ coincides with $\bigcup_n \Sigma_n^{\mathbb{HYP}(\mathfrak{N})}$ when $\mathfrak{N}$ is the standard model of arithmetics.

| **Admissible sets theory** | | |
|---|---|---|
| **Definability** | **Model theory** | **Computability** |
| $L_\alpha$ | $\omega$-logic, admissible fragments | Representability, computable principles, properties of maps |

## *History*

1965. A. Levy proposed notions of $\Delta_0$ and $\Sigma_1$ formulas.

1964. S. Kripke proposed axioms which are alike to KP.

1966. R. Platek proposed axioms which are alike to KP.

1968. C. Karp proposed to consider nontransistive admissible sets.

1974. J. Barwise proposed to consider admissible sets with urelements.

1975. The book of J. Barwise "Admissible sets and structures: An Approach to Definability Theory" was published.

1983. Yu. Ershov proposed dynamic logic on admissible sets.

1985. S. Goncharov, Yu. Ershov, D. Sviridenko proposed semantic programming which is based on $\Sigma$ formula.

1987. V. Rudnev built examples of admissible sets $\{\mathbb{A}_{\mathcal{I}}\}_{\mathcal{I}\in\mathcal{J}(L_T)}$ that class of Turing degrees of all $\mathbb{A}_{\mathcal{I}}$-computable subsets of $\omega$ coincides with $\mathcal{I}$ where $\mathcal{I}$ is an arbitrary ideal of Turing degrees.

1989-... R. Vaitsenavichyus, M. Korovina, I. Ashaev, V. Belyaev, A. Myasnikov proposed the method of representation of $\Sigma$ predicates on hereditarily finite set over $\mathfrak{M}$ as computable disjunction of $\exists$ formulas in the language of $\mathfrak{M}$.

1996. The book of Yu. Ershov "Definability and computability" was published (the second edition was published in 2000).

1996. M. Korovina, O. Kudinov proposed an approach to computability on reals as $\Sigma$-definability on proper hereditarily finite sets (this approach is equivalent to Grzegorczyk one).

Let $A, B \subseteq \omega$. We say that $A \leqslant_e B$ iff

$$\forall n \in \omega(n \in A \Leftrightarrow \exists u[\langle m, u\rangle \in W \& (D_u \subseteq B)])$$

for some c.e. $W$ where $D_u$ is the canonical numbering of finite subsets of $\omega$.

$A \equiv_e B \leftrightharpoons (A \leqslant_e B \& B \leqslant_e A)$.

$L_e \leftrightharpoons {}^{\mathcal{P}(\omega)}\!/_{\equiv_e}$. $\langle L_e, \leqslant\rangle$ is an upper semilattice with the least element. Moreover, $d_e(A) \sqcup d_e(B) = d_e(A \oplus B)$ where

$$A \oplus B \leftrightharpoons \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}.$$

$\mathcal{I} \subseteq L_e$ is called an *ideal* if the following conditions hold:
- $\mathbf{0} \in \mathcal{I}$;
- $\mathbf{a}, \mathbf{b} \in \mathcal{I} \Rightarrow \mathbf{a} \sqcup \mathbf{b} \in \mathcal{I}$;
- $\mathbf{b} \in \mathcal{I}, \mathbf{a} \leqslant \mathbf{b} \Rightarrow \mathbf{a} \in \mathcal{I}$.

Let $X \subseteq L_e$. Then $\langle X \rangle$ is an ideal generated by $X$. An ideal is called *principal* if it is generated by one element. We denote $\widehat{\mathbf{x}}$ for $\langle \{\mathbf{x}\} \rangle$.

**Theorem 2 (A. Morozov, V. Puzarenko; 2002)**

1. Given an admissible set $\mathbb{A}$, the collection of all $\mathbb{A}$-c.e. subsets of $\omega$ is closed under $\oplus$ and $e$-reducibility.
2. Given an ideal $\mathcal{I}$, there exists an admissible set $\mathbb{A}_{\mathcal{I}}$ such that the collection of $e$-degrees of all $\mathbb{A}_{\mathcal{I}}$-c.e. subsets of $\omega$ is equal to $\mathcal{I}$.

Let $\mathcal{I}_e(\mathbb{A})$ be the collection of $e$-degrees of all $\mathbb{A}$-c.e. subsets of $\omega$.

$\mathrm{Th}_\Sigma(\mathbb{A}) \leftrightharpoons \{\Phi \mid \Phi \text{ is } \Sigma_1 \text{ sentence}, \mathbb{A} \models \Phi\}$.

