ON THE INTENSIONALITY OF TURING DEFINITIONS

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ABSTRACT. Despite the importance of Turing definability being generally accepted within computability theory, there is no consensus relating to its wider significance, or agreement as to whether any such wider role for Turing definability should be expected, or looked for. At the same time there are differences of approach to the technical problem of actually identifying Turing invariant relations. These themes, and their relationships, are pursued in the context of some recent research.

1. Mathematics and meaning

> *"Although it is largely unrecognised, mathematics is, of course, part of our culture, because everything humans do is part of human culture. In a trivial sense mathematical objects and theorems are creations of human minds.*
>
> *The more profound mystery is why, after being created or discovered (either term is legitimate), they so beautifully fit the outside world."*
>
> — Martin Gardner, *The Night Is Large: Collected Essays 1938–1995,* St. Martin's Press, New York, 1996, p. 255.

> *"There is truth and then again there is truth . . . there really is no bottom to what is not known."*
>
> — Philip Roth, *The Human Stain,* Houghton Mifflin, Boston, Mass., 2000, p. 315.

The computability theorist is commonly motivated by an implacable sense of 'reality' attaching to the content of his or her research. And, of course, it is not

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[†] Partially supported by EPSRC Research Grant no. GR/M 91419, "Turing Definability", and by INTAS-RFBR Research Grant no. 97-0139, "Computability and Models".

1991 *Mathematics Subject Classification.* Primary 03D25, 03D30; Secondary 03D35.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

uncommon that a sense of intensionality in mathematics is initially confined within the investigative process. The exact relationship between mathematics — and in particular computability theory — and the ‘outside world’ referred to in the first quotation above, is often obscured by that process.

This paper focuses (briefly) on such questions as: What aspects of the real world *might* computability theory describe, even explain? And how does the tension between model and theory work itself out at a technical level, as regards the balance between specificity and organising principles? Attempts to extend what is known concerning Turing definable relations seem particularly relevant to both questions, and this is the context in which the second question in particular is examined.

So what are the basic features of this ‘outside world’? Any description would have to contend with *objects* — easy to interact with, hard to describe — and *algorithmic content*, a familiar component of the universe since Newton and before. In this sense, as has often been commented on, computability comes prepackaged, independent of any particular language necessary to its description. The need for *approximation* is an inevitable counterpart of this algorithmic content, of which a principal task of the scientist is its identification, and arises from the failure of attempts *in practice* to model the universe discretely. More generally, one is driven to accept the apparent importance of uncompleted infinities and the description of physical parameters via reals. Outside of science, there is a parallel recognition of the inadequacy of epistemological discreteness (specifically, language). Finally — and more contentiously — one has *incomputability* as an inconvenient fact of life. It is certainly the case that despite an apparent (causal) determinism, there is a breakdown of purely local scientific explanations of such phenomena as quantum mechanics or the origin of natural laws.

Given such essential ingredients, one finds it hard to reject the relevance of *logical structure* to some of the most fundamental problems confronting science and epistemology, and this is an approach increasingly observed outside of logic itself. What the logician can contribute is a wider understanding of the notion of *mathematical definability*, and of its potential importance in the modelling of globally determined material phenomena. Specifically, one would like to see a greater appreciation of the *Turing universe* as an appropriate structure within which to theoretically capture the problematic features of Martin Gardner’s outside world.

One can schematically represent the explanatory power potentially realisable (see below), showing computability theory arising naturally from empirically observable phenomena, and in return providing informative modelling of such phenomena (via suitably qualified variants of the Church-Turing thesis); the interplay between the mathematical information content – typically relations on reals – extractable and the observables themselves expressed via scientific theories; and the role which computability theory can play in organising and enriching the information content of those models, in such a way as to usefully mediate between model and modelled:[1]

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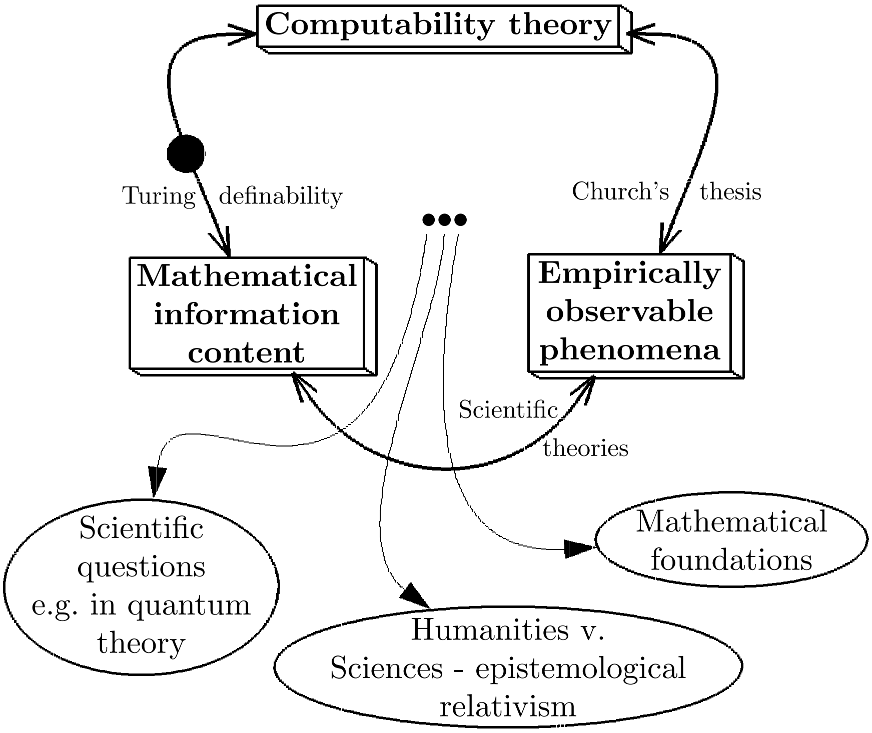
[1] For further background to the diagram which follows, see Cooper [1999].



2. The Turing universe

For an introduction to the Turing universe (via the degree structure based on Turing's [1939] notion of oracle machine) see Odifreddi [1989] or Soare [1987]. We use their standard notation and terminology, but as is quite usual now, translate all informal usage of terms derived from 'recursive' to the corresponding computability theoretic term.

