How Enumeration Reducibility Yields Extended Harrington Non-splitting

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1 Introduction

Sacks [16] showed that every computably enumerable (c.e.) degree $> \mathbf{0}$ has a c.e. splitting. Hence, relativising, every c.e. degree has a Δ_2 splitting above each proper predecessor (by 'splitting' we understand 'nontrivial splitting'). Arslanov [1] showed that $\mathbf{0}'$ has a d.c.e. splitting above each c.e. $\mathbf{a} < \mathbf{0}'$. On the other hand, Lachlan [11] proved the existence of a c.e. $\mathbf{a} > \mathbf{0}$ which has no c.e. splitting above some proper c.e. predecessor, and Harrington [10] showed that one could take $\mathbf{a} = \mathbf{0}'$. Splitting and nonsplitting techniques have had a number of consequences for definability and elementary equivalence in the degrees below $\mathbf{0}'$.

Heterogeneous splittings are best considered in the context of cupping and noncupping. Posner and Robinson [15] showed that every nonzero Δ_2 degree can be nontrivially cupped to $\mathbf{0}'$, and Arslanov [1] showed that every c.e. degree $> \mathbf{0}$ can be d.c.e. cupped to $\mathbf{0}'$ (and hence since every d.c.e., or even n-c.e., degree has a nonzero c.e. predecessor, every n-c.e. degree $> \mathbf{0}$ is d.c.e. cuppable). Cooper [4] and Yates (see Miller [13]) showed the existence of degrees noncuppable in the c.e. degrees. Moreover, the search for relative cupping results was drastically limited by Cooper [5], and Slaman and Steel [17] (see also Downey [9]), who showed that there is a nonzero c.e. degree $\mathbf{a}$ below which even Δ_2 cupping of c.e. degrees fails.

We prove below what appears to be the strongest possible of such nonsplitting and noncupping results.

Theorem 1. *There exists a computably enumerable degree* $\mathbf{a} < \mathbf{0}'$ *such that there exists no nontrivial cuppings of c.e. degrees above* $\mathbf{a}$ *in the* Δ_2 *degrees above* $\mathbf{a}$.

In fact, if we consider the extended structure of the enumeration degrees, Theorem 1 is a corollary of the even stronger result:

Theorem 2. *There exists a* Π_1 *e-degree* $\mathbf{a} < \mathbf{0}'_e$ *such that there exist no nontrivial cuppings of* Π_1 *e-degrees above* $\mathbf{a}$ *in the* Σ_2 *e-degrees above* $\mathbf{a}$.

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This would appear to be the first example of a structural feature of the Turing degrees obtained via a proof in the wider context of the enumeration degrees (rather than the other way round).

This can be seen as a first step towards using structural properties of the enumeration degrees to obtain definability of natural relations over the Turing degrees, where this cannot be got just within the Turing degrees. Whether such relations exist is completely open, of course.

For further information regarding splitting and non-splitting in the enumeration degrees, see [2] and [3].

Notation and terminology below is based on that of [7].

2 Requirements and Strategies

We assume a standard listing of all quadruples (Ψ, Θ, U, W) of enumeration operators Ψ and Θ, Σ_2 sets U and c.e. sets W. We will construct Π_1 sets A and E to satisfy the corresponding list of requirements:

$$\mathcal{N}_\Psi : E \neq \Psi^A$$

$$\mathcal{P}_{\Theta,U,W} : E = \Theta^{U,\overline{W}} \Rightarrow (\exists \Gamma, \Lambda)[\overline{K} = \Gamma^{U,A} \vee \overline{K} = \Lambda^{\overline{W},A}]$$

where $\Gamma^{U,A}$, for example, denotes an e-operator enumerating relative to the data enumerated from two sources U and A.

2.1 The simple $\mathcal{N}_\Psi$-Strategy

The naive strategy to satisfy one $\mathcal{N}$ requirement is the simple Friedberg-Muchnik strategy, we will denote it by $(\mathcal{N}_\Psi, FM)$. Select a witness x for $\mathcal{N}_\Psi$ and wait for $x \in \Psi^A$. Then extract x from E while restraining each $y \in A \upharpoonright use(\Psi, A, x)$, where the use function $use(\Psi, A, x)$ is defined in the usual way by $use(\Psi, A, x) = \mu y[x \in \Psi^{A \upharpoonright y}]$.

2.2 The naive $\mathcal{P}_\Theta$-Strategy

We are presented with three options to satisfy a fixed $\mathcal{P}$-requirement, say $\mathcal{P}_{\Theta,U,W}$. We start off by constructing an enumeration operator Γ meant to reduce $\overline{K}$ to the set $U \oplus A$ and monitor the length of agreement between the sets E and $\Theta^{U,\overline{W}}$. The hope is that a bounded length of agreement function will be enough evidence for an inequality between the sets E and $\Theta^{U,\overline{W}}$.

We progressively try to rectify Γ at each stage by ensuring that $n \in \overline{K} \Leftrightarrow n \in \Gamma^{U,A}$ for each n below $l(E, \Theta^{U,\overline{W}})$. The definition of the enumeration operator Γ involves axioms of the form $\langle n, U \upharpoonright u(n) + 1, A \upharpoonright \gamma(n) + 1\rangle$ with two types of markers $u(n)$ and $\gamma(n) \in A$. Given a suitable choice of a marker $\gamma(n) \in A$ if

n exits $\overline{K}$, Γ can be rectified via A-extraction. So if the length of agreement is unbounded we will construct the required reduction Γ.

Unfortunately using just any approximation to the Σ_2 set U might prevent us from being able to carry out this plan. We could have a bounded length of agreement function even if $E = \Theta^{U,\overline{W}}$. Moreover it is possible that at each stage s the approximation $U_s \restriction u(n) \nsubseteq U$, so that we will not have a valid axiom for n in Γ. Fortunately we can construct a special approximations to the sets U,$\overline{W}$ and $U \oplus \overline{W}$ which behave nicely.

2.3 The Approximations

Consider a $\mathcal{P}_{\Theta,U,W}$ requirement.

Definition 1. *We inductively say that a stage $s+1$ is P_Θ-expansionary if and only if $l(E[s+1], \Theta^{U,\overline{W}}[s+1])$ attains a greater value at stage $s+1$ than at any previous P_Θ-expansionary stage.*

To ensure that $E = \Theta^{U,\overline{W}}$ gives infinitely many expansionary stages, we need an approximation $(U \oplus \overline{W})_s = B_s$ to the set $U \oplus \overline{W} = B$, which should be a good approximation (basically one with sufficient thin stages, in the sense of Cooper [6]) as defined in [12]: i.e., it should have the following properties:

G1. $\forall n \exists s (B \restriction n \subset B_S \subset B)$ - such stages s are called good stages.
G2. $\forall n \exists s \forall t > s (B_t \subset B \Rightarrow B \restriction n \subset B_t)$.

On the other hand the approximation to the set U that is derived from this via $U_s = \{n | 2n \in B_s\}$ should still be a Σ_2 approximation, and the derived approximation to the set $\overline{W}$ should retain some stability − that is, under some computable condition we must be sure that if some element $n \notin \overline{W}_s$, then $n \notin \overline{W}_t$ for all stages $t > s$.

First we will choose a more convenient enumeration of the triples $\langle \Theta, U, W \rangle$. Fix an index a. Then all Σ_2 sets U can be listed by the sequence $\{W_e^{K \oplus W_a}\}_{e \in \omega}$, as the Σ_2 sets are exactly the sets that are c.e. in $K \equiv_T K \oplus W_a$. We obtain the required enumeration by listing all triples of natural numbers (i, e, a) and defining $\Theta = W_i$, $\overline{W} = \overline{W_a}$ and $U = W_e^{K \oplus W_a}$.

Given a requirement (e, a, i) we approximate $K \oplus W_a = C$ via a better approximation as defined in [12]. A better approximation to the set C is a computable sequence of finite characteristic functions α_s, such that:

B1. $\forall n \exists s_{1,n} (\chi_C \restriction n \subset \alpha_{s_{1,n}} \subset \chi_C)$.
B2. $\forall n \exists s_{2,n} \forall t > s_{2,n} (\{n | \alpha_t(n) = 1\} \subset C \Rightarrow \chi_C \restriction n \subset \alpha_t)$.

Consider C_s to be the standard approximating sequence to the c.e. set $C = K \oplus W_a$. And let $ap_s = \mu m[m \in C_s \backslash C_{s-1}]$, $ap_s = s$ if $C_s = C_{s-1}$. It is not hard to see that $\{\alpha_s\}$ is a better approximating sequence to C, where $\alpha_0 = \emptyset$, and if $s > 0$

$$\alpha_s = \begin{cases} 1 & \text{if } n \in C_s, \\ 0 & \text{if } n \notin C_S \text{ and } n < ap_s, \\ \text{not defined otherwise.} \end{cases}$$

As for all t we have that $\{n|\alpha_t(n) = 1\} \subset C$, the second property of a better approximating sequence can be improved to:

B2′. $\forall n \exists s_{2,n} \forall t > s_{2,n} (\chi_C \upharpoonright n \subset \alpha_t)$.

It follows that the value of ap_s will grow unboundedly as we progress with the approximation.

Now we can approximate U via $U_s = W_{e,s}^{\alpha_s}$, $\overline{W}$ via $\overline{W}_s = \{n|\alpha_s(2n+1) = 0\}$ and B via $B_s = U_s \oplus \overline{W}_s$.