**Proposition (2004)** For $B \subseteq \omega$, the following conditions are equivalent:

1. $B$ is $\mathbb{A}$-c.e.;
2. $B \leqslant_m \mathrm{Th}_\Sigma(\mathbb{A}, a)$ for some $a \in \mathrm{dom}(\mathbb{A})$;
3. $B \leqslant_e \mathrm{Th}_\Sigma(\mathbb{A}, a)$ for some $a \in \mathrm{dom}(\mathbb{A})$.

$\mathcal{I}_e(\mathbb{A}) = \langle d_e(\mathrm{Th}_\Sigma(\mathbb{A}, a)) \mid a \in \mathrm{dom}(\mathbb{A})\rangle.$

$\mathrm{Th}_\exists(\mathfrak{M}) \leftrightharpoons \{\Phi \mid \Phi \text{ is } \exists \text{ sentence}, \mathfrak{M} \models \Phi\}.$

**Proposition (2002)**
$\mathcal{I}_e(\mathbb{HIF}(\mathfrak{M})) = \langle d_e(\mathrm{Th}_\exists(\mathfrak{M}, a)) \mid a \in \mathrm{dom}(\mathfrak{M})^{<\omega}\rangle.$

## 2. REDUCIBILITIES ON ADMISSIBLE SETS

Let $\mathbb{A}$ be an admissible set and let $S$ be an arbitrary set. A map $\nu : \mathrm{dom}(\mathbb{A}) \twoheadrightarrow S$ (onto $S$) is called *$\mathbb{A}$-numbering* of $S$.

Let $S \subseteq \mathcal{P}(\mathrm{dom}(\mathbb{A}))$. $\mathbb{A}$-numbering $\nu$ is called *computable* if $\Gamma^*_\nu \leftrightharpoons \{\langle x, y\rangle \in \mathrm{dom}(\mathbb{A})^2 \mid y \in \nu(x)\} \in \Sigma^{\mathbb{A}}_1$.
$\mathbb{A}$-numbering $\nu$ is called *definable* if $\Gamma^*_\nu \in \bigcup_n \Sigma^{\mathbb{A}}_n$.

We will denote class of all the computable in $\mathbb{A}$ collections (having some computable $\mathbb{A}$-numbering) of subsets of $\omega$ as $\mathcal{S}_\omega(\mathbb{A})$.

(1985; Yu. Ershov) Let $\mathfrak{M} = \langle M, Q_1, \ldots, Q_k \rangle$ be an arbitrary model in some finite relation language. We say that $\mathfrak{M}$ is $\Sigma$ *definable* in an admissible set $\mathbb{A}$ (and we denote as $\mathfrak{M} \leqslant_\Sigma \mathbb{A}$) if there exists $\mathbb{A}$-numbering $\nu$ of $M$ such that $\nu^{-1}(=)$, $\nu^{-1}(Q_1)$, …, $\nu^{-1}(Q_k)$ are $\mathbb{A}$-c.

We say that $\mathfrak{M}$ is *definable* in $\mathbb{A}$ (and we denote as $\mathfrak{M} \leqslant \mathbb{A}$) if there exists $\mathbb{A}$-numbering $\nu$ of $M$ such that $\nu^{-1}(=)$, $\nu^{-1}(Q_1)$, …, $\nu^{-1}(Q_k)$ are in $\bigcup_n \Sigma_n^{\mathbb{A}}$.

(2002; A. Morozov) We say that an admissible set $\mathbb{A}$ $\mathbb{HYP}$-*reducible* to an admissible set $\mathbb{B}$ (and we denote as $\mathbb{A} \sqsubseteq_{\mathbb{HYP}} \mathbb{B}$) if there exists $\mathbb{B}$-numbering $\nu$ of $\mathrm{dom}(\mathbb{A})$ as from definition of $\mathbb{A} \leqslant_{\Sigma} \mathbb{B}$ such that $\{\langle x, y\rangle \mid \nu(x) = \{\nu(z) \mid z \in y\}\}$ is $\mathbb{B}$-c.

(2004) We say that an admissible set $\mathbb{A}$ is $\Sigma$ *reducible* to an admissible set $\mathbb{B}$ (and we denote as $\mathbb{A} \sqsubseteq_E \mathbb{B}$) if there exists $\mathbb{B}$-numbering $\nu$ of $\mathrm{dom}(\mathbb{A})$ such that $\nu^{-1}(C)$ is $\mathbb{B}$-c.e. for every binary $\mathbb{A}$-c.e. predicate $C$.