Corresponding to the ith oracle Turing machine, Φ_i denotes the ith partial computable (p.c.) functional $2^\omega \to 2^\omega$. We say A is *Turing reducible to* a B ($A \leq_T B$) if and only if $A = \Phi_i^B$, some $i \in \omega$. And A, B are said to be *Turing equivalent* ($A \equiv_T B$) if and only if $A \leq_T B$ and $B \leq_T A$.

Then the *degree of unsolvability* or *Turing degree* of A is defined by

$$\deg(A) = \{X \in 2^\omega \mid A \equiv_T X\}.$$

And $\leq$ is the induced partial ordering on $\mathcal{D}$ (= the set of all degrees), $\mathbf{0}$ = the least degree (consisting of all computable sets of numbers), and $\boldsymbol{\mathcal{D}}$ is the structure $\langle \mathcal{D}, \leq \rangle$. Also $W_i^A = \operatorname{dom} \Phi_i^A$ denotes the ith *computably enumerable in* A (A-*c.e.*) set ($W_i = W_i^\emptyset$ being the ith c.e. set).

The *jump* — or $n+1$ *th jump* — of a set A is defined by $A' = A^{(1)} = \{x \mid x \in W_x^A\}$ — or, $A^{(n+1)} = (A^{(n)})'$), respectively. The *jump operator* on degrees is defined by $\mathbf{a}' = \deg(A')$, $A \in \mathbf{a}$, where $\mathbf{a} < \mathbf{a}'$, and $\mathbf{a}'$ is the l.u.b. of the degrees of sets c.e. in $A \in \mathbf{a}$. We also write $\mathbf{a}^{(n+1)} = \deg(A^{(n+1)}) = (\mathbf{a}^{(n)})'$, and define the standard ω*-jump* of $\mathbf{a}$ by

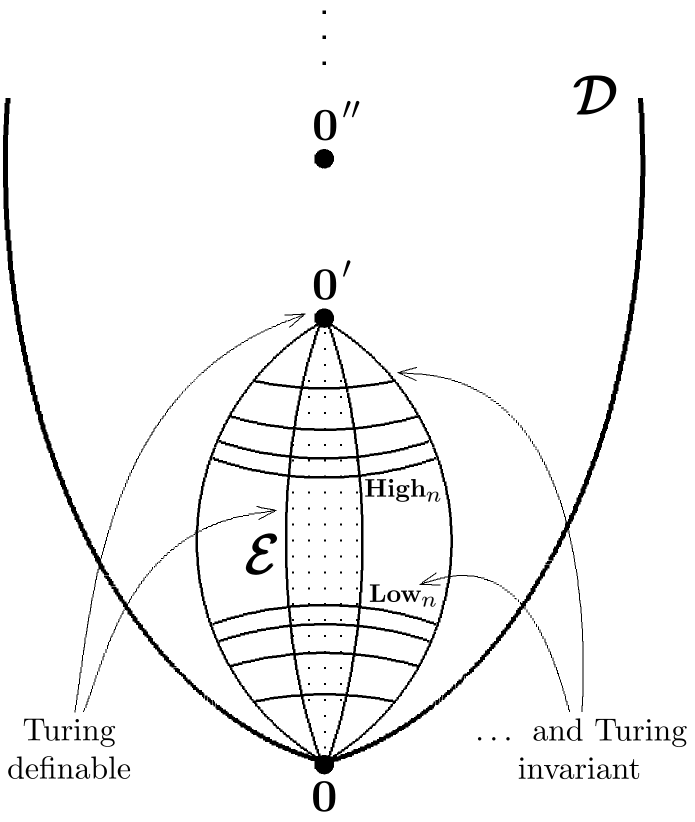
$$\mathbf{a}^{(\omega)} = \deg(\oplus_{n\in\omega} A^{(n)}), \quad A \in \mathbf{a}.$$

Then write $\boldsymbol{\mathcal{D}}'$ for the structure $\langle \mathcal{D}, \leq, ' \rangle$.

A relation on $\boldsymbol{\mathcal{D}}$ is said to be *Turing definable* if and only if it is describable in the first order theory of $\boldsymbol{\mathcal{D}}$. A relation on $\boldsymbol{\mathcal{D}}$ is *Turing invariant* if and only if it is unchanged by any automorphism $\boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{D}}$.

There is a close relationship between Turing definability and the strictly weaker (it turns out) notion of Turing invariance. It is worth noting that by Rogers [1967], given $\langle U, R\rangle$, with just one binary relation R, and $V \subseteq U$, then V is invariant if and only if V is definable in $\mathcal{L}_{\alpha\beta}$ (with predicate symbol for R), some α, β. So invariance does coincide with definability if the underlying language is sufficiently enriched.

What is more interesting, an increasingly close relationship between natural information content — in particular canonical relations on reals — and Turing definability has emerged. One can summarise the local situation diagrammatically:



Naturally arising information content in the Turing universe

In the diagram, the definable substructure $\boldsymbol{\mathcal{E}}$ (= the *computably enumerable degrees*) is just that part of $\boldsymbol{\mathcal{D}}$ containing c.e. sets. And the members $\mathbf{High}_n$, $\mathbf{Low}_n$, $n \geq 1$, of the *high/low hierarchy*, all Turing definable and hence invariant, are defined simply in terms of the jump by:

$$\mathbf{High}_n = \{\mathbf{a} \leq \mathbf{0}' \mid \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}\}, \quad \mathbf{Low}_n = \{\mathbf{a} \leq \mathbf{0}' \mid \mathbf{a}^{(n)} = \mathbf{0}^{(n)}\}.$$

3. Approaches to Turing definability

Proving invariance usually entails producing definitions in the structure in question. And one observes two more-or-less distinct approaches to establishing definability in computability theory, each with their own particular characteristics and aesthetic appeal.

The first of these approaches (call it ‘Approach I’) is via **coding techniques**, where one attempts to code information into degree structures (say) via generic techniques. The latter may originate quite independently of a given application, and depend, ideally, on a base level of structure common to many different contexts. It is usual to trace the origins of such techniques back to Simpson’s work at Berkeley in 1973–74, with the first major applications appearing in Jockusch and Simpson [1976] and Simpson [1977]. However it is worth noting that the theory of initial segments of the Turing degrees did produce global applications (see Lerman [1983]) having some of the flavour, if not the radical switch of perspective, of the later work.