Proposition 1. *$\overline{W}_s$ is a good approximation to $\overline{W}$. If n leaves the approximation on a stage t such that $2n+1 < ap_t$, i.e., $n \in \overline{W}_{t_0}$ and on some stage $t > t_0$ $n \notin \overline{W}_t$, then for all $s > t$ we would have $n \notin \overline{W}_s$.*

Proof. First note that if s is a stage such that $\alpha_s \subset \chi_C$, then $\overline{W}_s \subset \overline{W}$. Indeed if $n \in \overline{W}_s$, then $\alpha_s(2n+1) = 0$, hence $\chi_C(2n+1) = 0$ and $\chi_{W_a}(n) = 0$, which implies $\overline{W}(n) = 1$. So every better stage for α_s is a good stage for $\overline{W}_s$.

Then let t be a stage such that all elements of $B \upharpoonright n$ are already enumerated in B_t. Then at stages $s > t$, $ap_s > n$. If $n \notin \overline{W}_s$ and $2n+1 < ap_s$, then $\alpha_s(2n+1) = 1$. Hence $n \in C$ and it follows from the properties of a standard approximation to a c.e. set and the definition of α, that $n \notin \overline{W}_t$ at any $t > s$.

G1. Fix n. Let $s_{b1,2n+1}$ be the stage from the first property of the better approximation α_s. Then $\chi_C \upharpoonright 2n+1 \subset \alpha_{s_{b1,2n+1}} \subset \chi_C$. Then if $m < n$ and $m \in \overline{W} \upharpoonright n$, then $2m+1 < 2n+1$ and $2m+1 \notin C$, then $(\chi_C \upharpoonright 2n+1)(2m+1) = 0 = \alpha_{s_{b1,2n+1}}(2m+1)$, and hence $2m+1 \in \overline{W}_{s_{b1,2m+1}}$.

G2. Fix n. Let $s = s_{b2,2n+1}$. Then at stages $t > s$ $\overline{W} \upharpoonright n \subset \overline{W}_t$. If $m \in \overline{W} \upharpoonright n$, then $(\chi_C \upharpoonright 2n+1)(2m+1) = 0 = \alpha_t$, hence $m \in \overline{W}_t$.

The proposition follows. □

Proposition 2. *U_s is a good Σ_2 approximation to U.*

Proof. Again we first note that if $\alpha_s \subset \chi_C$, then $U_s \subset U$.

For each n, there is an m and an s such that $U \upharpoonright n = W_{e,s}^{\chi_C \upharpoonright m}$. Then if $t > \max(s_{b2,n}, s)$, then $\chi_C \upharpoonright m \subset \alpha_t$ and hence $U \upharpoonright n = W_{e,s}^{\chi_C \upharpoonright m} \subset W_{e,t}^{\alpha_t} = U_t$. This proves G2 and the fact that the approximation is Σ_2.

For G1, choose $t > \max(s_{b1,m}, s)$ to be a stage such that if $\alpha_t \subset C$ then $U_t \subset U$ and $U \upharpoonright n \subset U_t$. □

Proposition 3. *B_s is a good approximation to B.*

Proof. G1: Fix n. Choose s' to be the stage from the second property of a good approximation to U for $n/2$ and s'' to be the stage from the second property of a good approximation to $\overline{W}$ for $n/2$. Then let $s > \max(s', s'')$ be a stage such that $\alpha_s \subset \chi_C$. Then $U_t \subset U$ and $\overline{W}_t \subset \overline{W}$, hence $B_s \subset B$. On the other hand $s > s'$ and $U \upharpoonright n/2 \subset U_s$, $s > s''$ and $\overline{W} \upharpoonright n/2 \subset \overline{W}_s$. Hence $B \upharpoonright n \subset B_s$.

G2: Proved easily as well using the stages from property G2 of the better approximations to U and $\overline{W}$. □

As a consequence of the properties of a good approximation we have the following:

If $\Theta^{U,\overline{W}} = E$, then there will be infinitely many expansionary stages, as $lim_{\{s|s \text{ is a good stage}\}} \Theta^{U_s,\overline{W}_s} = \Theta^{U,\overline{W}}$.

Moreover if $n \in \Theta^{U,\overline{W}}$, then there is a stage s such that $\forall t > s(n \in \Theta^{U,\overline{W}}[t])$, and if $n \notin \Theta^{U,\overline{W}}$ then at good stages t we have $n \notin \Theta^{U,\overline{W}}[t]$. Of course, it could happen that the expansionary stages are not necessarily the good stages. And if $\Theta^{U,\overline{W}} \neq E$, we could still have infinitely many expansionary stages.

2.4 The Basic Module for one $\mathcal{P}$-requirement

We are now ready to define the basic module for a strategy to satisfy one $\mathcal{P}$-requirement. The strategy is going to construct an enumeration operator Γ and will be called a $(\mathcal{P}, \Gamma)$-strategy.

The $(\mathcal{P}, \Gamma)$-strategy tries to maintain the equality between $\overline{K}$ and $\Gamma^{U,A}$ at expansionary stages. It scans elements $n < l(\Theta^{U,\overline{W}}, E)$, fixing their axioms as appropriate. Each such element n will have a current U-marker $u(n)$, a current A-marker $\gamma(n)$ and a corresponding current axiom $\langle n, U \upharpoonright u(n)+1, A \upharpoonright \gamma(n)+1 \rangle$. If the current axiom is not valid and $n \in \overline{K}$ we will modify the current A-marker and a new current axiom will be enumerated in Γ. We will progress the approximation of the sets U, $\overline{W}$ and $U \oplus \overline{W}$ only at stages on which this strategy is active. Also we examine the axioms in Γ for an element $n < l(\Theta^{U,\overline{W}}, E)$, $n \in \overline{K}$ on each stage. In this way will be sure to catch the true approximation to the set $U \upharpoonright u(n)$, so that if $u(n)$ remains constant, so will the axiom for n after a certain stage. Note that if $U_s \upharpoonright u(n) = U \upharpoonright u(n)$ and the stage s is big enough so that all elements of $U \upharpoonright u(n)$ never leave the approximating sets, then at later stages t we have $U \upharpoonright u(n) \subset U_t \upharpoonright u(n)$ and the axiom will not be modified due to U.

If $n \notin \overline{K}$ we will make sure that $n \notin \Gamma^{U,A}$ only on expansionary stages s by extracting A-markers for axioms that are valid at s.

It may happen that the two strategies (P_Θ, Γ) and (P_Θ, Λ) influence each other by extracting markers from A. In order to prevent that we define two non-intersecting infinite computable sets A_G and A_L for the possible values of A-markers for (P_Θ, Γ) and (P_Θ, Λ), respectively. Each time (P_Θ, Γ) defines a new marker for some n, it defines $\gamma(n)$ big (bigger than any number that appeared in the construction until now and $\gamma(n) \in A_G$).

We will describe the module in a more general way, so that we can use it later in the construction involving all requirements.

The $(\mathcal{P}, \Gamma)$-strategy: At stage s all parameters will inherit their values from the previous true stage, unless otherwise specified. For this reason we omit the indices that specify the stage. At stage s we do the following:

1. If the stage is not expansionary, then $o = l$, otherwise $o = e$;
2. Choose $n < l(\Theta^{U,\overline{W}}, E)$ in turn ($n = 0, 1, \dots$) and perform the following actions:
   - If $u(n) \uparrow$, then define it new as $u(n) = u(n-1)+1$ (if $n = 0$, then define $u(n) = 1$). If $u(n)$ is defined, but $ap_s < u(n)$ skip to the next element.
   - If $n \in \overline{K}$
     - If $\gamma(n) \uparrow$, then define it anew and enumerate into Γ a new axiom $\langle n, (U \restriction u(n)+1, A_G \restriction \gamma(n)+1)\rangle \in \Gamma$.
     - If $\gamma(n) \downarrow$, but $\Gamma^{U,A}(n) = 0$, then define $\gamma(n)$ anew and define an axiom $\langle n, (U \restriction u(n)+1, A_G \restriction \gamma(n)+1)\rangle \in \Gamma$.
   - If $n \notin \overline{K}$, but $n \in \Gamma^{U,A}$ and the stage is expansionary, then look through all axioms defined for n and extract the A-marker for any axiom that is valid.

Note that if $n \notin \overline{K}$, then we will enumerate only finitely many axioms for n in Γ and hence extract only finitely many markers from A.

2.5 $\mathcal{N}_\Psi$ below $\mathcal{P}_\Theta$

Consider an $\mathcal{N}$-requirement, say $\mathcal{N}_\Psi$, working below one $\mathcal{P}$-requirement $\mathcal{P}_\Theta$. The simple strategy described in Section 2.1 will now not succeed. The A-restraint for N_Ψ following the extraction of x from E conflicts with the need to rectify Γ_Θ. We design a new strategy for $\mathcal{N}_\Psi$ denoted by $(\mathcal{N}_\Psi, \Gamma)$ which resolves this by choosing a threshold d, and trying to achieve $\gamma(n) > use(\Psi, A, x)$ for all $n \geqslant d$ at a stage previous to the imposition of the restraint.

Definition 2. *Let Φ be an enumeration operator and A a set. We will consider a generalised use function φ defined as follows:*

$$\varphi(x) = \max\left\{use(\Phi, A, y) | (y \leq x) \wedge (y \in \Phi^A)\right\}$$

We try to maintain $\theta(x) < u(d)$, in the hope that after we extract x from E, each return of $l(E, \Theta^{U,\overline{W}})$ will produce an extraction from $U \restriction \theta(x)$ which can be used to avoid an A-extraction in moving $\gamma(d)$.

