Non-uniformity and Generalised Sacks Splitting

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November 19, 1999

Abstract

We show that there do not exist computable functions $f_1(e,i)$, $f_2(e,i)$, $g_1(e,i)$, $g_2(e,i)$ such that for all $e, i \in \omega$,

(1) $(W_{f_1(e,i)} - W_{f_2(e,i)}) \leq_{\mathrm{T}} (W_e - W_i)$;
(2) $(W_{g_1(e,i)} - W_{g_2(e,i)}) \leq_{\mathrm{T}} (W_e - W_i)$;
(3) $(W_e - W_i) \leq_{\mathrm{T}} (W_{f_1(e,i)} - W_{f_2(e,i)}) \oplus (W_{g_1(e,i)} - W_{g_2(e,i)})$;
(4) $(W_e - W_i) \not\leq_{\mathrm{T}} (W_{f_1(e,i)} - W_{f_2(e,i)})$ unless $(W_e - W_i) \leq_{\mathrm{T}} \emptyset$; and
(5) $(W_e - W_i) \not\leq_{\mathrm{T}} (W_{g_1(e,i)} - W_{g_2(e,i)})$ unless $(W_e - W_i) \leq_{\mathrm{T}} \emptyset$.

It follows that the splitting theorems of Sacks and Cooper cannot be combined uniformly.

Splitting phenomena play an essential role in the study of the computably enumerable (c.e.) degrees and of the Turing degrees of the difference of c.e. sets (i.e. d.c.e. degrees).

Sacks [8] showed that for any c.e. degree $\mathbf{a} \neq \mathbf{0}$, there exist c.e. degrees $\mathbf{a_0}, \mathbf{a_1}$ such that $\mathbf{a_i} < \mathbf{a}$, $i = 0, 1$, $\mathbf{a_0} \vee \mathbf{a_1} = \mathbf{a}$. Cooper [4] extended this theorem to all levels of the n-c.e. hierarchy. In particular, for any d.c.e. degree $\mathbf{a} \neq \mathbf{0}$, there are d.c.e. degrees $\mathbf{a_0}, \mathbf{a_1} < \mathbf{a}$ such that $\mathbf{a_0} \vee \mathbf{a_1} = \mathbf{a}$. Cooper's proof is non-uniform, treating the c.e. degrees (as in the Sacks splitting theorem) and the properly d.c.e. degrees differently. A basic question is:

Is there a uniform proof of splitting in the d.c.e. degrees which simultaneously subsumes the splitting theorems of Sacks and Cooper?

Following Arslanov [1], giving the first difference between the elementary theories of the c.e. and the d.c.e. degrees, a number of authors reported further differences between these elementary theories and that of the n-c.e. degrees

*The second author was partially supported by NSF grant No. 69973048 and by NSF major grant No. 19931020 of P. R. CHINA

$(n > 1)$ (see [5], [6] and [7]). In this paper, we investigate some subtle distinctions between the c.e. degrees and the n-c.e. degrees, in particular showing that there is no uniform splitting theorem for the d.c.e. degrees.

Theorem. There are no computable functions $f_1(e,i)$, $f_2(e,i)$, $g_1(e,i)$ and $g_2(e,i)$ to satisfy the following properties: for all $e, i \in \omega$,

(1) $(W_{f_1(e,i)} - W_{f_2(e,i)}) \leq_{\mathrm{T}} (W_e - W_i)$;
(2) $(W_{g_1(e,i)} - W_{g_2(e,i)}) \leq_{\mathrm{T}} (W_e - W_i)$;
(3) $(W_e - W_i) \leq_{\mathrm{T}} (W_{f_1(e,i)} - W_{f_2(e,i)}) \oplus (W_{g_1(e,i)} - W_{g_2(e,i)})$;
(4) $(W_e - W_i) \not\leq_{\mathrm{T}} (W_{f_1(e,i)} - W_{f_2(e,i)})$ unless $(W_e - W_i) \leq_{\mathrm{T}} \emptyset$; and
(5) $(W_e - W_i) \not\leq_{\mathrm{T}} (W_{g_1(e,i)} - W_{g_2(e,i)})$ unless $(W_e - W_i) \leq_{\mathrm{T}} \emptyset$.