We say that $p$-reducibility ($\leqslant_p \in \{\leqslant_\Sigma, \sqsubseteq_\Sigma, \sqsubseteq_{\mathbb{HYP}}\}$) on admissible sets *preserves $P$* if $\mathbb{A} \leqslant_p \mathbb{B}$ implies $P(\mathbb{A}) \subseteq P(\mathbb{B})$.

| | $\mathcal{I}_e$ | $\mathcal{S}_\omega$ | Ord | Set structure |
|---|---|---|---|---|
| $\leqslant_\Sigma$ | $-$ | $-$ | $-$ | $-$ |
| $\sqsubseteq_\Sigma$ | $+$ | $+$ | $-$ | $-$ |
| $\sqsubseteq_{\mathbb{HYP}}$ | $+$ | $+$ | $+$ | $+$ |

**Examples.**

1. $\mathbb{HYP}(\mathfrak{N}) \leqslant_\Sigma \mathbb{HF}(\mathbb{HYP}(\mathfrak{N}))$ when $\mathfrak{N}$ is the standard model of arithmetics. However $\Pi^1_1 = \mathcal{I}_e(\mathbb{HYP}(\mathfrak{N})) \not\leqslant \mathcal{I}_e(\mathbb{HF}(\mathbb{HYP}(\mathfrak{N}))) = \Delta^1_1$.

2. $\mathbb{HYP}(S) \sqsubseteq_E \mathbb{HF}(S)$ when $S$ is some infinite model in the empty signature.

**Theorem (Yu. Ershov)**
$\mathfrak{M} \leqslant_\Sigma \mathbb{A} \Leftrightarrow \mathbb{HF}(\mathfrak{M}) \leqslant_\Sigma \mathbb{A} \Leftrightarrow \mathbb{HF}(\mathfrak{M}) \sqsubseteq_\Sigma \mathbb{A}$.

**Proposition** $\mathbb{HF}(\varnothing) \sqsubseteq_\Sigma \mathbb{A}$, for every admissible set $\mathbb{A}$.

**Proposition** Let $\mathbb{A}$, $\mathbb{B}$ and $\nu$ be admissible sets and a map such that $\nu : \mathbb{A} \sqsubseteq_\Sigma \mathbb{B}$. Then $\nu^{-1}(\Sigma_m^{\mathbb{A}}) \subseteq \Sigma_m^{\mathbb{B}}$, $\nu^{-1}(\Pi_m^{\mathbb{A}}) \subseteq \Pi_m^{\mathbb{B}}$, $\nu^{-1}(\Delta_m^{\mathbb{A}}) \subseteq \Delta_m^{\mathbb{B}}$, for any $m \geqslant 1$.

• Let $\mathbb{A}$ be an admissible set; then $J(\mathbb{A})$ is called a *jump*

if there are surjective maps

$\nu_0 : \mathrm{dom}(\mathbb{A}) \twoheadrightarrow \mathrm{dom}(J(\mathbb{A}))$

and

$\nu_1 : \mathrm{dom}(J(\mathbb{A})) \twoheadrightarrow \mathrm{dom}(\mathbb{A})$

such that

$\nu_0^{-1}(\Sigma_1^{J(\mathbb{A})}) \subseteq \Sigma_2^{\mathbb{A}}; \quad \nu_1^{-1}(\Sigma_2^{\mathbb{A}}) \subseteq \Sigma_1^{J(\mathbb{A})}.$

Given an admissible set $\mathbb{A}$, there exists the unique up to $\Sigma$-equivalence jump $J(\mathbb{A})$.

**Proposition** Let $\mathbb{A}$, $\mathbb{B}$ be admissible sets and let $\mathfrak{M}$ be a model in some finite signature. Then the following conditions hold:

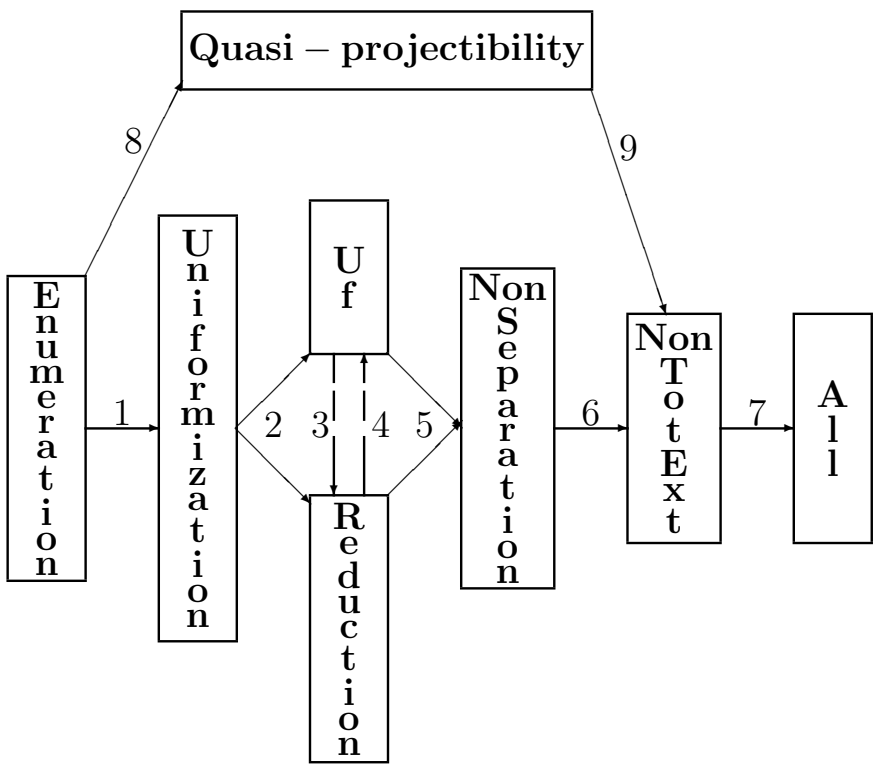
1. $\mathbb{A} \sqsubseteq_\Sigma J(\mathbb{A})$;