In the event that some such attempt to satisfy N_Ψ ends with a $\overline{W} \restriction \theta(x)$-change, then we will implement a backup strategy $(\mathcal{P}_\Theta, \Lambda)$ which is designed to allow lower priority $\mathcal{N}$-requirements to work below the $(\mathcal{P}_\Theta, \Gamma)$-activity, using the $W \restriction \theta(x)$-changes to move Λ-markers. Each time we progress the $(\mathcal{P}_\Theta, \Lambda)$-strategy, we cancel the current witness for (N_Ψ, Γ) – and if this happens infinitely often, N_Ψ might not be satisfied. This means that $(\mathcal{P}_\Theta, \Lambda)$ must be accompanied by an immediately succeeding copy (N_Ψ, Λ), say, designed to take advantage of the improved strategy for N_Ψ without any other $\mathcal{P}_{\Theta'}$ intervening between (N_Ψ, Γ) and (N_Ψ, Λ).

Module for $(\mathcal{N}_\Psi, \Gamma)$: Similarly, we will define this module in a more general way, so that it is valid for all $\mathcal{N}$-strategies working below one $\mathcal{P}$-strategy. To do this we will incorporate a parameter R which indicates that the strategy can safely assume $B \upharpoonright R$ does not change due to the activity of other $\mathcal{N}$-strategies. The precise definition of R is going to given later.

The basic module acts only at P_Θ-expansionary stages. If there are only finitely many expansionary stages, then P_Θ is trivially satisfied and N_Ψ moves to a truer path through the tree of outcomes.

If the threshold d enters the set K, then we shift its value to the next possible one, that is to the least $n > d$ such that $n \in \overline{K}$. Then we will *restart* this strategy by choosing a new witness and initializing strategies in the subtree of this strategy . As K is coinfinite this process will converge and we will eventually have a constant threshold d.

If at any stage a marker has been extracted from A for some element $n < d$ then we restart the strategy as well. Notice that once the threshold remains permanent, there are only finitely many axioms that can restart the strategy – hence this could happen finitely many times.

- **Initialization**
  1. If a threshold has not yet been defined or is cancelled, choose a new threshold d bigger than any defined until now, with $d > l(\Theta^{U,\overline{W}}, E)$.
  2. If a witness has not yet been defined or is cancelled, choose a new witness $x \in E$, $d < x$, bigger than any witness defined previously. Wait for $x < l(\Theta^{U,\overline{W}}, E)$. $(o = w)$
  3. For every element $y \leq x$, $y \in E$ enumerate into the list *Axioms* the current valid axiom from Θ, which has been valid the longest. That is, for each axiom $Ax_y \in \Theta$ for y let
  $$t_{Ax_y} = \mu r[\forall t(s \geq t \geq r \Rightarrow \text{ the axiom } Ax \text{ was valid at stage } t)],$$
  and then choose the axiom with least t_{Ax_y}.
  4. Extract all A-markers defined for the threshold d. We will modify the definition of $\theta(x)$ once again to incorporate the list *Axioms*, namely $\theta(x) = \max(\bigcup Axioms)$. Define $u(d)$ new, bigger than $\theta(x)$. Notice that this makes all axioms for elements $n \geq d$ invalid. Axioms defined at further stages will have the property that their U-part will include the U-part of the axioms in the list *Axioms*.
- **Honestification**

  Scan the list *Axioms*. If for any element $y \leq x$, $y \in E$, the listed axiom was not valid on some stage after this strategy was last active, then update the list *Axioms*, letting (o = h) and
  1. Extract from A all A-markers of axioms defined for the threshold d. Redefine $u(d)$ new, bigger than $\theta(x)$.
  2. Cancel all markers $u(n)$ for $n > d$ and $n \in \overline{K}$.

  Otherwise go to:

- **Waiting**

  If Γ is honest (that is, $u(d) > \theta(x)$), and all the axioms enumerated in *Axioms* have remained unchanged since the last stage, then wait for $x \in \Psi^A$ with $use(\Psi, A, x) < R$, returning at each successive stage to Honestification (o = w).
- **Attack**
  1. If $x \in \Psi^A$ and $u(d) > \theta(x)$, then extract x from E and restrain A on $use(\Psi, A, x)$. (o = g)
  2. Wait until the length of agreement has returned and $ap_s > \theta(x)$. Notice that the operator Γ will not be modified until such a stage is reached, since we require $ap_s > u(n)$ to consider an element n, and for all elements $n \geq d$ we have that $u(n) > \theta(x)$.
- **Result**

  Let $x' \leq x$ be the least element that has been extracted from E during the stage of the Attack. Call it the attacker. When the length of agreement returns $x' \notin \Theta^{U,\overline{W}}$. Hence all axioms for x' in Θ are not applicable, in particular the one enumerated in *Axioms*, say $\langle x', U_{x'}, \overline{W}_{x'} \rangle$. At least one element has been extracted from $U_{x'}$ or $\overline{W}_{x'}$.

  If no element has been extracted from $\overline{W}_{x'}$ then the attack is successful and the activity at (P_Θ, Γ) lifts the γ-markers of all elements greater than d above the restraint to maintain $A \upharpoonright use(\Psi, A, x)$ unchanged. Note that this change affects all axioms defined in Γ for $n \geq d$, because we insured that all possibly valid axioms in Γ, old and current, contain as a subset $U_{x'}$. If the change in $U_{x'}$ is permanent, then this will lead to success for N_Ψ. Otherwise the attack is unsuccessful and we are forced to capriciously destroy Γ by extracting markers $\gamma(d)$ from A and to start over with a bigger witness. This is necessary in order to provide a safe working space for the backup strategies $(\mathcal{P}, \Lambda)$ and $(\mathcal{N}_\Psi, \Lambda)$.
  1. *Unsuccessful attack*. Extract all A-markers for axioms defined for the threshold d from A. Cancel $\gamma(n)$ for $n \geq d$. Remove the restraint on A. Cancel the current witness x. Return to Initialization at the next stage (choosing a new big enough witness) (o = g).
  2. *Successful attack*. All valid axioms in Γ for $n \geq d$ are with $\gamma(n) > use(\Psi, A, x)$. (o = f) Return to Result at next stage. Note that we will keep returning to Result at all further stages. Hence if it later on turns out that $\overline{W}$ does change, we will re-evaluate the attack as unsuccessful and proceed with a new cycle of this strategy - choosing a new witness.

Analysis of Outcomes – P_Θ has two possible outcomes:

[l] - there is a stage after which $l(\Theta^{U,\overline{W}}, E)$ remains bounded by its previous expansionary value, say L. Then P_Θ is trivially satisfied. In this case we implement a simple "Friedberg- Muchnik" strategy for N_Ψ working with boundary $R = \infty$. Notice that in this case the only elements that can initialize the subtree below outcome l are the ones $< L$, hence finitely many times.

[e] - infinitely many expansionary stages, on which (N_Ψ, Γ) acts:

The possible outcomes of the (N_Ψ, Γ)-strategy are:

[w]- There is an infinite wait at Waiting for $\Psi^A(x) = 1$. Then N_Ψ is satisfied because $E(x) = 1 \neq \Psi^A(x)$ and the Γ_Θ-strategy remains intact. Successive strategies work below $R = \infty$.

[f] - There is a stage after which Success applies exclusively. At sufficiently large stages $\overline{K} \upharpoonright d$ has its final value. So there is no injury to the outcomes below f, $\Psi^A(x) = 1$, N_Ψ is satisfied, leaving the Γ_Θ-strategy intact. Successive strategies work below $R = \infty$.

[h] - There are infinitely many occurrences of Honestification, precluding an occurrence of Attack. Then there is a permanent witness x, which has unbounded $lim_{sup}\theta(x)$. This means that $\Theta^{U,\overline{W}}(y) = 0$, for some $y \leq x$,$y \in E$, thus P_Θ is again satisfied. In this case we also implement a simple "Friedberg-Muchnik" strategy for N_Ψ working below $R = \gamma(d)$.

[g]- We implement the unsuccessful attack step - infinitely often. As anticipated we must activate the $\Lambda_{\Theta,\Psi}$-strategy for P_Θ. N_Ψ is not satisfied, but we have a copy of N_Ψ designed to take advantage of the switch of strategies for P_Θ below N_Ψ. It works below $R = x$.

Module for the (P_Θ, Λ)-strategy: Notice that the outcome g is visited in two different cases - at the beginning of an attack and when the attack turns out to be unsuccessful. The first case starts a nonactive stage for the subtree below g, allowing the other $\mathcal{N}$-strategies to synchronize their attacks. The second case starts an active stage for the strategies in the subtree below g.

The (P_Θ, Λ) (call it) acts only on active stages in a similar but less complicated way than (P_Θ, Γ). Also notice that this strategy is visited only on expansionary stages.

1. Choose $n < l(\Theta^{U,\overline{W}}, E)$ in turn ($n = 0, 1, \dots$) and perform following actions:
   - If $w(n) \uparrow$, then define it new as $w(n) = w(n-1) + 1$. If $w(n)$ is defined, but $ap_s < w(n)$ skip to the next element.
   - If $n \in \overline{K}$
     - If $\lambda(n) \uparrow$, then define it anew and define an axiom $\langle n, (\overline{W} \upharpoonright w(n) + 1, A_L \upharpoonright \lambda(n) + 1) \rangle \in \Lambda$.
     - If $\lambda(n) \downarrow$, but $\Lambda^{\overline{W},A}(n) = 0$ then define $\lambda(n)$ anew and define an axiom $\langle n, \overline{W} \upharpoonright w(n) + 1, A_L \upharpoonright \lambda(n) + 1 \rangle \in \Lambda$. Note that in this case the old axiom is not valid. If this is due to a change in A below the old $\lambda(n)$, then this axiom will never be valid again. If this is due to a change in $\overline{W}$ then the axiom was enumerated on a stage t such that $ap_t > w(n)$. Some element that at stage t was assumed to be in $\overline{W} \upharpoonright w(n)$ has now left $\overline{W}$. Proposition 1 from the section on the approximations tells us that this element will never again appear in the approximations of $\overline{W}$ and hence the axiom will never again be valid.
   - If $n \notin \overline{K}$, but $n \in \Lambda^{\overline{W},A}$ then extract $\lambda(n)$ from A.