Our notation and terminology are standard, and generally follow Soare [1987]. In particular, we write $[s]$ after functionals and formulas to denote their evaluation at the end of stage s. During the course of a construction, we may define a parameter as $fresh$, meaning that it is defined as the least number which is greater than any number mentioned so far.

To prove the theorem, we will build a d.c.e. set A uniformly from two given d.c.e. sets X, Y to satisfy the following requirements:

$\mathcal{P}_e$ $A \neq \xi_e$
$\mathcal{R}_e$: $(X = \Phi_e(A) \& Y = \Psi_e(A) \& A = \Theta_e(X,Y)) \rightarrow (A \leq_{\mathrm{T}} X$ or $A \leq_{\mathrm{T}} Y)$

where $e \in \omega$, $\{\xi_e, \Phi_e, \Psi_e, \Theta_e) : e \in \omega\}$ is an effective enumeration of all quadruples $(\xi, \Phi, \Psi, \Theta)$ such that ξ is a partial computable (p.c.) function, and that Φ, Ψ and Θ are p.c. functionals.

A $\mathcal{P}$-Strategy: A $\mathcal{P}$-strategy will satisfy a $\mathcal{P}$-requirement, $A \neq \xi$ say. It is a Friedberg-Muchnik procedure:

1. Choose a *candidate*, a say, which has never been enumerated into A, and enumerate a into A.

2. Wait for a stage at which $\xi(a) \downarrow = 1 = A(a)$, then extract a from A, and stop.

Clearly the $\mathcal{P}$-strategy satisfies the $\mathcal{P}$-requirement $A \neq \xi$.

An $\mathcal{R}$-Strategy: For an $\mathcal{R}$-requirement, $\mathcal{R}$ say (we drop the index), we define the length functions $l(X, \Phi(A))$, $l(Y, \Psi(A))$ and $l(A, \Theta(X,Y))$ as usual. We define L as follows:

$$\begin{aligned} L = \max\{x : \ & (\forall y < x)[\Theta(X,Y;y) \downarrow \\ & = A(y) \ \& \ l(X,\Phi(A)) > \theta(y) \ \& \ l(Y,\Psi(A)) > \theta(y)]\}. \end{aligned}$$

We say that s is $\mathcal{R}$-*expansionary*, if $L[s] > L[v]$ for all $v < s$. If there are only finitely many $\mathcal{R}$-expansionary stages, then $\mathcal{R}$ is satisfied. Suppose that there are infinitely many $\mathcal{R}$-expansionary stages. To satisfy $\mathcal{R}$, we will build p.c. functionals Γ and $\widehat{\Gamma}$ such that either (a) or (b) below holds.

(a) $\Gamma(X)$ is total and $\Gamma(X) =^* A$;

(b) $\widehat{\Gamma}(Y)$ is total and $\widehat{\Gamma}(Y) =^* A$.

Satisfying $(\mathcal{R}, \mathcal{P})$**:** In this case, a $\mathcal{P}$-strategy will satisfy its $\mathcal{P}$-requirement, $A \neq \xi$ say, while its priority ordering is given to an $\mathcal{R}$-strategy which is building p.c. functionals $\Gamma(X)$ and $\widehat{\Gamma}(Y)$. The $\mathcal{P}$-strategy will proceed as follows.

1. Appoint a possible candidate, a say, which is fresh.

2. Wait for a stage at which

   (a) $l(A, \Theta(X, Y)) > a$

   (b) $l(X, \Phi(A)) > \theta(a)$

   (c) $l(Y, \Psi(A)) > \theta(a)$. Then

   - enumerate a into A;
   - create a *Hard Link* $(\mathcal{R}, \mathcal{P})$.

3. (Travel the Hard Link $(\mathcal{R}, \mathcal{P})$) Travel the Hard Link $(\mathcal{R}, \mathcal{P})$ at the next stage at which

   (a) $l(A, \Theta(X, Y)) > a$

   (b) $l(X, \Phi(A)) > \theta(a)$

   (c) $l(Y, \Psi(A)) > \theta(a)$.

   Let s^- be the stage at which the Hard Link $(\mathcal{R}, \mathcal{P})$ was created. There are three cases.