2. $\mathbb{A} \sqsubseteq_\Sigma \mathbb{B} \Rightarrow J(\mathbb{A}) \sqsubseteq_\Sigma J(\mathbb{B})$;

3. $\mathfrak{M} \leqslant \mathbb{A}$ iff $\mathfrak{M} \leqslant_\Sigma J^n(\mathbb{A})$ for some $n \in \omega$.

# 3. COMPUTABLE PRINCIPLES ON ADMISSIBLE SETS

**Theorem** There are correlations between principles on admissible sets:




Moreover, all the transitions but 3 are strong.

# 4. CONSTRUCTION

**Theorem (2004)** Let $\mathbb{A}$ be an admissible set and let $P$ be one of the following properties: uniformization, reduction, separation, total extension, existence of a universal function, enumerability via $\omega$, quasi-projectibility in $\omega$. Then there exists a model $\mathfrak{M}_{\mathbb{A}}$ in the signature $\{Q^2\}$ such that the following conditions hold:

1. $\mathbb{A} \equiv_{\Sigma} \mathbb{HF}(\mathfrak{M}_{\mathbb{A}})$;

2. $n - P(\mathbb{A}) \Leftrightarrow n - P(\mathbb{HF}(\mathfrak{M}_{\mathbb{A}}))$.

**Example.** Consider $\mathbb{HYP}(\mathfrak{N})$ when $\mathfrak{N}$ is the standard model arithmetics. The admissible set satisfies quasi-projectibility in $\omega$, uniformization, reduction, existence of a universal $\Sigma$ function and does not satisfy other principles. In particular, it is not enumerable via $\omega$, nevertheless it is recursively listed. By theorem 2, there exists a hereditarily finite set satisfying the same properties and being $E$-equivalent to $\mathbb{HYP}(\mathfrak{N})$.

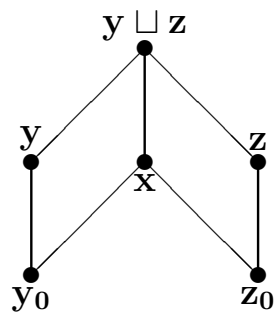
# 5. SEMILATTICES UNDER $E$-REDUCIBILITY

Let $D_E(\mathbb{A})$ be class of all the admissible sets which are $E$-equivalent to $\mathbb{A}$.
$\mathcal{R}_\alpha = \{D_E(\mathbb{A}) \mid \mathbb{A}$ is admissible, $\mathrm{card}(\mathbb{A}) = \alpha\}$,
$\mathcal{R}_{\leqslant\alpha} = \{D_E(\mathbb{A}) \mid \mathbb{A}$ is admissible $, \mathrm{card}(\mathbb{A}) \leqslant \alpha\}$,
$\mathcal{R}_{\alpha,\mathcal{I}} = \{D_E(\mathbb{A}) \mid \mathbb{A}$ is admissible, $\mathrm{card}(\mathbb{A}) = \alpha$, $\mathcal{I}_e(\mathbb{A}) = \mathcal{I}\}$,
$\mathcal{R}_{\leqslant\alpha,\mathcal{I}} = \{D_E(\mathbb{A}) \mid \mathbb{A}$ is admissible, $\mathrm{card}(\mathbb{A}) \leqslant \alpha$,
$\mathcal{I}_e(\mathbb{A}) = \mathcal{I}\}$,
where $\mathcal{I}$ is an ideal of $e$-degrees and $\alpha$ is an infinite cardinal.