Module for (N_Ψ, Λ): This strategy is similar to that for (N_Ψ, Λ), and it has a threshold ($\widehat{d}$, say) and a witness ($\widehat{x}$). It acts mainly on active stages. The only action that is performed on a nonactive stage is the attack.

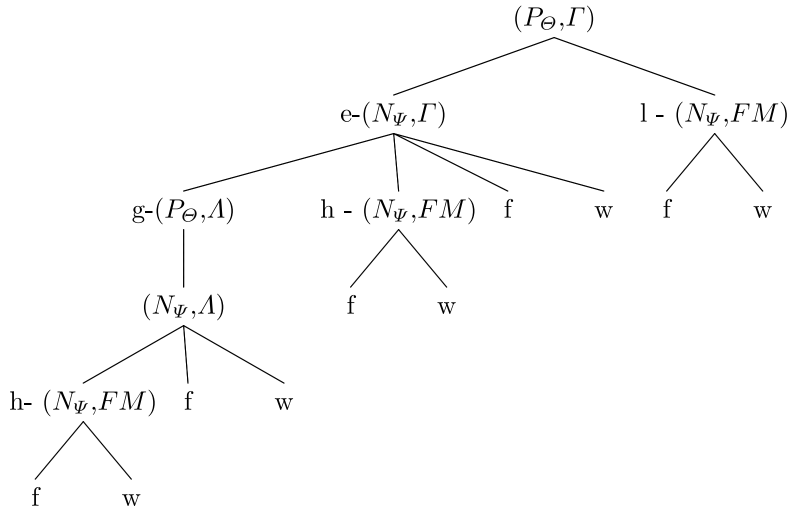
On active stages: If $\widehat{d}$ has entered $\overline{K}$ then shift the value of $\widehat{d}$ to the next possible value and restart the strategy.

If a λ-marker for some element $n < \widehat{d}$ has been extracted from A, since the last stage at which this strategy was active then restart the strategy.

- **Initialization**
  1. Choose a new threshold $\widehat{d}$, bigger than any defined previously, such that $\widehat{d} > l(\Theta^{U,\overline{W}}, E)$.
  2. Choose a new witness $\widehat{x} \in E$, such that $\widehat{d} < \widehat{x}$, bigger than any witness defined until now. Note that the (N_Ψ, Γ)-strategy has just cancelled its witness and will define its next witness x after $\widehat{x}$ has been defined so $\widehat{x} < x$.
  3. Wait for $\widehat{x} < l(E, \Theta^{U,\overline{W}})$. $(o = w)$
  4. For every element $y \leq \widehat{x}$, $y \in E$ enumerate in the list *Axioms* the current valid axiom from Θ, that was valid longest. Extract $\lambda(\widehat{d})$ from A and cancel $w(n)$ for $n \geq \widehat{d}$ and $n \notin K$. Define $w(\widehat{d})$ to be greater than $\theta(\widehat{x})$.
- **Honestification**
  On active stages:
  If for $y \leq \widehat{x}$, $y \in E$, the corresponding axiom in *Axioms* is not valid, then update the list and let (o = h), then
  1. Extract $\lambda(\widehat{d})$ from A. And redefine $w(\widehat{d})$ to be bigger than its previous value and $\theta(\widehat{x})$.
  2. Cancel all markers $w(n)$ for $n \geq \widehat{d}$, $n \notin K$.
- **Waiting**
  If Λ is honest, i.e. $w(\widehat{d}) > \theta(\widehat{x})$, and the list *Axioms* has remained the same since the last stage on which this strategy was active, then wait for $\widehat{x} \in \Psi^A$, with $use(\Psi, A, \widehat{x}) < R$ returning at each successive step to Honestification (o = w).
- **Attack**
  1. Wait for a nonactive stage $(o = w)$. This synchronizes the attacks of the two strategies (N_Ψ, Γ) and (N_Ψ, Λ).
  2. If Λ is honest, then extract $\widehat{x}$ from E.
- **Result**
  The next stage at which this strategy will be accessible will be an unsuccessful attack for (N_Ψ, Γ), hence if the strategy does not get initialized due to a $\overline{K} \upharpoonright \widehat{d}$-change, there will be a $\overline{W} \upharpoonright \theta(\widehat{x})$- change:
  Notice that $\widehat{x} < x$, so that the attacker $x' \leq \widehat{x} < x$ for this strategy is the same as the attacker for (N_Ψ, Γ). This outcome is visited on unsuccessful attacks, which means that $\overline{W} \upharpoonright \theta(x')$ has changed, but $\theta(x') \leq \theta(\widehat{x})$, hence $\overline{W} \upharpoonright \theta(\widehat{x})$ has changed as well. Hence at the next accessible stage we can simple assume:
  Successful attack: Return to Result at the next stage. $(o = f)$

Analysis of Outcomes: The possible outcomes of the (N_Ψ, Λ)-strategy are [w], [f], and [h], exactly corresponding to the outcomes [w],[f] and [h] of (N_Ψ, Γ). Note that in each of these outcomes we will *either* have satisfied the requirement P_Θ, and can implement a simple "Friedberg-Muchnik" strategy to satisfy N_Ψ, *or* have that N_Ψ is satisfied while the Λ-strategy for P_Θ remains intact.

The tree of outcomes at this point looks as follows:



It is worth noticing that the outcomes on the tree, strictly speaking, are outcomes relating to strategies, rather than outcomes telling us exactly how the requirement is satisfied. And these subsume the case relating to the $\mathcal{P}$-requirements when $E \neq \Theta^{U,\overline{W}}$ but there are infinitely many expansionary stages. This case only needs to be specially factored in when one considers in the verification what the strategies deliver.

3 All Requirements

When all requirements are involved the construction becomes more complicated. We will start by describing the tree of outcomes.

The requirements are ordered in the following way:

$$\mathcal{N}_0 < \mathcal{P}_0 < \mathcal{N}_1 < \mathcal{P}_1 \dots$$

Each $\mathcal{P}$-requirement has at least one node along each path in the tree. Each $\mathcal{N}$-requirement has a whole subtree of nodes along each path, the size of which depends on the number of $\mathcal{P}$-requirements of higher priority.

Consider the requirement $\mathcal{N}_i$. It has to clear from A the markers of i $\mathcal{P}$-requirements $\mathcal{P}_0, \mathcal{P}_1, \dots \mathcal{P}_{i-1}$. Each of them can follow one of the three strategies (N_Ψ, Γ_i), (N_Ψ, Λ_i) or (N_Ψ, FM_i). There will be nodes for each of the possible combinations in the subtree.

We distinguish between the following strategies:

1. For every $\mathcal{P}_i$-requirement we have two different strategies: $(\mathcal{P}_i, \Gamma)$ with outcomes $e <_L l$ and $(\mathcal{P}_i, \Lambda)$ with one outcome s.
2. For every $\mathcal{N}_i$-requirement, where $i > 0$, we have strategies of the form $(\mathcal{N}_i, S_0, \dots S_{i-1})$, where $S_j \in \{\Gamma_j, \Lambda_j, FM_j\}$. The requirement $\mathcal{N}_0$ has one strategy $(\mathcal{N}_0, FM)$. The outcomes are f, w and for each $j < i$ if $S_j \in \{\Gamma_j, \Lambda_j\}$ there is an outcome h_j, if $S_j = \Gamma_j$, there is an outcome g_j. They are ordered according to the following rules:
   - For all j_1 and j_2, $g_{j_1} <_L h_{j_2} <_L f <_L w$.
   - If $j_1 < j_2$ then $g_{j_2} <_L g_{j_1}$ and $h_{j_1} <_L h_{j_2}$.

   Let $\mathbb{O}$ be the set of all possible outcomes and $\mathbb{S}$ be the set of all possible strategies.

Definition 3. *The tree of outcomes is a computable function $T : D(T) \subset \mathbb{O}^* \to \mathbb{S}$ which has the following properties:*

1. $T(\emptyset) = (\mathcal{N}_0, FM)$

2. $T(\alpha) = S$ and O_S is the set of outcomes for the strategy S, then for every $o \in O_S$, $\alpha\hat{\ }o \in D(T)$.

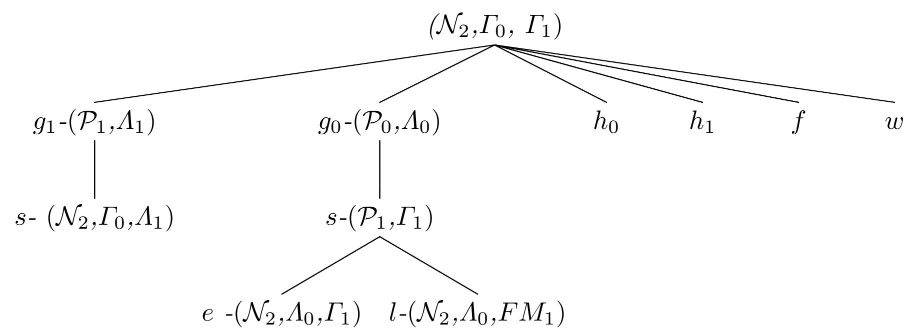
3.If $S = (\mathcal{N}_i, S_0, S_1, \dots, S_{i-1})$, then
$T(\alpha\hat{\ }g_j) = (\mathcal{P}_j, \Lambda_j)$ and $T(\alpha\hat{\ }g_j\hat{\ }s) = (\mathcal{P}_{j+1}, \Gamma_{j+1}) \dots T(\alpha\hat{\ }g_j\hat{\ }s\hat{\ }o_{j+1}\hat{\ }o_{i-2}) = (\mathcal{P}_{i-1}, \Gamma_{i-1})$, where $o_k \in \{e_k, l_k\}$ for $j+1 \leq k \leq i-2$.