   Case 3a. There is an $x < \theta(a)[s^-]$ such that either $x \in X_{s^-} - X$ or $x \in Y_{s^-} - Y$. Then extract a from A and let $r = s^-$ be the A-restraint.

   [In this case, either $X \neq \Phi(A)$ or $Y \neq \Psi(A)$, and we get a global win for $\mathcal{R}$.]

   Case 3b. There is an $x < \theta(a)[s^-]$ such that $x \in X - X_{s^-}$. Then we say that a is confirmed as a candidate of the $\mathcal{P}$-strategy, and we say that a is a 0-state candidate.

   Case 3c. Otherwise. Then we say that a is confirmed as a candidate of the $\mathcal{P}$-strategy, and we define a as a 1-state candidate.

   We say that the $\mathcal{P}$-strategy *is ready at stage* s, if:

   (a) there is an a which was confirmed as a candidate for the $\mathcal{P}$-strategy;

   (b) $\xi(a) \downarrow= 1 = A(a)$.

4. Wait for a stage at which the $\mathcal{P}$-strategy is ready. Then extract the candidate a from A and stop.

The Building of Γ and $\widehat{\Gamma}$. We proceed as follows.

1. At the stage at which we define a as a 0-state candidate for some $\mathcal{P}$-strategy below the $\mathcal{R}$-strategy, we have that for any $x \geq a$, $\Gamma(X;x)$ has never been defined so far. In this case, for any $x \leq a$, if $\Gamma(X;x)\uparrow$, then define $\Gamma(X;x)\downarrow= A(x)$ with $\gamma(x) = \theta(a)$.
2. If a is built as a 1-state candidate for some $\mathcal{P}$-strategy, then for any $y \leq a$, if $\widehat{\Gamma}(Y;y)\uparrow$, then define $\widehat{\Gamma}(Y;y)\downarrow= A(y)$ with $\widehat{\gamma}(y) = \theta(a)$.

Satisfying $(\mathcal{R}, \mathcal{P}_0, \mathcal{P}_1)$.

A $\mathcal{P}_1$-strategy will satisfy a $\mathcal{P}$-requirement, $A \neq \xi_1$. It is similar to the $\mathcal{P}$-strategy above, the only difference is that if a $\mathcal{P}_1$-strategy builds a 0-state candidate, a_1 say, and there is no 0-state candidate for the $\mathcal{P}_0$-strategy, then the $\mathcal{P}_1$-strategy will turn it over to the $\mathcal{P}_0$-strategy, and it will build a new candidate for its own next time. To understand the idea of a general $\mathcal{P}$-strategy, we describe the $\mathcal{P}_1$-strategy as follows.

1. Appoint a possible candidate, a_1 say, which is fresh.
2. (Test a possible candidate) Wait for a stage at which
   (a) $l(A, \Theta(X, Y)) > a_1$
   (b) $l(X, \Phi(A)) > \theta(a_1)$
   (c) $l(Y, \Psi(A)) > \theta(a_1)$. Then
   - enumerate a_1 into A;
   - create a Hard Link $(\mathcal{R}, \mathcal{P}_\infty)$.
3. (Travel the Hard Link $(\mathcal{R}, \mathcal{P}_\infty)$) Travel the Hard Link $(\mathcal{R}, \mathcal{P}_1)$ at the next $\mathcal{R}$-expansionary stage, s say. Let s^- be the stage at which the current Hard Link $(\mathcal{R}, \mathcal{P}_1)$ was created. Then there are three cases.

   Case 3a. There is an $x < \theta(a_1)[s^-]$ such that either $x \in X_{s^-} - X_s$ or $x \in Y_{s^-} - Y_s$. Then
   - extract a_1 from A;
   - let $r = s^-$ be the A-restraint.

   Case 3b. There is an $x < \theta(a_1)[s^-]$ such that $x \in X_s - X_{s^-}$. Then

- If there is no 0-state candidate for the $\mathcal{P}_0$-strategy, then define a_1 as a 0-state candidate of the $\mathcal{P}_0$-strategy and go back to Step 1.
- Otherwise. Then define a_1 as a 0-state candidate of the $\mathcal{P}_1$-strategy.

Case 3c. Otherwise. Then define a_1 as a 1-state candidate for the $\mathcal{P}_1$-strategy.