Further landmarks in the development and refinement of the coding approach were Nerode and Shore [1980], [1980a], Harrington and Shelah [1982], Slaman and Woodin [1986] and Nies, Shore and Slaman [1996], while there were a number of other significant contributions by these and other researchers.

Main distinguishing characteristics of Approach I:

The content from computability theory of unadulterated coding is relatively simple, and so that the techniques are applicable to a broad spectrum of structures, and relations within those structures. One typically finds a multiplicity of applications, each ‘minor perturbations’ of others. So in the first instance, the approach tends to bring out *similarities* between structures and relations on them rather than differences.

Another dividend is a better understanding of *how much* computability theory is needed for a given definition.

There is also the promise of ‘easier’ proofs. However, the reader’s expectations may be frustrated by a need for complex combinatorial structures of implementation, with a resulting distancing from the object being defined. And tension between code and object may emerge, particularly at local level, leading to a reliance on parameters, or hybrid ‘pull down’ arguments and convergence with approach II. The usual asset-stripping in relation to local degree theory may not suffice for more sophisticated applications of codings, and the difficulties encountered can throw up new and interesting technical developments which feed back into approach II. Underlying all this is the bi interpretability conjecture, of course, and, at the local level particularly, one may end up doggedly straining to develop fairly uninformative fragments of bi interpretability. That being said, there is no doubt that the achievements and major successes of the approach are some of the crowning glories of the subject. Bi interpretability can bring out definability of whole hierarchies of relations, and only at the lower levels do special problems lead one to look for more appropriate techniques.

A curiosity-led investigation of the structural, and informational, *content* of Turing relations characterises Approach II.

This is the historically earlier approach to definability, and can be traced back to Kleene and Post [1954] and a time when structure-theory was not yet properly regarded as the raw material of Turing definitions. The landmarks in the development and maturation of approach II comprised the mainstream of the subject for around two decades, of which one must at least mention seminal papers by Spector [1956], Shoenfield [1966], leading on to those of Yates [1970], [1976], Lerman [1971], Lachlan [1968] and others, bringing intial segments to a point where they could be applied in getting important global results, and those of Friedberg [1957] and Muchnik [1956] applying priority for the first time, followed by Sacks [1963] (infinite injury) and Lachlan [1975] ($\emptyset'''$-priority and the tree framework, later developed by Harrington, Soare and others). Another important strand can be traced through from Post's Theorem, via Martin [1966] to the work of Soare, Harrington and others on lattice invariant classes of degrees, which brings out the close relationship between degree and information content, and the latters increasingly fruitful role in pinning down new definable classes at the local level.

In retrospect, it was by no means inevitable that the unavoidable shift from an implicit to explicit approach to Turing definability, which one can trace back to Lerman's [1980] lecture at Logic Colloquium '79, would be accompanied by a more reductionist view of the subject, as expressed by the coding approach. But until there is a fuller understanding of the level of bi interpretability pertaining, it is still possible to ignore the evidence for the *necessity* for the less convergent approach II, and for that matter, for those outside the subject to reasonably expect the imminent exhaustion of the mysteries remaining at the subject's core.

Characteristics of Approach II:

As already noted, approach II is distinguished by a higher degree of interest in the *content* of computability theory — and particularly in *constructions* rather than results. There is a concomitant belief that the *intensionality* of a given Turing definition does matter. Definability typically emerges, maybe in unexpected ways, from degree-theoretic *practice*, giving definitions which contribute to the *understanding* of relations. And one expects that the approach will deliver definitions of optimal *logical* (but not necessarily computability theoretic) complexity. A definition satisfying these last two criteria may be said to be *natural*, and natural definitions are what we rely on approach II for.

On the other hand, there is usually a deficiency of uniformity of definitions from relation to relation, and from structure to structure, giving an impression of pathology, and an inevitability of *piecemeal* development of definitions.

Since both approaches have their strengths and weaknesses, and interact fruitfully, it makes no sense to ask which is the *better* approach. But they are aesthetically quite different, and associated with contrasting mind-sets. Obviously, a particular researcher may vary in his or her attachment to a particular approach, and with other factors existing, may not appear to be at all supportive of the of the broad correlations listed (with a minimum of qualification and circumlocution) below.

Correlations with approach I:

Here, the Turing universe is approached as a purely mathematical object, and any question of *meaning* is downplayed. Attention to underlying philosophical questions is avoided, and at best delegated to 'experts' working within their own academic disciplines. There is a disregard for — even lack of understanding of — the wider significance of the subject. And an irritation with those who draw attention to this. This is part and parcel of the customary preference for recondite terminology, and a reluctance to move on from that established during the recursion theoretic era. A sense of worth of the research activity is frequently achieved via peripheral careerist structures — with its negative dynamics — rather than solidly based on meaningful content and real-world connections. Alternatively narcissism (and what area of academia is free from this?) hegemonically defended, becomes a substitute for a more securely based motivation. It is true that one can recognise such characteristics occuring quite generally, so it becomes a question of degree. And one might ask whether it *matters*, so long as the mathematics created is good, and one immediately thinks of a number of outstandingly impressive contributions. One can only speculate about the extent to which a lack of interest in wider aspects, and a careerist agenda, potentially overflows into technical convergence, and a failure to fully adapt technically proficient research to ultimately realisable ends. While sterile "one shot" problem solving is also symptomatic of the sort of narrow mathematical motivation one might reasonably associate (in the face of quite striking counterexamples of course) with approach I. What is noticeable is the extent to which the agenda associated with leading proponents of approach I, largely (but not entirely) products of the recursion theoretic paradigm established by Kleene, has become an orthodoxy. Persuasive advocacy of a regenerated terminology has produced at least a cosmetic change, but one still has a situation in which curiosity-led research is subjected to austere mathematical objectives. It is not surprising that one is currently faced by a subject dominated by aging but technically proficient researchers, which limits the opportunities for modest new contributions to the body of knowledge on which major breakthroughs must ultimately depend. What kind of subject is it in which so many deep and technically challenging questions are deprioritised just because of their lack of immediately recognisable relationship to a narrowly defined agenda, potentially undermining the inductive structure of progress in the area? It is not surprising that such a scenario sometimes appears as a microcosm of that in the outside world, with its unresolved dichotomy between scientism and the humanities, and the frequent misapplication of reductive criteria in relation to basic research.