All this means that, under an outcome g_j the strategy $\mathcal{P}_j$ starts its work on building the second possible functional Λ_j, and all strategies $\mathcal{P}_k$ for $k > j$ start their work from the beginning, i.e., start building the functional Γ_k again.

$T(\alpha\hat{\ }g_j\hat{\ }s\hat{\ }o_{j+1}\hat{\ }o_{i-1}) = (\mathcal{N}_i, S_0, \dots, \Lambda_j, \dots, S_{i-1})$, where $S_k = \Gamma_k$ if $o_k = e_k$ and $S_k = FM_k$ if $o_k = l_k$ for every k such that $j < k < i$.

Then there is a copy of the strategy $\mathcal{N}_i$ which starts work with the old strategies S_l for $l < j$ and the new strategies S_k for $k \geq j$.

To illustrate this complicated definition here is a picture of this part of the tree in the simpler case of only two $\mathcal{P}$- requirements.



The tree under outcome h_j is built in a similar fashion.

$T(\alpha\hat{\ }h_j) = (\mathcal{P}_{j+1}, \Gamma_{j+1}) \dots T(\alpha\hat{\ }h_j\hat{\ }o_{j+1}\hat{\ }o_{i-2}) = (\mathcal{P}_{i-1}, \Gamma_{i-1})$, *where* $o_k \in \{e_k, l_k\}$ *for* $j+1 \leq k \leq i-2$.

Hence all strategies S_k for $k > j$ start their work from the beginning, building a new functional Γ_k.

$T(\alpha\hat{\ }h_j\hat{\ }o_{j+1}\hat{\ }o_{i-1}) = (\mathcal{N}_i, S_0, \dots, FM_j, \dots, S_{i-1})$, *where* $S_k = \Gamma_k$ *if* $o_k = e_k$ *and* $S_k = FM_k$ *if* $o_k = l_k$ *for every* k *such that* $j < k < i$.

$T(\alpha\hat{\ }f) = (\mathcal{P}_i, \Gamma_i)$.

$T(\alpha\hat{\ }w) = (\mathcal{P}_i, \Gamma_i)$.

Say $S = (\mathcal{P}_i, \Gamma)$, *and* $\alpha = \alpha'\hat{\ }f$ *or* $\alpha = \alpha'\hat{\ }w$, *and* $T(\alpha') = (\mathcal{N}_i, S_0, S_1, \dots, S_{i_1})$. *It follows that this is not the case described and* $(\mathcal{P}_i, \Gamma)$ *appears for the first time, and then*

$T(\alpha\hat{\ }e) = (\mathcal{N}_{i+1}, S_0, \dots, S_{i-1}, \Gamma_i)$,

$T(\alpha\hat{\ }l) = (\mathcal{N}_{i+1}, S_0, \dots, S_{i-1}, FM_i)$.

3.1 Interaction between strategies

In order to prevent unwanted interaction between the different strategies on different nodes we will do the following:

Different $\mathcal{P}$-strategies define and extract different A-markers at stages of the construction. Extraction of markers for one $\mathcal{P}$-strategy may influence the validity of axioms for another $\mathcal{P}$-strategy. Again we deal with this problem by separating the A-markers for the different possible strategies. We have countably many different nodes in the tree of outcomes, whose values are $\mathcal{P}$-strategy . For each such node α we define an infinite computable set A_α, from which the strategy $T(\alpha)$ can choose A-markers. If $\alpha \neq \beta$ then $A_\alpha \cap A_\beta = \emptyset$.

Similarly we define separate non-intersecting sets D_α and X_α for the different nodes on the tree which are labelled with $\mathcal{N}$- strategies, from which they choose their thresholds and witnesses.

As usual we give higher priority to nodes that are to the left or higher up in the tree of strategies. This is achieved by two forms of initialization.

1. On each stage initialization is performed on all nodes that are bigger than the last node visited on that stage. We initialize a $(\mathcal{P}, \Gamma)$-node by setting $\Gamma = \emptyset$, all markers being undefined .
   A $(\mathcal{P}, \Lambda)$-node is initialized by setting $\Lambda = \emptyset$, again, with all markers undefined.
   An $(\mathcal{N}, S_1, S_2, \dots, S_i)$-node is initialized by cancelling all thresholds and the witness. The next time it is visited, it starts from Initialization.
2. The second case when initialization is performed is the following:
   Every strategy α with $T(\alpha) = (\mathcal{N}_i, S_1, \dots, S_j, \dots, S_{i-1})$ has a threshold d_j for each strategy $S_j \neq FM_j$. The active P_j-strategy at α will be the biggest P_j-strategy $\beta \subset \alpha$.
   For each $j \in \{i-1, i-2, \dots 0\}$ in that order we perform $Check_j$ at the beginning of each α-true stage s. Suppose the previous α-true stage is $s-$
   $Check_j$: If an A_j-marker for an element $n \leq d_j$ has been extracted by the active $\mathcal{P}_j$-strategy at a stage t such that $s- < t \leq s$ then:

(a) Initialize all strategies below $\alpha's$ outcomes that assume that d_j is permanent, i.e. below outcomes w, f, h_k where $k < i$ and g_l where $l \leq j$.
(b) If $d_j \in K$ then shift the values of the thresholds d_j and d_l, where $l \leq j$.
(c) If at stage $s-$ the strategy α had not yet started an attack, or if the attack initiated a nonactive stage for strategies below outcomes g_l where $l \leq j$, then cancel the current witness and start from *Initialization*.
(d) If at stage $s-$ the strategy α had started an attack, initiating a nonactive stage for strategies below g_l where $l > j$, then continue to evaluate *Result*.

It will be useful to define a notion of dependency between the different $\mathcal{N}$-strategies. This is important for the synchronization of the attacks.

Definition 4. *1. A node α with $T(\alpha) = (\mathcal{N}_i, S_0, S_1, \dots S_{i-1})$ depends on node $\beta \subset \alpha$, if $\alpha \supseteq \beta\hat{\ }g_j$ for some j.*
2. A node α is independent if it is not dependent on any node $\beta \subset \alpha$.

In case α is dependent, let ins_α be the biggest node on which it depends. The α must time its attacks with the attacks performed by β. That is, whenever α is ready to attack, it waits for a β-nonactive stage, and attacks on that stage. All the rest of the activity by α is performed only on active stages. Note that if $\beta\hat{\ }g_j$ is on the true path, then there will be infinitely many β-nonactive stages on which $\beta\hat{\ }g_j$ is visited.

3.2 The construction

Suppose α is a $\mathcal{P}$-node. We will denote with M_α, m_α, Z_α and z_α: Γ_α, γ_α, U_α and u_α respectively if α is a (P_Θ, Γ)-strategy and Λ_α,λ_α,$\overline{W}_\alpha$ and w_α respectively if α is a (P_Θ, Λ)-strategy. The same notation is valid for an $\mathcal{N}$-requirement and the corresponding S_i-strategy.

On each stage s of the construction we build inductively a string $\delta_s \in D(T)$ of length s, by visiting nodes from the tree starting from the root and acting according to their corresponding strategies. Each visited node will select its outcome, determining the next node to be visited and a right boundary R.

$\delta_s(0) = \emptyset$ and $R_\emptyset = \infty$.

Let $\delta_s \upharpoonright n = \alpha$. If $n = s$ then $\delta_s = \delta_s \upharpoonright n$, go to the next stage. Otherwise let $s-$ be the previous α-true stage.

1. $T(\alpha) = (\mathcal{P}_i, \Gamma)$ on active stages we perform the actions as stated in the main module. $\delta_s(n+1) = l$ at non-expansionary stages. At expansionary stages $\delta_s(n+1) = e$.
   At nonactive stages no actions are performed. The outcome is o_{s-}. The right boundary $R_{\alpha\hat{\ }\delta_s(n+1)}$ is the same as the boundary for α.
2. $T(\alpha) = (\mathcal{P}_i, \Lambda)$ on active stages we perform the actions as stated in the main module. $\delta_s(n+1)$ is the symbol s.
   At nonactive stages no actions are performed, $\delta_s(n+1)$ is the symbol s.
   The right boundary $R_{\alpha\hat{\ }\delta_s(n+1)}$ is the same as the boundary for α.

3. $T(\alpha) = (\mathcal{N}_i, S_0, \ldots, S_{i-1})$

   – **Initialization**

     On active stages:
     Each strategy $S_j \neq FM_j$ picks a threshold if it is not already defined. The different thresholds must be in the following order:

     $$d_{i-1} < d_{i-2} < \cdots < d_0.$$

     Strategy S_j picks its threshold so that it is bigger than any threshold it has picked before and the length of agreement $l(\Theta_j^{U_j,\overline{W}_j}, E)$.
     After all thresholds have been chosen, the strategy picks a witness $x \in E$, bigger than any witness used until now and such that $d_0 < x$. Then waits until $l(E, \Theta_j^{U_j,\overline{W}_j}) > x$ for all $j < i$. Let $\delta(n+1) = w$, working below $R = R_\alpha$.
     On the first stage on which $l(E, \Theta_j^{U_j,\overline{W}_j}) > x$ for all $j < i$, extract all A-markers for all axioms defined for all thresholds d_j in the corresponding active operator S_j, cancel all j-markers $n \geq d_j$ and let $z_j(d_j) > \theta_j(x)$, where $z_j = u_j$ if $S_j = \Gamma_j$ and $z_j = w_j$ if $S_j = \Lambda_j$.
     For every element $y \leq x$, $y \in E$, enumerate in the list $Axioms_j$ the current valid axiom from Θ_j that has been valid longest.
     Go to honestification at the next stage. Notice that this guarantees that any axiom $\langle n, Z_n, A_n \rangle$ enumerated in S_j for an element $n \geq d_j$, $n \in \overline{K}$, will have the property that for any $y \leq x$, $x \in E$, with axiom $\langle y, U_y, W_y \rangle \in Axioms_j$, we will have that $Z_y \subset Z_n$, where $Z = U$, if $S_j = \Gamma_j$ and $Z = W$ if $S_j = \Lambda_j$.
     $\delta(n+1) = h$, working below $R = R_\alpha$.