A general $\mathcal{P}$-strategy, $\mathcal{P}_e$ say, below an $\mathcal{R}$-strategy is similar to the $\mathcal{P}_1$-strategy above. The difference is that if Case 3b occurs, and there is an $i < e$ for which a $\mathcal{P}_i$-strategy has not been appointed a 0-state candidate, then the $\mathcal{P}_e$-strategy will turn the 0-state candidate to the highest such a $\mathcal{P}_i$-strategy.

We say that a $\mathcal{P}_e$-strategy is *ready at stage* s, if:

(a) the $\mathcal{P}_e$-strategy has built a candidate;

(b) for every candidate a, $\xi(a) \downarrow = 1 = A(a)$.

If a $\mathcal{P}_e$-strategy is ready at stage s, then extract the greatest candidate, a say (of the $\mathcal{P}_e$-strategy) from A.

Satisfying $(\mathcal{R}, \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \cdots)$**:** By the strategies above, if there are infinitely many 0-state candidates which are built during the course of the construction, then $\Gamma(X)$ is built infinitely often, and by the appointment of the possible candidates, $\Gamma(X)$ is total. And by the definition of the candidates, if a 0-state candidate, a say, which is extracted from A, then we will get a permanent X-permission for this extraction, i.e. an $x \leq \gamma(a)$ will be extracted from X. (Otherwise, we will get a finite inequality $X \neq \Phi(A)$). Because $\Gamma(X)$ is built only at the stages at which a 0-state candidate is built, so for any 1-state candidate a, $\Gamma(X; a) = A(a)$. We have that $\Gamma(X) = A$. If there are only finitely many 0-state candidates which are built during the course of a construction, then $\Gamma(X)$ is a finite function, and $\widehat{\Gamma}(Y)$ is built as a total function, and $\widehat{\Gamma}(Y) =^* A$. Therefore in any case, $\mathcal{R}$ is satisfied.

By the strategies above, every $\mathcal{P}$-strategy will get at least one permanent candidate. Let a be the greatest candidate for a $\mathcal{P}$-strategy. Then if $\xi(a) \neq 1$, then $\xi(a) \neq 1 = A(a)$, otherwise, we will create an inequality $\xi(a) \downarrow = 1 \neq 0 = A(a)$. $\mathcal{P}$ is satisfied.

The Priority Tree: We first look at some definitions.

Definition 1. (i) We define *the priority ranking of the requirements* as follows $\mathcal{R}_0 < \mathcal{P}_0 < \mathcal{R}_1 < \mathcal{P}_1 < \mathcal{R}_2 < \mathcal{P}_2 < \cdots$.

(ii) The **possible outcomes** of an $\mathcal{R}$-strategy are $0 <_{\mathrm{L}} 1$ to denote infinite and finite actions respectively.

(iii) The **possible outcomes** of a $\mathcal{P}$-strategy is 1.

Definition 2. Given a node ξ:

(i) We say that $\mathcal{R}_e$ is *satisfied at* ξ if either there is an $\mathcal{R}_i$-strategy α such that $\alpha\hat{\ }\langle 0\rangle \subseteq \xi$, or there is an $\mathcal{R}_e$-strategy α such that $\alpha \subset \xi$.

(ii) We say that $\mathcal{P}_e$ is *satisfied at* ξ, if there is a $\mathcal{P}_e$-strategy β such that $\beta \subset \xi$.

We now define the priority tree T.

Definition 3. (i) Define the root node $\emptyset$ as an $\mathcal{R}_0$-strategy.

(ii) The immediate successor of a node are the possible outcomes of the corresponding strategy.

(iii) A node ξ will work for the highest priority requirement which is not satisfied at ξ.

For a $\mathcal{P}$-strategy β, let $c(\beta)$ be the greatest candidate of β, and let $a(\beta)$ be the greatest possible candidate of β. If β is initialised, then both $a(\beta)$ and $c(\beta)$ are cancelled (if they are defined). If there is an $\mathcal{R}$-strategy α such that $\alpha\hat{}\langle 0\rangle \subseteq \beta$, then we say that β is a 0-type strategy, in which case, let $\tau(\beta)$ be the unique α such that $\alpha\hat{}\langle 0\rangle \subseteq \beta$. If a $\mathcal{P}$-strategy β is not a 0-type strategy, then we say that β is a 1-type strategy. For an $\mathcal{R}$-strategy α, we will build p.c. functional $\Gamma_\alpha(X)$.