Of course, approach II comes with its own, more familiar, occupational hazards.

Correlations with approach II:

The first of these is technique (in terms of constructions) and structural pathology for its own sake. The global applications used to justify the approach appear not just in unexpected ways, but from unexpected sources — often from researchers from other areas. The basis for such a situation being an approach II lack of interest in, and of knowledge of, the *mathematical* big picture — an understandable response to the frequent sacrifice of interesting computability theoretic content to the taxonomy of extension theorems, elementary differences, and so on. One also recognises, particularly in the formative years of the subject, a prevalence of unexplicated extra-mathematical motivations. In pure mathematics it may be that vague unanchored intuitions suffice to motivate a research area, only much later these being substantiated by important real-world applications or explanatory functions, but a subject can die while it awaits development from its embrionic state. There can be a creeping decoherence of the mathematical activity with the initial motivations, reducing computability theory (in particular) to the level of a memetic parasite. While there is a developing susceptibility to outside negative views of the subject, leading to withdrawal and isolation, or an ineffective opportunism of response.

But the impacting of differing approaches is not just technically productive. The resulting theory itself suggests that such differences are necessary to the defining process. And that complete ideational harmony, even at the level of individual thought processes, is unwanted in the face of important enough questions.

We conclude with an examination of the two approaches in relation to a particular problem.

4. An example - the relation of c.e. in

In the rest of this paper we look briefly at an example which brings out the strengths of the second approach. The reader will immediately bring to mind impressive applications of approach I, which deal mathematically with relations quite beyond the sort of intuitive grasp achievable in our example, but these are already well known.

The notion of 'computably enumerable' presents a key test to any approach to establishing Turing definability, particularly since it arises from very basic information content — such as axiomatisable theories, diophantine equations — exhibiting levels of algorithmic closure derived from mathematically fundamental ingredients, and with close associations with real-world phenomena. This was one of the major questions raised by Rogers in the 1960's (see Rogers [1967], Cooper [2000]).

Approaching the problem via generic techniques, Slaman and Woodin [1986] made the first direct impact on the problem by showing that definability of $\boldsymbol{\mathcal{E}}$ is achievable — but using a finite number of additional parameters, and a hybrid argument, relying on specific features of Sacks splitting.

Another impressive achievement was Shore and Slaman's [1999] definition of the *one* c.e. degree $\mathbf{0}'$, attained by superimposing new work within the framework of the 1989 proof on an earlier Slaman and Woodin derivation of the definability of $\mathbf{0}''$. However, one would be hard put to it to extract a meaningful definition of $\mathbf{0}'$ from this proof, and it adds little to ones *understanding* of the role of the relation of c.e. in. On the other hand, it has claims to providing the 'easiest', if not the most informative, proof of definability of the Turing jump, in that its computability theoretic content is relatively simple. One would expect there to be a full approach I solution to the problem of defining $\boldsymbol{\mathcal{E}}$ and the relation of 'c.e. in', of hopefully not much greater Frankensteinian hybridicity, but it seems that the basic details for such a solution must break new ground.

The approach via the local theory has had an ultimately successful, but somewhat tortuous, history. For many years the search for a definition of $\mathbf{0}'$ had focused on its role as the base of an upper cone of complete degrees, rather than that as the top of a cone of c.e. degrees, and reducing the search to a testing of candidates potentially converting this striking property of $\mathbf{0}'$ into a structural one — invariably with negative results. It is an interesting open question whether such a property of the complete degrees does exist or not — one suspects not.

Things look very different when the focus switches away from structure relating to being CEA ('computably enumerable and above') to that relating to being *not* c.e. And it was this pointed to a plausible local definition of the relation of 'c.e. in', achieved via a local construction involving the theory of the d.c.e. degrees and Lachlan nonsplitting techniques (see Cooper [1990]), although it was not until Cooper [2001] that a more satisfactory refinement of the original definition appeared in print.

Given the local definition, global definability follows via that of the Turing jump, for which one has a similar definition but needing a different construction and an added ingredient of pseudo-jumps (not used in the original proof of the local definability of c.e.). The ultimate objective is to bring these related defining properties within the unifying pseudo-jump framework, giving us an intuitively informative definition of 'c.e. in', involving a single $\emptyset'''$ priority argument.

5. Pseudo-jump operators

Jockusch and Shore [1983], [1984] observed that much of the basic theory of the Turing jump, such as the Friedberg completeness theorem, applied generally to other uniformly CEA objects, including a number of which were associated with constructions bringing with them interesting and diverse degree theoretic structure.

We say that J^n is an n*-CEA operator* if and only if there exist $j_0, j_1, \ldots, j_{n-1} \in \omega$ such that $J^k(A) \leq_T W_{j_k}^{J^k(A)}$, each $k < n$, $A \subseteq \omega$, and J^n is inductively defined by

$$J^0(A) = A, \qquad J^{k+1}(A) = W_{j_k}^{J^k(A)},\ (k < n).$$

As usual, if $D = W_i - W_j$, some $i, j \geq 0$, we say D is a *d-c.e. set* (a difference of two c.e. sets).

Then making a special choice of the indices j_0, j_1 in relation to i, j in the above definitions, Jockusch and Shore [1984] verified that if $D = W_i - W_j$ is a d-c.e. set then $A \oplus (W_i^A - W_j^A)$ is a 2-CEA operator. The underlying idea being to first notice that $W_i - W_j$ is CEA the c.e. set

$$\{\langle x, s\rangle \mid x \in (W_{i,s+1} - W_{i,s})\} \cup \{\langle x, s+1\rangle \mid x \in (W_{i,s+1} - W_{i,s}) \cap W_j\},$$

and then choose j_0, j_1 so that for each set X of numbers

$$\begin{aligned} W_{j_0}^X &= X \oplus [\{\langle x, s\rangle \mid x \in (W_{i,s+1}^X - W_{i,s}^X)\} \cup \\ &\qquad\qquad \{\langle x, s+1\rangle \mid x \in (W_{i,s+1}^X - W_{i,s}^X) \cap W_j^X\}], \\ W_{j_1}^{X \oplus Y} &= X \oplus \{x \mid \exists s\,[\langle x, s\rangle \in Y \,\&\, \langle x, s \pm 1\rangle \notin Y]\}, \end{aligned}$$

and define the operator J^2 by $J^2(A) = W_{j_1}^{J^1(A)}$, where $J^1(A) = W_{j_0}^A$. All one needs to do then is verify that: (a) that $J^2(A) = A \oplus (W_i^A - W_j^A)$, and (b) that J^2 is a 2-CEA operator.