   – **Honestification**

     On active stages:
     Scan all strategies from the list $S_0 \ldots S_{i-1}$ in turn $(j = 0, 1, \ldots i-1)$. Perform $Honestification_j$ from the main module for each $S_j \neq FM_j$. If the outcome of $Honestification_j$ is w go on to the next strategy. If it is h, then for all $k > j$ extract all A-markers of axioms defined for d_k and define $z_k(d_k) > \theta_k(x)$. Cancel all k-markers for $n \geq d_k$. The outcome is $\delta(n+1) = h_j$ working below $R = \min(R_\alpha, \gamma(d_j)))$. Start from $Honestification$ at the next stage.

   – **Waiting**.

     If all outcomes of all $Honestification_j$-modules are w, i.e all enumeration operators are honest, then wait for $x \in \Psi_i^A$ with $use(\Psi_i, A, x) < R_\alpha$. $f(n+1) = w$, working below $R = R_\alpha$. Return to $Honestification$ at the next stage.

   – **Attack**

     (a) If α is dependent, then wait for a ins_α - nonactive stage. $\delta(n+1) = w$, working below $R = R_\alpha$.

(b) If $x \in \Psi^A$, all operators are honest, then extract x from E and restrain A on $use(\Psi, A, x)$. This starts a nonactive stage for the strategies below the most recently visited outcome g_j (if none has been visited until now, then g_0) working below the boundary it worked in before. The only thing that they can do at this stage is attack with their own witnesses.

– **Result**

Wait until the length of agreement has returned for all strategies and they have been visited at an expansionary stage s with $ap_s^{U_j,W_j} > \theta_j(x)$. Let the attacker x' at this stage be the least element extracted during the attack with entry $\langle x', U_j(x'), \overline{W}_j(x')\rangle$ in the list $Axioms_j$, and let L be the largest restraint imposed on A during the attack. Note that if α is dependent then the attacker for this strategy and the $ins(\alpha)$ is the same. Scan all strategies $S_0, \dots S_{i-1}$ in turn, starting with S_0 and perform the corresponding $Result(j)$ on each.

Result(j):

- If $S_j = FM_j$ or $S_j = \Lambda_j$, then go to $Result(j+1)$.
- If $S_j = \Gamma_j$ and one of the following two conditions is true:
  (a) There was a change in $\overline{W}_j(x')$, i.e $\overline{W}_j(x') \not\subseteq \overline{W}_j[s]$.
  (b) For some $k < j$ an A-marker $m(n)$ of an element n, where $n < d_k$ was extracted by the active $\mathcal{P}_k$-strategy and $m(n) < L$.

  Then extract $\gamma_j(d_j)$ from A. In addition for $k < j$ cancel the thresholds d_k. For $l > j$ cancel all A-markers for d_l and extract them from A. Cancel the witness. Start from Initialization at the next stage. $\delta(n+1) = g_j$, working below $R = \min(x, R_\alpha)$.
- Otherwise the attack is successful. Go to $Result(j+1)$.

Result(i) is reached only in case all attacks were successful.
Then $\delta(n+1) = f$, working below $R = R_\alpha$. Return to $Result(0)$ at the next stage.

3.3 The true path

The true path is defined to be the leftmost path of nodes on the tree that are visited infinitely many times. Such a path exists, because the tree is finitely branching. More precisely it has the following properties:

Definition 5. *The true path f^{tp} is a maximal linearly ordered subset of $D(T)$ such that:*

1. $\forall n \overset{\infty}{\exists} s(f^{tp} \restriction n \subseteq \delta_s)$.
2. $\forall n \exists s_n \forall s > s_n(\delta_s \not<_L f^{tp} \restriction n)$.

We prove that the strategies along the true path satisfy their requirements. To do this we have to first establish that these nodes eventually do not get initialized. The leftmost property of the true path deals with the first case of initialization, but not the second. So we prove the following lemma.

Lemma 1. *For every n there is a stage s_n such that $f^{tp} \upharpoonright n$ does not get initialized after stage s_n.*

Proof. We will prove this by induction on the number n.

The first case $n = 0$ is trivial, as the root of the tree is never initialized.

Assume that the lemma is true for numbers $\leq n$. Let s_1 be a stage such that at stages $t > s_1$, $f^{tp} \upharpoonright n$ is not initialized and $\delta_t \not<_L f^{tp} \upharpoonright (n+1)$. We will consider the different cases depending on the type of the strategy $f^{tp} \upharpoonright n$.

1. $T(f^{tp} \upharpoonright n) = (\mathcal{P}_i, S_i)$, where $S_i \in \{\Gamma_i, \Lambda_i\}$. In this case the strategy $f^{tp} \upharpoonright (n+1)$ will not be initialized at further stages and $s_{n+1} = s_1$.
2. $T(f^{tp} \upharpoonright n) = (\mathcal{N}_i, S_0, \dots, S_{i-1})$. Now there are different cases according to the outcome o along the true path. Starting from the most left, we will examine each of them.

   If $o = g_{i-1}$, then after stage s_1 the threshold d_{i-1} can only change if $d_{i-1} \notin \overline{K}$, and in this case we will choose the next threshold to be $d_{i-1}+1$. As $\overline{K}$ is infinite there will be a stage $s_2 \geq s_1$ after which d_{i-1} will remain fixed. Let s_3 be a stage after which $\overline{K} \upharpoonright (d_{i-1}+1)$ remains unchanged, i.e. no numbers $z \leq d_{i-1}$ enter K after stage s_3. At this stage there are finitely many axioms enumerated in Γ_i for elements $z \leq d_{i-1}$, $z \in K$. And at further stages no new axioms are enumerated for these elements. The markers of these finitely many axioms are the only ones whose extraction from A will force $f^{tp} \upharpoonright n\hat{\ }g_{i-1}$ to be initialized. Let s_4 be a stage by which all of the finitely many markers that get extracted from A are already extracted. Then at stages $t > s_{n+1} = \max(s_1, s_2, s_3, s_4)$, $f^{tp} \upharpoonright (n+1)$ will not be initialized.

   If $o = g_j$ where $0 \leq j < i-1$ then similarly after some stage $s_2 \geq s_1$ all thresholds d_k for $k \geq j$ will remain fixed, as in order to cancel d_k we need to pass through an outcome g_l with $l > k$ and hence to the left of g_j. But this will not happen according to our choice of stage s_1. Let s_3 be a stage after which $K \upharpoonright (\max(d_j, \dots d_{i-1}) + 1)$, does not change. Again, by that time there are finitely many axioms enumerated in each of the operators S_k, $j \leq k \leq i-1$, for elements $z \notin \overline{K}$. Hence there are finitely many markers whose extraction from A could initialize $f^{tp} \upharpoonright (n+1)$. Let s_4 be a stage by which all of these finitely many markers that ever get extracted from A are already extracted. Then after stage $s_{n+1} = \max(s_1, s_2, s_3, s_4)$, we have that $f^{tp} \upharpoonright (n+1)$ will not be initialized.

   If $o \in \{w, s, h_j | j < i\}$, then after some stage $s_2 \geq s_1$ all thresholds remain unchanged and there is a certain stage s_3 after which $K \upharpoonright (\max(d_0, \dots, d_{i-1}) + 1)$ remains unchanged, and a stage s_4, after which no more markers for elements less than or equal to d_j will be extracted from A. Then after stage $s_{n+1} = \max(s_1, s_2, s_3, s_4)$, $f^{tp} \upharpoonright (n+1)$ will not be initialized. □

We turn our attention to the $\mathcal{P}$-requirements. First we establish that $\mathcal{P}$-strategies along the true path will succeed in finding a true axiom for each of the elements $n \in \overline{K}$.

Proposition 4. *Let* $\Theta_j(U_j, \overline{W}_j) = E$ *and* $\alpha \subset f^{tp}$.

1. *Suppose* $\alpha = (\mathcal{P}_j, \Gamma_j)$. *And suppose that for some element* $n \in \overline{K}$ *the current* U_j*-marker and the* γ_j*-marker for each* $m \leq n$ *is not changed by any other strategy after stage* t_0. *Then* α *will stop changing the current marker eventually, and then* $n \in \Gamma_j^{U_j,A}$.
2. *Suppose* $\alpha = (\mathcal{P}_j, \Lambda_j)$. *And suppose that for some element* $n \in \overline{K}$ *the current* W_j*-marker and the* λ_j*-marker for all* $m \leq n$ *are not changed by any other strategy after stage* t_0. *Then* α *will stop changing the current marker eventually and then* $n \in \Lambda_j^{\overline{W}_j,A}$.

Proof. The proof is by induction on n. Suppose the lemma is true for all $m < n$. Then:

1. Suppose $u(n)$ remains the same after stage t_0, the axioms for elements $m < n$ do not change anymore and any such element $m \in K$ is already in K_{t_0} and all markers that get extracted due to m are already extracted. We will use what we know about the approximation to the set U (we are omitting the index j as we will be talking only about the sets $U_j, \overline{W}_j, \Theta_j$), namely that it is Good and Σ_2. There will be a stage $t_1 > t_0$ such that:

Good: $(\forall s > t_1)[s \in G \Rightarrow U \upharpoonright u(n) = U_s \upharpoonright u(n)]$.