An upper semilattice $\langle L, \leqslant \rangle$ is called *distributive* if

$$x \leqslant y \sqcup z \Rightarrow \exists y_0 \leqslant y \exists z_0 \leqslant z [x = y_0 \sqcup z_0]$$

for any $x, y, z \in L$.



**Theorem (2004)** Let $\mathcal{I}$ and $\alpha$ be an $e$-ideal and an infinite cardinal such that $\alpha \geqslant \mathrm{card}(\mathcal{I})$. Then the following conditions hold:

1. $\langle \mathcal{R}_\alpha, \sqsubseteq_E \rangle$, $\langle \mathcal{R}_{\leqslant\alpha}, \sqsubseteq_E \rangle$, $\langle \mathcal{R}_{\leqslant\alpha,\mathcal{I}}, \sqsubseteq_E \rangle$, $\langle \mathcal{R}_{\alpha,\mathcal{I}}, \sqsubseteq_E \rangle$ are upper semilattices of cardinality not greater than $2^\alpha$;

2. if $\mathcal{I}$ is at most countable ideal then $\mathrm{card}(\mathcal{R}_{\omega,\mathcal{I}}) = 2^\omega$ and hence $\mathrm{card}(\mathcal{R}_\omega) = 2^\omega$;

3. $\langle \mathcal{R}_\alpha, \sqsubseteq_E \rangle$, $\langle \mathcal{R}_{\leqslant\alpha}, \sqsubseteq_E \rangle$ and $\langle \mathcal{R}_{\alpha,\mathcal{I}}, \sqsubseteq_E \rangle$ have the least elements;

4. $\langle \mathcal{R}_{\leqslant\alpha,\mathcal{I}}, \sqsubseteq_E \rangle$ has the least element iff $\mathcal{I}$ is a principal ideal or $\alpha = \omega$;

5. if $\beta$ is a cardinal such that $\alpha < \beta$ then there exist embeddings

$\langle \mathcal{R}_{\leqslant\alpha}, \sqsubseteq_E, \sqcup, \mathbf{0}\rangle \hookrightarrow \langle \mathcal{R}_{\leqslant\beta}, \sqsubseteq_E, \sqcup, \mathbf{0}\rangle$,

$\langle \mathcal{R}_{\alpha}, \sqsubseteq_E, \sqcup, \mathbf{0}\rangle \hookrightarrow \langle \mathcal{R}_{\beta}, \sqsubseteq_E, \sqcup, \mathbf{0}\rangle$,

$\langle \mathcal{R}_{\leqslant\alpha,\mathcal{I}}, \sqsubseteq_E, \sqcup\rangle \hookrightarrow \langle \mathcal{R}_{\leqslant\beta,\mathcal{I}}, \sqsubseteq_E, \sqcup\rangle$ and

$\langle \mathcal{R}_{\alpha,\mathcal{I}}, \sqsubseteq_E, \sqcup, \mathbf{0}\rangle \hookrightarrow \langle \mathcal{R}_{\beta,\mathcal{I}}, \sqsubseteq_E, \sqcup, \mathbf{0}\rangle$;

6. $\langle L_e, \leqslant, \sqcup, \setminus, \mathbf{0}\rangle \hookrightarrow \langle \mathcal{R}_{\omega}, \sqsubseteq_E, \sqcup, J, \mathbf{0}\rangle$ and hence

$\langle \mathcal{R}_{\leqslant\alpha}, \sqsubseteq_E\rangle$ and $\langle \mathcal{R}_{\alpha}, \sqsubseteq_E\rangle$ are not distributive;

7. if $\alpha > \omega$ then $\langle \mathcal{R}_{\leqslant\alpha}, \sqsubseteq_E\rangle$ is not a lattice; furthermore, if $\mathcal{I}$ is nonprincipal then $\langle \mathcal{R}_{\leqslant\alpha,\mathcal{I}}, \sqsubseteq_E\rangle$ is not a lattice, too.

Let $\mathbf{0}_{\alpha,\mathcal{I}}$ be the least element of $\langle \mathcal{R}_{\alpha,\mathcal{I}}, \sqsubseteq_E\rangle$ where $\mathcal{I}$ is an ideal and $\alpha$ is an infinite cardinal with $\alpha \geqslant \mathrm{card}(\mathcal{I})$.