Suppose that β is a 0-type strategy, let $\alpha\hat{}\langle 0\rangle \subseteq \beta$. We say that α *is delayed by* β if β is the least strategy γ such that $c(\gamma)\downarrow$ and $c(\gamma)$ is a 1-state candidate. If α is delayed by β, then we are building a p.c. functional $\widehat{\Gamma}_\beta(Y)$. If an $\mathcal{R}$-strategy α is initialised, then $\Gamma_\alpha(X)$ is set to be totally undefined. If a $\mathcal{P}$-strategy β is initialised, then set $\widehat{\Gamma}_\beta(Y)$ to be totally undefined.

The Construction

The construction will proceed as follows.

Stage $s = 0$. Set $A = \emptyset$, initialise all strategies.

Stage $s > 0$. We say that ξ *is visited at stage* s, if ξ *is eligible to act at a substage of stage* s. We first allow the root node ξ to be eligible to act at substage $t = 0$.

Substage t. Let ξ be eligible to act at substage t. If $t = s$, then initialise any strategy γ with $\xi <_{\mathrm{L}} \gamma$. If $t < s$, then there are three cases.

Case A. $\xi = \alpha$ is an $\mathcal{R}$-strategy. Run the following steps.

1. s is not α-expansionary, or there is a β such that

   (a) $\alpha\hat{}\langle 0\rangle \subseteq \beta$,

   (b) $a(\beta)\downarrow$ or $c(\beta)\downarrow$

   Let $b(\beta) = \max\{a(\beta), c(\beta)\}$.

   (c) either $l(A, \Theta_\alpha(X, Y)) \not> b(\beta)$, or $l(X, \Phi_\alpha(A)) \not> \theta_\alpha(b(\beta))$ or $l(Y, \Psi_\alpha(A)) \not> \theta_\alpha(b(\beta))$, then let $\beta\hat{}\langle 1\rangle$ be eligible to act next.

2. If there is a *Red Link* (α, β) which was created and which has neither been cancelled nor been travelled, then let β_0 be the $<$-least such β and let β be eligible to act next.

3. There is a $\mathcal{P}$-strategy β such that $\alpha\hat{\ }\langle 0\rangle \subseteq \beta$ and a Hard Link (α, β) which was created previously, has neither been cancelled nor been travelled. Then let β_0 be the least such β, and let β be eligible to act next.

4. There is a $\mathcal{P}$-strategy β such that $\alpha\hat{\ }\langle 0\rangle \subseteq \beta$, $a(\beta)\downarrow$, and either $c(\beta)\uparrow$ or $c(\beta) \neq a(\beta)$, then let β_0 be the least such β, and let β_0 be eligible to act next.

5. Otherwise. Then let $\alpha\hat{\ }\langle 0\rangle$ be eligible to act next.

Case B. $\xi = \beta$ is a 0-type strategy. Let $\alpha = \mathrm{top}(\beta)$, and let β^* be the strategy $\gamma \subset \beta$ such that α is delayed by γ at stage s (if such a γ exists). Run the following:

Program β

1. (Travel the Red Link (α, β)) Let $c(\beta) = c$, if $\beta^*\downarrow$, then for any $x \leq c$, if $\widehat{\Gamma}_{\beta^*}(Y; x)\uparrow$, then define is as $A(x)$ with $\widehat{\gamma}_{\beta^*}(x) = \theta_\alpha(c)$, initialise any $\xi \not\leq \beta$ and go to stage $s+1$; if $\beta^*\uparrow$, then for any $x \leq c$, if $\Gamma_\alpha(X; x)\uparrow$, then define it as $A(x)$ with $\gamma_\alpha(x) = \theta_\alpha(a)$, initialise any $\xi \not\leq \beta$ and go to stage $s+1$.