6. A jump and join theorem

A combination of the Posner-Robinson [1981] cupping theorem and the Friedberg completeness theorem, within the pseudo-jump framework, is common to the known definitions of the jump. For instance, Cooper [2001] uses a basic jump-join theorem for 2-CEA operators derived from a d-c.e. set, which says: If J^2 is a 2-CEA operator derived from a d-c.e. set, then if $C \geq_T \emptyset'' \oplus X$ and $X \not\leq_T \emptyset'$, one can find an A such that

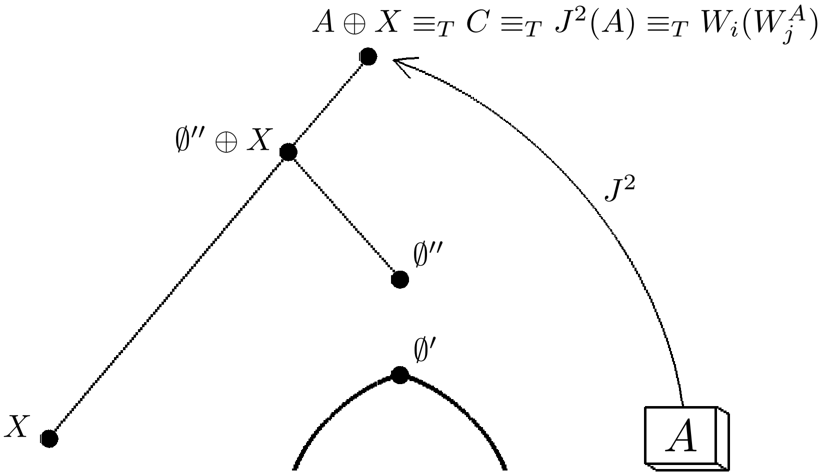
$$X \oplus A \equiv_T C \equiv_T J^2(A).$$

To prove this, let $(\forall A \subseteq \omega)[J^2(A) = W_i^{W_j^A} = W_i(W_j^A)]$, where from the previous section, one can assume that W_i, W_j are given by equations of the form

$$\begin{aligned} W_j^X =& X \oplus [\{\langle x, s\rangle \mid x \in (W_{i',s+1}^X - W_{i',s}^X)\} \\ & \qquad\qquad \cup \{\langle x, s+1\rangle \mid x \in (W_{i',s+1}^X - W_{i',s}^X) \cap W_{j'}^X\}], \\ W_i^{X\oplus Y} =& X \oplus \{x \mid \exists s\,[\langle x, s\rangle \in Y \,\&\, \langle x, s\pm 1\rangle \notin Y]\}, \end{aligned}$$

with $J^2(A) = A \oplus (W_{i'}^A - W_{j'}^A)$.

Then without changing the degree of X, one can assume that X is $\emptyset'$-immune (i.e., has no infinite $\emptyset'$-c.e. subsets), and then construct an A satisfying the picture:



To do this, one defines $A = \cup_{n\geq 0}\,\sigma_n$, where the σ's$\subset A$ are chosen to code C into A with help from X, to force certain $\langle x, s\rangle \in W_j^{\sigma'}$, $\sigma' \supset \sigma_n$, and to ensure that, for each $n \in \omega$,

$$n \in J^2(A) \Leftrightarrow n \in W_i(W_j^\sigma \restriction M)[|\sigma|],$$

σ corresponding to n, and M depending on the construction.

Although the basic jump-join theorem is sufficient for a natural Turing definition of the jump, a local version is needed to provide that of the relation of 'computably enumerable in', which applies to 2-CEA operators derived from dynamically *special* d.c.e. sets.

7. Extending the jump and join theorem

If one wants to use the special 2-CEA operator used to define the jump, one needs a local jump-join theorem for 2-CEA operators derived from a d-c.e. set of the form: If J^2 is a 2-CEA operator derived from a special d-c.e. set, and $X \leq_T \emptyset'$ is not of c.e. degree, then one can find an $A \leq_T \emptyset''$ such that $X \oplus A \equiv_T J^2(A)$.

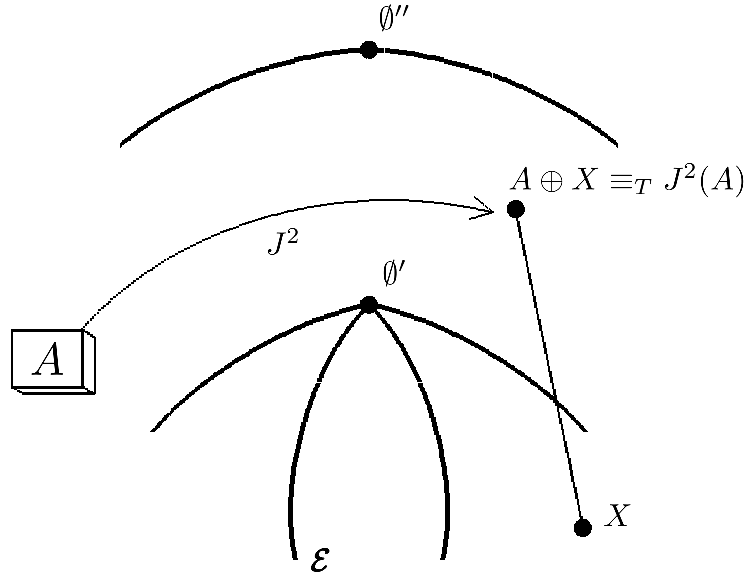
It would be nice if such a local jump-join theorem held for J^2 in the generality of the basic jump-join theorem, but it is easy to see that if for instance $J^2(A)$ is always low over A, then there is plenty of scope to build an X for which $X \oplus A \not\equiv_T J^2(A)$ for any candidate A. And in fact, Slaman (private communication) has such a counterexample consisting of a suitable operator and a *self-restraining* set X. Of course, the special operator we want the local jump-join theorem to apply to (see section 8 below) is very different to that of Slaman, but if one is to avoid a return to the original *ad hoc* method of defining 'c.e. in', one must extract a suitable general condition on J^2 from the special operator in question. Since $J^2(A)$ will be properly d.c.e. over A, it is not surprising that the required prerequisite is that derived from a suitably non-standard construction of a properly d.c.e. set, leading to a class of operators containing the one needed.