Σ_2: $(\forall s > t_1)[U \upharpoonright u(n) \subseteq U_s]$.

We proved that $\{U_s \oplus \overline{W}_s\}_{s<\infty}$ is a good approximation to $U \oplus \overline{W}$ and hence that if $x \in \Theta^{U,\overline{W}}$, then there is a stage t_{Σ_2} such that $\forall s > t_{\Sigma_2}(x \in \Theta_s^{U_s,\overline{W}_s})$ and if $x \notin \Theta^{U,\overline{W}}$ then on good stages s we have $x \notin \Theta_s^{U_s,\overline{W}_s}$. It follows that as $E = \Theta^{U,\overline{W}}$ for any number x there will be a stage $t_l > t_{\Sigma_2}$ such that on all good stages $s > t_l$, $l(\Theta^{U,\overline{W}}[s], E[s]) > x$. The last thing to mention is that ap_s grows unboundedly.

So there will be a good stage $t_2 > t_1$ at which $n < l(\Theta^{U,\overline{W}}, E)$ and $u(n) < ap_{t_2}$. At this stage we will examine the current axiom for n in Γ, say $\langle n, U_n, A_\alpha \upharpoonright m+1\rangle$. If it is valid, then $U_n \subset U_{t_2} = U \upharpoonright u(n)$. And hence at all stages $s > t_2$, we have $U_n \subset U_s$. If it is not valid, then we will enumerate a new axiom $\langle n, U_{t_2} \upharpoonright u(n), A_\alpha \upharpoonright \gamma(n)+1\rangle$, and for this axiom we will have that at all stages $s > t_2$, $U_{t_2} \upharpoonright u(n) \subset U_s$. In both cases the marker $\gamma(n)$ will not be moved at any later stage. And the axiom remains valid forever, and hence $n \in \Gamma^{U,A}$.

2. Suppose $w(n)$ remains constant after stage t_0, the axioms for elements $m < n$ do not change anymore and any $m \in K$ is already in K_{t_0} and all markers that get extracted due to m are already extracted. We can find a stage $t_1 > t_0$ such that:

Good: $(\forall s > t_1)[s \in G \Rightarrow \overline{W} \upharpoonright w(n) = \overline{W}_s \upharpoonright w(n)]$.

Stable: $(\forall s > t_1)[ap_s > w(n)]$.

In fact after stage t_1 the approximation to $\overline{W} \upharpoonright w(n)$ will remain constant. Then on the next α-true stage $t_2 > t_1$ we will examine the current axiom for n in Λ, say $\langle n, \overline{W}_n, A_\alpha \upharpoonright \lambda(n)+1\rangle$. If it is valid then it will be valid forever. If it isn't valid, then we will enumerate a new axiom $\langle n, \overline{W}_{t_2} \upharpoonright w(n), A_\alpha \upharpoonright \lambda(n)+1\rangle$, and this axiom will remain valid forever. □

We now need to establish that the if the influence of the $\mathcal{N}$-strategies on a $\mathcal{P}$-strategy is infinitary, then the $\mathcal{P}$-strategy is satisfied trivially by $E \neq \Theta^{U,\overline{W}}$.

Proposition 5. *1. Let $\alpha \subset f^{tp}$ be the biggest $(\mathcal{P}_j, \Gamma)$-strategy. If $\gamma_j(n)$ moves off to infinity, then the following condition holds:*

If $\Theta_j^{U_j,\overline{W}_j} = E$ then there is an outcome g_j along the true path.

2. Let $\alpha \subset f^{tp}$ be the biggest $\mathcal{P}_j$-strategy. It builds an operator M_j with A-markers denoted by m_j. If $m_j(n)$ moves off to infinity then $\Theta_j^{U_j,\overline{W}_j} \neq E$.

Proof. 1. Assume that $\Theta_j^{U_j,\overline{W}_j} = E$. Let n be the smallest element whose γ_j-marker moves off to infinity, and let s_0 be a stage after which the markers for $n' < n$ do not change and are already extracted from A, if they ever get extracted.

If $n \in K$ then there will be a stage s at which n enters K. After that stage no more axioms for n are enumerated in Γ_j, hence the marker $\gamma_j(n)$ will remain constant. Hence $n \notin K$.

There are finitely many permanent thresholds $d_j \leq n$. If $\gamma_j(n)$ moves off to infinity, then this must be due to a marker of some threshold moving off to infinity.

Hence, according to our choice of n as the least element with unbounded γ_j-marker, $n = d_j$ for some strategy along the true path. The thresholds that belong to strategies to the left are not accessible after a certain stage and the threshold that belong to strategies to the right of the true path are cancelled at every true stage.

Suppose the strategy is $(\mathcal{N}_i, S_0, \dots S_{i-1})$ along the true path.

The true outcome can not be g_k with $k > j$, because then d_j would not be permanent.

Outcomes f and w do not move neither $u(d_j)$ nor $\gamma_j(d_j)$ infinitely often, hence $\gamma_j(d_j)$ would be bounded.

Outcomes g_k with $k < j$ and h_k for $k < j$ are followed by a new strategy $(\mathcal{P}_j, \Gamma_j)$ and hence are also impossible according to our assumption.

Outcome h_j will prove that $\Theta_j^{U_j,\overline{W}_j}(x) \neq E(x)$.

Outcomes h_k for $k > j$, do not move $\gamma_j(d_j)$.

Hence the only possible outcome is g_j.

2. Assume, in order to get a contradiction, that $\Theta_j^{U_j,\overline{W}_j} = E$. Let n be the least element whose marker moves off to infinity. If $M_j = \Gamma_j$, then according to the previous case there will be a strategy $\mathcal{N}$ along the true path with true outcome g_j, followed by another $\mathcal{P}_j$-strategy. This contradicts α being the biggest one.

Hence $M_j = \Lambda_j$. Let s_0 be a stage after which the markers for $n' < n$ do not change and are extracted from A, if they ever get extracted.
If $n \in K$ then there will be a stage s at which n enters K and after which the $\lambda_j(n)$ remains the same. Hence $n \notin K$.
And as in the previous case it is clear that $n = d_j$ for some threshold and some strategy $(\mathcal{N}_i, S_0, \dots \Lambda_j \dots S_{i-1})$ along the true path.
The true outcome cannot be g_k with $k > j$, because then d_j would not be permanent.
Outcomes f and w do not move $w_j(d_j)$ or $\lambda_j(d_j)$ infinitely often, hence $\lambda_j(d_j)$ would be bounded.
Outcomes g_k with $k < j$ and h_k for $k < j$ are followed by a new $\mathcal{P}_j$-strategy and hence are impossible according to our assumption.
Outcome h_j is impossible due to our assumption that $E = \Theta_j^{U_j, \overline{W}_j}$.
Outcomes h_k, for $k \geq j$, are only accessible at stages at which $w_j(d_j)$ does not change. Hence if they are true then $\lambda_j(d_j)$ will eventually come to rest.

This means there are no possible outcomes, which gives the required contradiction. □

Corollary 1. *Every $\mathcal{P}_j$-requirement is satisfied.*

Proof. If $\Theta_j^{U_j, \overline{W}_j} \neq E$, the the requirement is trivially satisfied. Suppose we have $\Theta_j^{U_j, \overline{W}_j} = E$, so that there are infinitely many expansionary stages. Let $\alpha \subset f^{tp}$ be the biggest $(\mathcal{P}_j, M_j)$-strategy along the true path. By Propositions 4 and 5 all markers m_j used to build the operator M_j are bounded.

For each n we prove that $\overline{K}(n) = M_j^{Z_j, A}(n)$, where $Z_j = U_j$ if $M_j = \Gamma_j$ and $Z_j = \overline{W}_j$ if $M_j = \Lambda_j$.

If $n \notin \overline{K}$ then $z_j(n)$ remains constant after the stage at which n exits $\overline{K}$ and is extracted from $M_j^{Z_j, A}$ at least once for every α-true expansionary stage $t > s$ after $ap_s > z_j(n)$. Hence every axiom in M_j for n is eventually invalidated.

If $n \in \overline{K}$, then Proposition 4 proves that $n \in M_j^{Z_j, A}$. □

We turn our attention to the $\mathcal{N}$-requirements, examining first the interactions between them.

Lemma 2. *Let $\alpha \subset f^{tp}$ be an $\mathcal{N}_i$ requirement along the true path. And let s be a stage after which α is not initialized anymore. Then*

1. *None of the nodes to the right or to the left of α extract elements from A that are less than R_α after stage s.*
2. *None of the $\mathcal{N}_j$-nodes above α extract elements from A that are less than R_α after stage s.*
3. *Suppose $\beta \subset \alpha$ is a $\mathcal{P}_j$-node such that there is another $\mathcal{P}_j$-node β', with $\beta \subset \beta' \subset \alpha$. Then β does not extract elements from A that are less than R_α after stage s.*

Hence after stage s the only strategies above α that extract elements from A that are less than the right boundary are the $\mathcal{P}$- strategies that are still active.

Proof. 1. The nodes to the left of α are not accessible and do not extract any elements at all. Strategies to the right are initialized every time we visit α. $\mathcal{P}$- strategies choose their markers bigger than R_α and $\mathcal{N}$-strategies work with new thresholds whose markers are defined after this visit and hence are bigger than any number mentioned, in particular bigger than R_α.

2. Notice that if $\beta \subset \alpha$, then $R_\alpha \leq R_\beta$.
We prove this case with induction on the length of α.
$l(\alpha) = 0$ is trivial.
Let α be of length $n > 0$ and let $\beta = (N_j, S_0 \dots S_{j-1})$ where $j \leq i$ be the greatest $\mathcal{N}$-node above α.
According to the inductive hypothesis none of the $\mathcal{N}$-nodes above β extract elements less than $R_\beta \geq R_\alpha$.
We have a few cases depending on the true outcome o of β.
Note that if β extracts elements from A, then they are markers of thresholds.