2. If $c(\beta)\downarrow$ and $c(\beta) \notin A$, then let $\beta\hat{\ }\langle 1\rangle$ be eligible to act next.

3. If a Hard Link (α, β) which was created at stage, s^- say, has neither been cancelled nor been travelled. Then travel the Hard Link (α, β) by the following cases: for $a = a(\beta)$

   Case 3a. There is an $x < \theta_\alpha(a)[s^-]$ such that either $x \in X_{s^-} - X_s$ or $x \in Y_{s^-} - Y_s$. Then

   - extract a from A;
   - let $r(\beta) = s^-$ be the A-restraint;
   - initialise any ξ with $\beta <_{\mathrm{L}} \xi$ and go to stage $s+1$.

   Case 3b. There is an $x < \theta_\alpha(a)[s^-]$ such that $x \in X_s - X_{s^-}$.

   Substep (i). if β^* is defined. Then

   - define a as a 0-state candidate of β^*;
   - initialise all $\xi \not\leq \beta^*$.

   Substep (ii). if β^* is not defined. Then

   - define a as a 0-state candidate of β, initialise any $\xi \not\leq \beta$.

Substep (iii). for any $x < a$, if $\Gamma_\alpha(X;x)\uparrow$, then define $\Gamma_\alpha(X;x) = A(x)$ with $\gamma_\alpha(x) = \theta_\alpha(a)$, and go to stage $s+1$.

Case 3c. Otherwise. Then define a as a 1-state candidate of β, if $\beta^*\downarrow$, then let $\beta_* = \beta^*$, if $\beta^*\uparrow$, then let $\beta_* = \beta$, for any $x \leq a$, if $\widehat{\Gamma}_{\beta_*}(Y;x)\uparrow$, then define $\widehat{\Gamma}_{\beta_*}(Y;x)\downarrow = A(x)$ with $\widehat{\gamma}_{\beta_*}(x) = \theta_\alpha(a)$, initialise any $\xi \not\leq \beta$ and go to stage $s+1$.

4. (Create a Hard Link) If $a(\beta) \neq c(\beta)$, then
   - enumerate $a(\beta)$ into A;
   - create a Hard Link (α, β);
   - initialise any $\xi \not\leq \beta$ and go to stage $s+1$.

   We say that β is *ready at stage* s, if:

   (a) $c(\beta)\downarrow = a(\beta) = a$;

   (b) $\xi_\beta(a)\downarrow = 1 = A(a)$.

5. If β is ready at stage s, then let $a = c(\beta)$, extract a from A, create a Red Link (α, β), initialise all $\xi \not\leq \beta$ and go to stage $s+1$.
6. If $c(\beta)\downarrow = a(\beta)$, then let $\beta\hat{\ }\langle 1\rangle$ be eligible to act next.
7. Otherwise. Then
   - define $a(\beta)$ as fresh;
   - initialise any $\xi \not\leq \beta$ and go to stage $s+1$.

Case C. $\xi = \beta$ is a 1-type strategy. Then run the following steps:

1. If $a(\beta)\downarrow \notin A$, then let $\beta\hat{\ }\langle 1\rangle$ be eligible to act next.
2. If $a(\beta)\downarrow = a$, and $\xi_\beta(a)\downarrow = 1 = A(a)$, then extract a from A, initialise any $\xi \not\leq \beta$ and go to stage $s+1$.
3. If $a(\beta)\downarrow$, then let $\beta\hat{\ }\langle 1\rangle$ be eligible to act next.
4. Otherwise. Then
   - define $a(\beta)$ as fresh;
   - enumerate $a(\beta)$ into A;
   - initiliase any $\xi \not\leq \beta$ and go to stage $s+1$.

The Verification

We now verify the correctness of the construction.

Definition 4. (i) Let δ_s be the last node which is visited at stage s.

(ii) We define *the true path TP of the construction* as $TP = \liminf_s \delta_s$.

Lemma 1. Given $\xi \in TP$:

(i) ξ is initialised only finitely many times.

(ii) ξ is visited infinitely often.

Proof. This is immediate from the construction.

Lemma 2. If there is no α such that $\alpha\hat{\ }\langle 0\rangle \in TP$, then all $\mathcal{R}_e$ and all $\mathcal{P}_e$ are satisfied.

Proof. A finite injury argument.

Lemma 3. If there is an α such that $\alpha\hat{\ }\langle 0\rangle \in TP$, then all $\mathcal{P}_e$ are satisfied.