We conclude with a sketch of a module for sufficient requirements to enable a description of the framework of the proof of the local jump-join theorem, but avoiding the technicalities corresponding to the infinite outcome dealt with by the specialness of $J^2(A)$, for which we refer the reader to Cooper [ta].

Let $J^2(A) = W_{j_1}^{W_{j_0}^A} = W_{j_1}(W_{j_0}^A)$, all $A \subseteq \omega$, and again assume that

$$\begin{aligned} W_{j_0}^X = {} & X \oplus [\{\langle x,s\rangle \mid x \in (W_{i,s+1}^X - W_{i,s}^X)\} \\ & \qquad\qquad \cup \{\langle x,s+1\rangle \mid x \in (W_{i,s+1}^X - W_{i,s}^X) \cap W_j^X\}], \\ W_{j_1}^{X\oplus Y} = {} & X \oplus \{x \mid \exists s\,[\langle x,s\rangle \in Y \,\&\, \langle x,s\pm 1\rangle \notin Y]\}, \end{aligned}$$

where $J^2(A) = A \oplus (W_i^A - W_j^A)$. In fact, we can assume J^2 to be defined by $J^2(A) = W_i^A - W_j^A$, and then construct a set A satisfying the picture:



Again, we define $A = \cup_{n\geq 0}\,\sigma_n$, with the σ's $\subset A$ chosen to code X into A, and to force appropriate membership relations between numbers n and $J^2(A)$.

This time σ_n $(= \lim_s \sigma_n^s)$ is chosen relative to computably enumerable sets $S_{i,n}$, $S_{j,n}$ via an infinite injury priority argument, with the true path, represented by $\{\sigma_n\}_{n\geq 0}$, retrievable from $A \oplus X$ and from $J^2(A)$.

We define $S_{i,n}$, $S_{j,n}$ by :

$$S_{i,n} = \{\sigma \supseteq \sigma_n \mid n \in W_i^\sigma\}, \quad S_{j,n} = \{\sigma \in S_{i,n} \mid n \in W_j^\sigma\}.$$

We say that σ, τ *force* $n \in J^2(A)$ if and only if $\sigma \in S_{i,n}$, and for no $\tau' \approx \tau$ is $\tau' \in S_{j,n}$. And say σ, τ *force* $n \notin J^2(A)$ if and only if $\tau \approx$ some $\tau' \in S_{i,n} \Rightarrow \tau \in S_{j,n}$.

The overall requirements are that $\{\sigma_n\}_{n\geq 0}$ is retrievable from $A \oplus X$, $\{\sigma_n\}_{n\geq 0}$ is retrievable from $J^2(A)$, and $\sigma_{n+1}(|\sigma_{n+1}|-1) = X(n)$. From which it will follow that $A \oplus X$ is retrievable from $\{\sigma_n\}_{n\geq 0}$.

The prioritised requirements are:

$\mathcal{R}_n :$ $(\exists \tau \subset A)[\sigma_{n+1}, \tau \Vdash n \in J^2(A)$ or $\sigma_{n+1}, \tau \Vdash n \notin J^2(A)]$.

These will ensure that the forcing relation does its job and, if σ_{n+1} is sufficiently informative about how it does it, that $J^2(A)$ is retrievable from $\{\sigma_n\}_{n\geq 0}$.

The aim is to select the least π, and then $\sigma \supseteq \sigma_n{}^\frown 0^{\langle \pi \rangle}{}^\frown 1$, $\tau \supseteq \sigma^\frown X(n)$, for which *either*

(a) π is to the right of X, $\sigma \supset \sigma_n{}^\frown 0^{\langle \pi \rangle}{}^\frown 1$, and σ, τ force $n \in J^2(A)$ at stage $s+1$, *or*

(b) π is to the left of X, $\sigma = \sigma_n{}^\frown 0^{\langle \pi \rangle}{}^\frown 1$, and σ, τ force $n \notin J^2(A)$ at stage $s+1$,

and then to define $\sigma_{n+1}^{s+1} = \sigma^\frown X(n)$, and choose $A^{s+1} \supset \tau_{n+1}^{s+1} = \tau$. And finally, to maintain satisfaction of $\mathcal{R}_n$ at later stages $t+1$ via π, τ still satisfying (a) or (b), with $\tau \subset A^{t+1}$.

That is, we want to get that there exist π, τ permanently satisfying either (a) or (b). And to do this, one first needs to notice that if no such π, τ exist permanently satisfying clause (a), then for each π to the right of X, and each $\sigma \supseteq \sigma_n{}^\frown 0^{\langle \pi \rangle}{}^\frown 1$, $\tau \supseteq \sigma^\frown X(n)$, one *either* has σ permanently $\notin S_{i,n}$, *or* $\tau \approx$ some $\tau' \in S_{j,n}$. And if no such π, τ exist permanently satisfying clause (b), then for each π to the left of X one has for each $\tau \supseteq \sigma_n{}^\frown 0^{\langle \pi \rangle}{}^\frown 1^\frown X(n)$ that $\tau \approx$ some $\tau' \in S_{i,n}$, but τ permanently $\notin S_{j,n}$.