# 6. MINIMAL ELEMENTS: GENERAL CASE

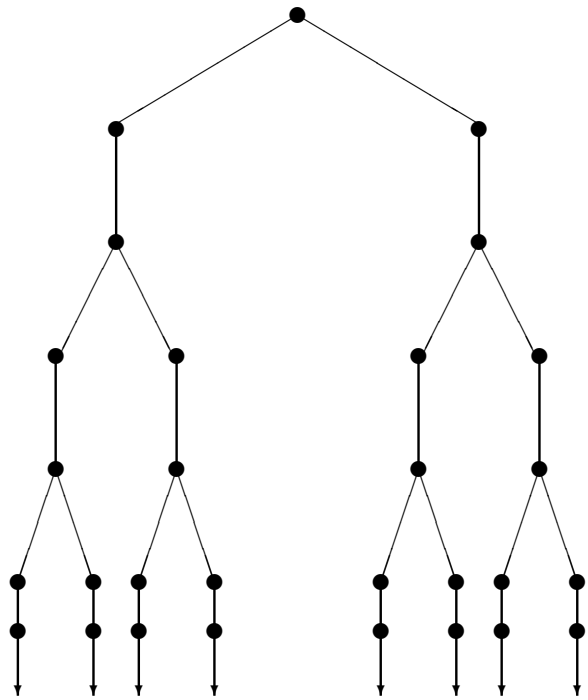
Let $\mathcal{I}$ and $\alpha$ be an ideal and an infinite cardinal such that $\alpha \geqslant \mathrm{card}(\mathcal{I})$. Set a model $\mathfrak{M}_{\alpha,\mathcal{I}}$ in the signature $\{P^2\}$ as a disjunct union of woods of kind $\mathcal{D}_S$, $S \in \bigcup \mathcal{I}$ (we take exactly $\alpha$ woods of kind $\mathcal{D}_S$, for every such $S$):

• every vertex has either one or two immediate succesors;

• a vertex $x$ has two immediate succesors iff $\mathrm{height}(x) \in S$;

$$\bullet P(x,y) \Leftrightarrow \begin{cases} x \text{ is a root} \& (x = y); \\ x \text{ is not a root} \& (x \text{ is an immediate} \\ \text{successor of } y). \end{cases}$$

Then $\mathbb{HF}(\mathfrak{M}_{\alpha,\mathcal{I}}) \in \mathbf{0}_{\alpha,\mathcal{I}}$.



$\mathcal{D}_S$ is represented on the picture where $S = \{0, 2, 4\}$.

**Theorem** (2002) Let $\mathbb{A}$ be an admissible set such that $\mathbb{A} \in \mathbf{0}_{\alpha,\mathcal{I}}$. Then $S \subseteq \mathcal{P}(\omega)$ is computable in $\mathbb{A}$ iff $S \cup \{\varnothing\} = \{\Theta_n(R, A) \mid n \in \omega,\, d_e(R) \in \mathcal{I}\}$ for some computable sequence $\{\Theta_n\}_{n\in\omega}$ of enumeration operators and $A \subseteq \omega$ with $d_e(A) \in \mathcal{I}$.

We say that $\mathcal{I}$ satisfies the property of universal function existence if collection of all the graphs of functions $f$ with $d_e(\Gamma_f) \in \mathcal{I}$ is computable in admissible sets from $\mathbf{0}_{\alpha,\mathcal{I}}$.

**Theorem** (2004) Let $P$ be one of the following properties: existence of a universal function, separation, total extension. Then there exists an admissible set $\mathbb{A} \in \mathbf{0}_{\alpha,\mathcal{I}}$ such that $P(\mathbb{A}) \Leftrightarrow P(\mathcal{I})$.

## 7. MINIMAL ELEMENTS: CASE OF NONPRINCIPAL IDEAL

Let $\mathcal{I}$ and $\alpha$ be a nonprincipal ideal and an infinite cardinal such that $\alpha \geqslant \mathrm{card}(\mathcal{I})$.

**Theorem** (2004) Let $\mathbb{A}$ be an admissible set such that $\mathbb{A} \in \mathbf{0}_{\alpha,\mathcal{I}}$. Then the reduction principle does not hold in $\mathbb{A}$.

Let $\mathbb{A}$ be an admissible set and let $\nu$ be $\mathbb{A}$-numbering of $S$. $\nu$ is called *positive* if $\eta_\nu \leftrightharpoons \{\langle x, y\rangle \in \mathrm{dom}(\mathbb{A})^2 \mid \nu(x) = \nu(y)\}$ is $\mathbb{A}$-c.e. $\nu$ is called *decidable* if $\eta_\nu$ is $\mathbb{A}$-c. $\nu$ is called *single-valued* if there is a total $\mathbb{A}$-c. function $f$ such that $\eta_\nu = \{\langle x, y\rangle \mid f(x) = f(y)\}$. $\nu$ is called *definable* if $\eta_\nu \in \bigcup_n \Sigma_n^{\mathbb{A}}$.