Proof. Let α be the unique ξ such that $\xi\hat{\ }\langle 0\rangle \in TP$, let $e_0 = e(\alpha)$. Then every $\mathcal{P}_e$ for $e > e_0$, will be satisfied by a 0-type strategy β such that $\alpha\hat{\ }\langle 0\rangle \subseteq \beta \in TP$. The proof for this is an easy finite injury argument.

Lemma 4. If there is a τ such that $\tau\hat{\ }\langle 0\rangle \in TP$, then either $A \leq_{\mathrm{T}} X$ or $A \leq_{\mathrm{T}} Y$.

Proof. Let α be the unique τ such that $\tau\hat{\ }\langle 0\rangle \in TP$. There are two cases.

Case A. There is a β such that $\alpha\hat{\ }\langle 0\rangle \subseteq \beta$ and $\widehat{\Gamma}_\beta(Y)$ is built infinitely often.

Let s_0 be the strategy at which β is initialised. We note that for any γ with $\beta <_{\mathrm{L}} \gamma$, if $a = a(\gamma)$ is defined at a stage $s_1 > s_0$, then let s_2 be the least stage $> s_1$ at which β is visited, then $a \in A$ if and only if $a \in S_{s_2}$.

By the construction, for any $\beta \subset \gamma_1 \subset \gamma_2$, any s, let $c_1 = c(\gamma_1)[s]$, $c_2 = a(\gamma_2)[s]$ or $c_2 = c(\gamma_2)[s]$, the enumeration and extraction of c_2 will never injure $\widehat{\Gamma}_\beta(Y;c_1)[s]$. By the construction, for any $\beta \subset \gamma$, for any $s > s_0$, if $c = a(\gamma)$ or $c = c(\gamma)$, then c is cancelled at stage s only if there is a strategy γ' such that $\beta \subseteq \gamma' \subset \gamma$, and $c(\gamma')$ is extracted from A. By the definition of the candidate, for any $\beta \subseteq \gamma$, any stage $s_1 (> s_0)$ say, if $c(\gamma)$ is extracted from A at stage s_1. Then $\widehat{\Gamma}_\beta(Y;c)$ will be redefined as 0 at the next α-expansionary stage, s_2 say. The point is that $\widehat{\Gamma}_\beta(Y;c)[s_2]$ will never be redefined at any stage $> s_2$, because there is a $y < \widehat{\gamma}_\beta(c)[s_1]$ which is in Y at the stage at which we define $\widehat{\Gamma}_\beta(Y;c)$ for the first time, and which has been extracted from Y since stage s_1. Therefore $\widehat{\Gamma}_\beta(Y)$ is total and $\widehat{\Gamma}_\beta(Y) = A_\beta$, where $A_\beta = \{a \in A : a = a(\gamma) \text{ or } a = c(\gamma) \text{ for some } \gamma \text{ with } \beta \subset \gamma\}$. By the argument above, we note that $A - A_\beta$ is a computable set. Therefore in this case $A \leq_{\mathrm{T}} Y$.

Case B. Otherwise.

Let A_α be the set of elements of A which are enumerated by strategies below $\alpha\hat{\ }\langle 0\rangle$. Clearly $A - A_\alpha$ is computable. We now prove that $A_\alpha \leq_{\mathrm{T}} X$.

Let s_0 be the least stage at which α is initialised. By the construction, if a 0-state candidate c is built for some $\mathcal{P}$-strategy $\beta \supseteq \alpha\hat{\ }\langle 0\rangle$ at a stage, $s_1 > s_0$ say, then for any $x < c$, if x has not been built as a 0-state candidate for any $\mathcal{P}$-strategy $\supseteq \alpha\hat{\ }\langle 0\rangle$, then $x \in A$ if and only if $x \in A_{s_1}$. By the construction, if a possible candidate a say, which will be proven to be a 0-state candidate is enumerated into A at a stage $s_1 > s_0$, then $\Gamma_\alpha(X; a)$ has never been defined. And by the same argument as that in Case A, X via Γ_α can recoganise every extraction of the 0-state candidate of a $\mathcal{P}$-strategy $\supseteq \alpha\hat{\ }\langle 0\rangle$. Therefore $A_\alpha \leq_{\mathrm{T}} X$ and so $A \leq_{\mathrm{T}} X$.

Lemma 4 follows.

This completes the proof of the theorem.

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