There are two cases to consider:

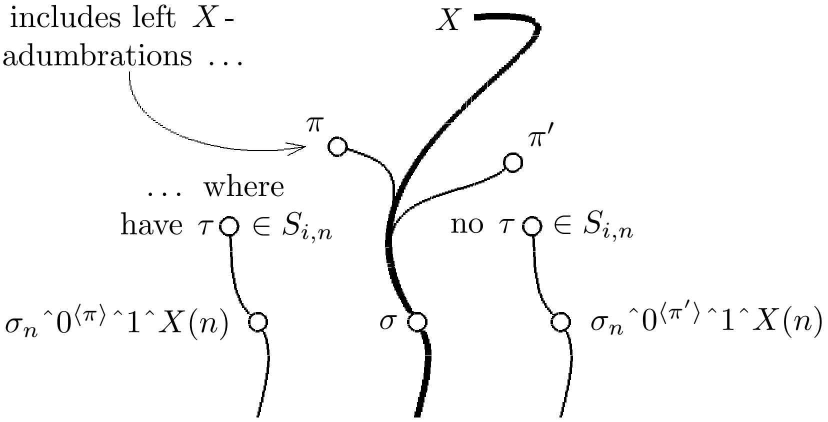
I. There exists some $\sigma \subset X$ — for which, for each $\pi \supset \sigma$ to the right of X, there does not exist a $\tau \supseteq \sigma_n{}^\frown 0^{\langle \pi \rangle}{}^\frown 1^\frown X(n)$ with $\tau \in S_{i,n}$, and

II. Otherwise.

Assume first that case I applies.

Call a τ, for which there is some $\pi \supset \tau$, for which $\pi \supset \sigma$, and for which there exists some $\tau' \in S_{i,n}$ with $\tau' \supseteq \sigma_n{}^\frown 0^{\langle \pi \rangle}{}^\frown 1^\frown X(n)$, a *left* X*-adumbration.*

We then notice that the set of all left X-adumbrations is c.e., and there exist only finitely many strings minimally to the right of a given τ.

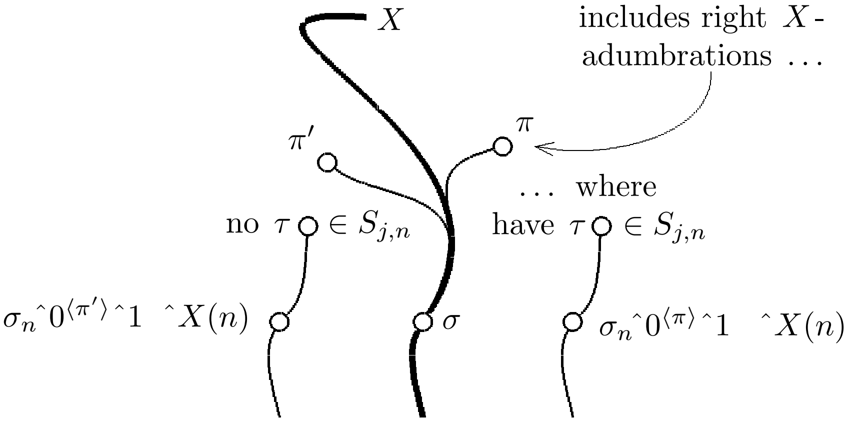


We can then verify that for each $\tau \supset \sigma$ we have that

$\tau \subset X \ \Leftrightarrow\ \tau$ is a left X-adumbration, and no $\tau' \supset \sigma$ minimally to the right of τ is a left X-adumbration.

And hence, since $X \equiv_T$ the set of all left X-adumbrations, we get a contradiction to X not being of c.e. degree.

On the other hand, assuming case II applies, we get infinitely many strings $\sigma \subset X$, with some $\pi \supset \sigma$ to the right of X, for which $\sigma_n\hat{}0^{\langle\pi\rangle}\hat{}1\hat{}X(n) \subset$ some $\tau \in S_{i,n}$. So we can modify the definition from case I, and call a τ for which there is some $\pi \supset \tau$ for which there exists a $\tau' \in S_{j,n}$ with $\tau' \supset \sigma_n\hat{}0^{\langle\pi\rangle}\hat{}1\hat{}X(n)$ a *right X-adumbration.*



And then note that the set of all right X-adumbrations is c.e., and for each τ have :

$\tau \subset X \ \Leftrightarrow\ \tau$ is a right X-adumbration, and no τ' minimally to the left of τ is a right X-adumbration, and hence $X \equiv_T$ the set of all right X-adumbrations, again contradicting the assumption that X is not of c.e. degree.

For the satisfaction of the background requirements one needs to verify the following Turing reductions:

$$\begin{tikzcd}
J^2(A) \arrow[r, bend left=60, "(4)", "\geq_T"'] & \{\sigma_n\}_{n\leq 0} \arrow[l, bend left=60, "(2)", "\leq_T"'] \arrow[r, bend left=60, "(1)", "\geq_T"'] & A \oplus X \arrow[l, bend left=60, "(3)", "\leq_T"']
\end{tikzcd}$$

For (1), $A\restriction(|\sigma_{n+1}|-1)$ and $X(n)$ will be computable from σ_{n+1}, using $\sigma_{n+1} \approx A$ and $X(n) = \sigma_{n+1}(|\sigma_{n+1}|-1)$.

For (2), one can decide $n \in J^2(A)$ using σ_n and σ_{n+1}, since the π for which $\mathcal{R}_n$ is permanently satisfied will be retrievable from σ_n and A, and

$$n \in W_i^A - W_j^A \Leftrightarrow \sigma_{n+1}\restriction(|\sigma_{n+1}|-2) \supset \sigma_n\hat{\ }0^{\langle\pi\rangle}\hat{\ }1.$$

For (3), with careful choice of τ_{n+1}^{s+1} we can determine σ_{n+1} from σ_n using A and X, since to retrieve a *true* stage $s+1$ (that is a stage $s+1$ at which one defines $\sigma_{n+1}^{s+1} = \sigma_{n+1}$) one can use σ_n and A to find a suitable π, and then check whether π is to the left of X or not to see which of cases (a) or (b) apply at stage $s+1$. If π is to the left of X, so that (b) applies, one has $\sigma_{n+1} = \sigma_{n+1}^{s+1} = \sigma_n\hat{\ }0^{\langle\pi\rangle}\hat{\ }1\hat{\ }X(n)$. And if π is to the right of X, so that (a) applies, one can follow through the module until one finds $\sigma_n\hat{\ }0^{\langle\pi\rangle}\hat{\ }1 \subset \sigma \subset A$ with $\sigma \in S_{i,n}$ in 1(a), with help from A. And then $\sigma_{n+1} = \sigma_{n+1}^{s+1} = \sigma\hat{\ }X(n)$.

So from (2) and (3) one gets that given σ_n, one can decide $n \in J^2(A)$ using A and X.