**Theorem** (2004) Let $\mathbb{A}$ be an admissible set such that $\mathbb{A} \in \mathbf{0}_{\alpha,\mathcal{I}}$. Then there is no positive computable $\mathbb{A}$-numbering of collection of all the $\mathbb{A}$-c.e. sets.

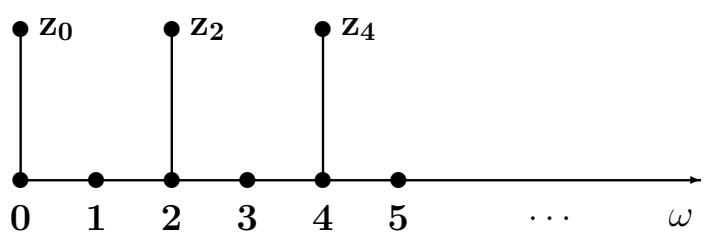
**Theorem** (2000) 1. There exists an admissible set $\mathbb{A}$ such that there is a single-valued computable $\mathbb{A}$-numbering of collection of all the $\mathbb{A}$-c.e. sets.
2. There exists an admissible set $\mathbb{A}$ such that there is a decidable computable $\mathbb{A}$-numbering of collection of all the $\mathbb{A}$-c.e. sets but there is no single-valued computable $\mathbb{A}$-numbering of such collection.

# 8. MINIMAL ELEMENTS: CASE OF PRINCIPAL IDEAL

Let $C \subseteq \omega$. We set a model $\mathfrak{N}_C$ in the signature $\{0, s^2, f^2\}$ by follows:

$\mathrm{dom}(\mathfrak{N}_C) = \omega \uplus \{z_n \mid n \in C\}$ so that $z_n \neq z_{n'}$ when $n \neq n'$;

$0 \in \omega$ is the least element of $\omega$ and $s$ is the succesor function on $\omega$;
$f$ corresponds $n$ to $z_n$ when $n \in C$.




A model which is represented in the picture corresponds to $C = \{0, 2, 4\}$.

Notice that $\mathbb{HF}(\mathfrak{N}_C) \in \mathbf{0}_{\omega,\mathcal{I}}$ when $\mathcal{I} = \widehat{\mathrm{d}_e(C)}$.

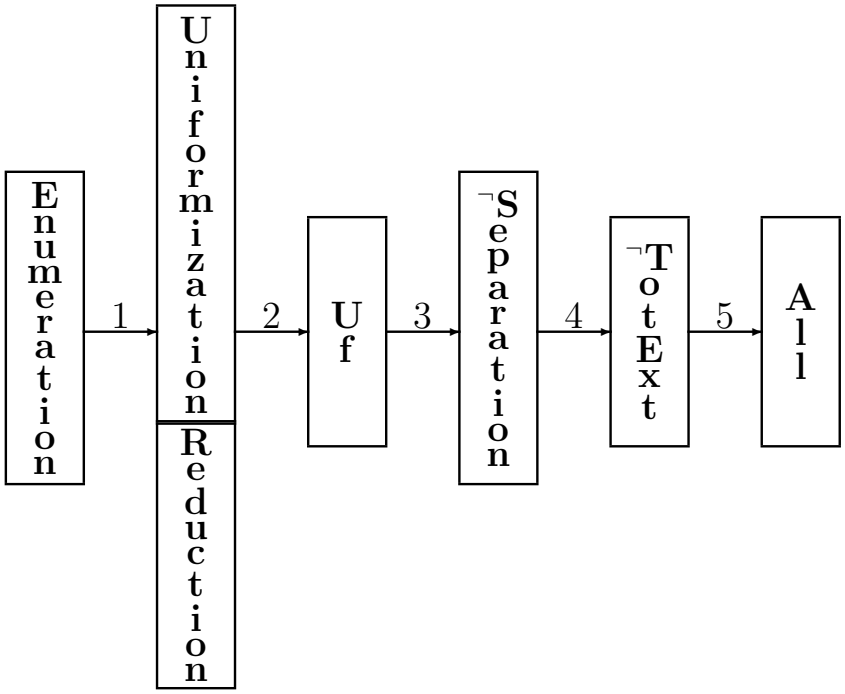
**Proposition**
Let $B, C \subseteq \omega$. Then $B \leqslant_e C$ iff $\mathbb{HF}(\mathfrak{N}_B) \sqsubseteq_E \mathbb{HF}(\mathfrak{N}_C)$.

**Proposition**
Let $C \subseteq \omega$. Then $\mathbb{HF}(\mathfrak{N}_{J_e(C)}) \equiv_E J(\mathbb{HF}(\mathfrak{N}_C))$.