   Let $o = g_{j-1}$ with boundary $R = \min(x, R_\beta) = x$. Then $S_{j-1} = \mathit{\Gamma}_{j-1}$. At every β-true stage on which it has this outcome, thresholds d_l , for $l < i-1$ are cancelled and then redefined to be bigger than x. Their markers are chosen on a later stage and hence are bigger than R.
Also all A-markers for d_{j-1} are extracted from A. Any new axiom that enters $\mathit{\Gamma}_{j-1}$ for d_{j-1}, will have an A-marker defined after this stage and hence will be bigger than x.

   Similarly $o = g_l$ with boundary $R = x$ where $l < j-1$. Then after stage s outcomes to the left will not be accessible. Every time we visit β and it has this outcome, the thresholds d_k for $k < l$ are cancelled and redefined bigger than x. All markers for thresholds d_r, $r \geq l$ are cancelled and extracted from A. Hence any new axiom that enters S_r will have A-marker $m(d_r) > x$.

   Say $o = h_0$ with $R = \min(m(d_0), R_\beta)$. Then after stage s outcomes to the left are not accessible, and the witness and all thresholds remain constant. At every β-true stage on which we have this outcome all A-markers for d_l where $l < j$ are extracted from A and are redefined at the next stage to be bigger than R.

   Say $o = h_l$ with $R = \min(m(d_l), R_\beta)$.
Then after stage s the outcomes to the left are not accessible and the markers for the thresholds d_k with $k < l$ are not extracted by β.
The A-markers for d_r where $r \geq l$ are redefined whenever we visit h_l, hence are bigger than the right boundary.

   If the true outcome of β is f or w and α is not initialized anymore, then β does not extract any more elements at all.

3. Let γ be the $\mathcal{N}_k$-node that initiates the cancellation of the $\mathcal{P}_j$-strategy β. Hence the outcome along the true path for γ is $o = h_l$ for $l \leq j$ or $o = g_l$ for $l \leq j$.

Notice that in both cases d_j remains unchanged after stage s, otherwise α would be initialized. Every time we visit h_l or g_l, all A-markers for d_j are extracted from A and a new marker is redefined at the next stage to be bigger than $R_\alpha \leq R_{\gamma\hat{}o} \leq m(d_l)$. Hence if β extracts an element less than R_α it must be a marker of an element $n < d_j$. In which case α would be initialized. □

We claimed that the right boundary R moves off to infinity. Here we give a formal proof.

Proposition 6. *For every node α along the true path $lim_s R_{\alpha,s} = \infty$*

Proof. We prove this statement by induction on the length of the node $\beta \subset f^{tp}$.

The case $length(\beta) = 0$ is trivial because then $T(\beta) = \mathcal{N}_0$, and $R_\beta = \infty$.

If β is a successor of a $\mathcal{P}$-strategy α, then it preserves the boundaries that it receives from α and by the inductive hypothesis the statement is true for R_β.

The only case that remains to be examined is when β is a successor of an $\mathcal{N}$-strategy α. Let $\alpha \subset f^{tp}$ be the strategy $(\mathcal{N}_i, S_0, \ldots, S_{i-1})$ and let the lemma be true for R_α. We will examine the different possibilities for the true outcome o of α. Let s be a stage such that $\beta = \alpha\hat{}o$ is not initialized at any stage $t \geq s$.

1. $o = w$ or $o = f$. Then the right boundary $R_{\beta,t} = R_{\alpha,t}$ on all $t > s$ and is unbounded according to the induction hypothesis.
2. If $o = g_j$ where $j < i$ then the boundary is $R_{\beta,t} = \min(x[t], R_{\alpha,t})$. Every time we visit g_j, x is redefined to be bigger. And by induction $lim_t R_{\alpha,t} = \infty$ and hence $lim_t \min(R_{\alpha,t}, x[t]) = \infty$.
3. $o = h_j$ where $j < i$. Then the right boundary $R_{\beta,t} = \min(m_j(d_j)[t], R_{\alpha,t})$ where $m_j = \lambda_j$ if $S_j = \Lambda_j$ and $m_j = \gamma_j$ if $S_j = \Gamma_j$ will grow unboundedly, because every visit of h_j is accompanied by redefining $m_j(d_j)$. □

Given an $\mathcal{N}_i$-strategy $\alpha = (\mathcal{N}_i, S_0 \ldots S_{i-1})$ along the true path with true outcome f, it is easy to verify in the case that an attack is j-successful whether $S_j = \Gamma_j$ or $S_j = FM_j$. If $S_j = \Lambda_j$ though the success depends on the actions of the instigator for α. We prove that our construction ensures j-success for the last witness x of α.

Lemma 3. *Suppose $\alpha \subset f^{tp}$, $T(\alpha) = (\mathcal{N}_i, S_0 \ldots S_{i-1})$ and $S_j = \Lambda$. And let α begin an attack with witness x at stage $s > s_{|\alpha|}$. Then at the next stage, at which α is accessible, there is a $\overline{W}_j \upharpoonright \theta_j(x_\alpha)$-change or α's outcome is a g-outcome or else α is restarted.*

Proof. Suppose $\Lambda_{i_1} \ldots \Lambda_{i_k}$ are all the Λ-strategies among $S_0 \ldots S_{i_1}$. Then we have a list on nodes $\beta_{i_1} \ldots \beta_{i_k}$ such that $\beta_{i_1}\hat{}g_{i_1} \subset \ldots \beta_{i_k}\hat{}g_{i_k} \subset \alpha$ and $ins_\alpha = \beta_{i_k} \ldots ins_{\beta_{i_2}} = \beta_{i_1}$.

At stage s all of the strategies $\beta_{i_1} \ldots \beta_{i_k}$ start an attack with the same attacker x'. As α is eventually visited again this attack turns out to be unsuccessful for each of the listed strategies so they eventually have outcome $o = g$.

Suppose β_{i_1} has outcome g_k. Here the choice of stage s guarantees that $k \leq i_1$ as any other g-outcome would initialize α. If g_k is chosen due to the first clause of the $Result_k$ then there is a change in $\overline{W}_k \upharpoonright \theta_k(x')$. Then if $k < i_1$ the next time we visit α and we re-evaluate the $Result_k$ the first clause will be valid and α would have outcome at least g_k and this proves the lemma.

If it is chosen due to the second clause of $Result_k$, then for some $l < k \leq i_1$ the active $\mathcal{P}_l$-strategy at β which is necessarily a Γ strategy and is the same as the active $\mathcal{P}_l$-strategy at α (call it δ) extracted an A-marker $m(n) < L$ for $n < d_{l,\beta}$. If $n < d_{l,\alpha}$ then on the next α-true stage $Check_l$ would restart α or α would have outcome g_m for some $m > l$ thus proving the lemma. If $n > d_{l,\alpha}$, then due to the activity at α the axiom for n in Γ_l, say $\langle n, U_n, A \upharpoonright m(n)\rangle$, has the property that $U_l(x') \subset U_n$, and this would be proof that $\overline{W}_l(x') \nsubseteq \overline{W}$, since axioms are extracted only on expansionary stages. Hence, when we next visit α it would have outcome at least g_l thus proving the lemma.

The only other possibility is that β_{i_1} has outcome g_{i_1} due to the first clause in $Result_{i_1}$, thus we have obtained proof that there is a change in $\overline{W}_{i_1} \upharpoonright \theta_{i_1}(x')$ and can move on inductively to β_{i_2}. □

Corollary 2. *Every $\mathcal{N}_i$-requirement is satisfied.*

Proof. Let α be the last $\mathcal{N}_i$ requirement along the true path. We will prove that it satisfies $\mathcal{N}_i$.

α has true outcome w or f or else there will be a successive copy of $\mathcal{N}_i$ along the true path.

In the first case there is a stage s_1 after which α has only outcome w without passing through any other outcome. Let $s_2 \geq s_1$ be a stage after which $\alpha\hat{\ }w$ is not initialized or restarted. Then the thresholds remain constant after stage s_2 and so does the witness x. α waits forever for $\Psi_i^A(x) = 1$ below R_α, and R_α grows unboundedly by Proposition 6, giving $\Psi_i^A(x) \neq E(x)$.

Let the true outcome be f. Then after a certain stage the witness x remains constant. Indeed every time we cancel a witness, we initialize $\alpha\hat{\ }f^{tp}$ either by restarting α or by passing through a g-outcome. Let s be the stage of the attack. Then $x \in \Psi_i^A$ with $use(\Psi_i, A, x) < R_{\alpha,s}$. Hence by Lemma 2 none of the strategies to the left, right and above, except for the active $\mathcal{P}$-strategies at α, will extract elements from A.

The active $\mathcal{P}$-strategies at α do not extract any markers below the restraint either. Indeed if a marker $m(n) < use(\Psi_i, A, x)$ for some element $n < d_j$ is extracted from A, then the witness x would be cancelled. On the other hand when α attacked with x, it received all required permissions.

It is clear that if $S_j = \Gamma_j$ the permission is correct and permanent. Otherwise we would have an outcome to the left of f. By Lemma 3 if $S_j = \Lambda_j$ the permission is correct, otherwise the witness would be cancelled.

Strategies below $\alpha\hat{\ }f$ are accessible for the first time after their initialization on the stage of the attack. Hence all their A-markers would be defined after the stage of the attack and would be greater than the restraint on A. Every new threshold is bigger than the thresholds used by α, and their markers will also have been moved above the restraint. Hence strategies below $\alpha\hat{}f$ will never injure $x \in \Psi_i^A$.

Hence $\mathcal{N}_i$ is satisfied. □

This completes the proof of the theorem. □

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