**Proposition** Let $\mathbb{A}$ be an admissible set which is quasi-projectible in $\omega$. Then the following conditions hold:
1. $\mathcal{I}_e(\mathbb{A})$ is a principal ideal.
2. If $C \subseteq \omega$ is such that $\mathcal{I}_e(\mathbb{A}) = \widehat{\mathrm{d}_e(C)}$ then $\mathbb{A} \equiv_E \mathbb{HF}(\mathfrak{N}_C)$. In particular, $\mathbb{A} \in \mathbf{0}_{\omega,\mathcal{I}_e(\mathbb{A})}$.
3. $\mathbb{A}$ does not satisfy the total extension property.

**Theorem** (2003) Let $C \subseteq \omega$ and let $P$ be one of the following properties: enumerability via $\omega$, uniformization, reduction, existence of a universal function, separation. Then $P(\mathbb{HF}(\mathfrak{N}_C)) \Leftrightarrow P(\widehat{\mathrm{d}_e(C)})$.

**Theorem 13** (I. Kalimullin, V. Puzarenko; 2003) The following correlations are between principal ideals:



## 9. APPLICATION

**Theorem 14** (2003) Let $\mathbb{R}$ be a field of reals and let $\mathbb{A}$ be an admissible set. Then $\mathbb{R}$ is definable in $\mathbb{A}$ iff $\mathcal{P}(\omega)$ is definable in $\mathbb{A}$ i.e. there is a definable $\mathbb{A}$-numbering of $\mathcal{P}(\omega)$.

**Theorem 15** (Yu. Ershov) Let T be one of the following theories: infinite sets without structures, dense linear order, algebraically closed fields. Then $\mathbb{R}$ is not definable in hereditarily finite sets $\mathbb{HF}(\mathfrak{M})$ over $\mathfrak{M} \models \mathrm{T}$.

**Theorem 16** (2003) Let $\mathbb{R}$ be a field of reals and let $\mathbb{A}$ be an admissible set with $\mathrm{Ord}(\mathbb{A}) > \omega$. Then $\mathbb{R}$ is $\Sigma$-definable in $\mathbb{A}$ iff $\mathcal{P}(\omega) \subseteq \mathrm{dom}(\mathbb{A})$.

**Theorem 17** (2004) There exists an admissible set $\mathbb{A}$ such that $\mathbb{R} \leqslant \mathbb{A}$ but $\mathbb{R} \not\leqslant_{\Sigma} \mathbb{A}$.

## 10. OPEN PROBLEMS

1. Is there an admissible set satisfying the reduction property but without a universal $\Sigma$-function?

2. Is there an admissible set that any definable in $\mathbb{A}$ model is $\Sigma$-definable (this condition is equal to existence of an admissible set $\mathbb{A}$ with $J(\mathbb{A}) \equiv_E \mathbb{A}$)?

3. Is there an admissible set that its full $\exists$-diagram is enumerable but its full $\Sigma$-diagram is not (this condition is equal to existence of an admissible set $\mathbb{A}$ with $\mathbb{A} \not\sqsubseteq_E \mathbb{HF}(\varnothing)$ and $\mathbb{A} \leqslant_\Sigma \mathbb{HF}(\varnothing)$)?

4. Is there an admissible set $\mathbb{A}$ such that there exists a positive computable $\mathbb{A}$-numbering of collection of all the $\mathbb{A}$-c.e. sets but there is no decidable computable $\mathbb{A}$-numbering of the collection?

5. To describe admissible sets of height $\omega$ such that $\mathbb{R}$ is $\Sigma$-definable in them.

## 11. REFERENCES

1. Puzarenko V. Decidable computable $\mathbb{A}$-numberings// Algebra and logic, **41**, №5(2002).

2. Puzarenko V. General numberings and definability of $\mathbb{R}$ in admissible sets// Vestnik NGU, Ser.: mathem., mechanics, inform., **3**, №2(2003), 107–117 (in russian).

3. Morozov A., Puzarenko V. $\Sigma$-subsets of naturals// Algebra and logic, **43**, №3(2004).

4. Kalimullin I., Puzarenko V. Computable principles on admissible sets// Siberian Advances in Mathematics (2004).

5. Puzarenko V. On computability on special models// Siberian mathematical journal, in 2005.

## 12. RESUME

1. A reducibility on admissible sets is proposed. It allowed a lot of question about correlations between separate computable principles on admissible sets to reduce to corresponding ones on hereditarily finite sets.

2. The properties of minimal classes of admissible sets were studied. This allowed to build all the possible but one variants of admissible sets under computable principles.

3. A class of admissible sets without friedberg numbering is built.

4. A criterium of definability of $\mathbb{R}$ in admissible sets have been got.