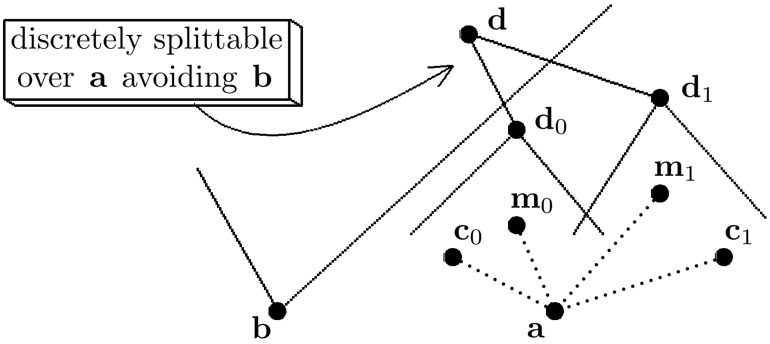
To verify (4), one can determine σ_{n+1} from σ_n using $J^2(A)$, since, using $A \leq_T J^2(A)$, to retrieve a true stage one can find π from σ_n and A, and then verify whether $n \in J^2(A)$ so as to see if one ends up permanently in case (a) or (b). And as for (3), if (b) applies one has $\sigma_{n+1} = \sigma_n\hat{\ }0^{\langle\pi\rangle}\hat{\ }1\hat{\ }X(n)$. And if (a) applies, one can follow through the module until one finds $\sigma_n\hat{\ }0^{\langle\pi\rangle}\hat{\ }1 \subset \sigma \subset A$ with $\sigma \in S_{i,n}$ in 1(a), with help from A, in which case $\sigma_{n+1} = \sigma\hat{\ }X(n)$.

So from (1) and (4) one gets that given σ_n, one can compute $A\restriction(|\sigma_{n+1}|-1)$ and $X(n)$ using $J^2(A)$.

8. A particular 2-CEA operator

We briefly reiterate the description of the structure to be associated with the special operator needed for the Turing definitions we want.

Given $\mathbf{a},\mathbf{b},\mathbf{d}$, we say $\mathbf{d}$ is *splittable over* $\mathbf{a}$ *avoiding* $\mathbf{b}$ if and only if $\mathbf{a},\mathbf{b}\le\mathbf{d}$, $\mathbf{b}\not\le\mathbf{a}$ imply there exist $\mathbf{d}_0,\mathbf{d}_1<\mathbf{d}$ for which $\mathbf{a}<\mathbf{d}_0,\mathbf{d}_1$, $\mathbf{b}\not\le\mathbf{d}_0$ and $\mathbf{d}_1$, and $\mathbf{d}=\mathbf{d}_0\cup\mathbf{d}_1$. Further, $\mathbf{d}$ is *discretely* splittable over $\mathbf{a}$ avoiding $\mathbf{b}$ if and only if each such $\mathbf{d}_i$, and for each $\mathbf{c}_i<\mathbf{d}_i$, $\mathbf{a}$ has a minimal cover $\mathbf{m}_i\le\mathbf{d}_i$ but not $\le\mathbf{c}_i$. $\mathbf{d}$ is *relatively unsplittable* if and only if not discretely splittable over $\mathbf{a}$ avoiding $\mathbf{b}$, some $\mathbf{a},\mathbf{b}$.



Then one can verify (see Cooper [2001]), that on the one hand there is no relatively unsplittable c.e. degree, while on the other there does exist a relatively unsplittable d-c.e. degree, achieved by constructing a d-c.e. set $D=W_i-W_j$ and sets $A,B\le_T D$ such that $\deg(D)$ is not discretely splittable over $\deg(A)$ avoiding $\deg(B)$.

Consequently, one gets a 2-CEA operator J such that for each C one has $J(C)=C\oplus(W_i^C-W_j^C)$ and $\deg(J(C))$ is relatively unsplittable, since $\deg(J(C))$ is not discretely splittable over $\mathbf{a}$ avoiding $\mathbf{b}$, some $\mathbf{a},\mathbf{b}\ge\deg(C)$.

As in Cooper [2001], using J with the basic jump-join theorem for 2-CEA operators derived from a d-c.e. set one can get a Π_6 Turing definition of $\mathbf{0}'$ as the largest degree satisfying

$$(\forall\mathbf{a},\mathbf{b})[\,\mathbf{x}\cup\mathbf{a}\text{ is discretely splittable over }\mathbf{a}\text{ avoiding }\mathbf{b}\,]. \qquad (\dagger)$$

Relativising one gets a natural first order definition of the Turing jump in $\boldsymbol{\mathcal{D}}$, and of course, answering Rogers [1967], [1967a], the degree theoretic invariance of the jump. It follows that one can eliminate the jump from many pre-existing results for the structure $\boldsymbol{\mathcal{D}}'$, often greatly improving those previously available for $\boldsymbol{\mathcal{D}}$ (see Cooper [2001]). Many of these results have since been obtained directly using coding methods, and where feasible improved by a factor of one, see Nies, Shore & Slaman [1996].

But we can get more out of this same operator J via the *local* jump-join theorem for 2-CEA operators derived from a d-c.e. set. It immediately follows that a degree $\mathbf{x}$ is computably enumerable if and only if it satisfies

$$(\forall\mathbf{a},\mathbf{b}\le\mathbf{0}'')[\,\mathbf{x}\cup\mathbf{a}\text{ is discretely splittable over }\mathbf{a}\text{ avoiding }\mathbf{b}\,]. \qquad (\dagger\dagger)$$

This is because every c.e. $\mathbf{x}$ satisfies (†) above, and so must also satisfy (††), and if $\mathbf{x} \leq \mathbf{0}'$ is *not* c.e., one can use the local jump-join theorem as in the proof of the definability of $\mathbf{0}'$ to get $\mathbf{a}, \mathbf{b}$ negating (††).

Hence one gets that $\mathbf{x}$ is computably enumerable if and only if it satisfies

$$(\forall \mathbf{a}, \mathbf{b})[\, \mathbf{x} \cup \mathbf{a} \text{ is discretely splittable over } \mathbf{a} \text{ avoiding } \mathbf{b}\,],$$

since we again have that every c.e. $\mathbf{x}$ satisfies (†), and if $\mathbf{x}$ is not c.e. *either* $\mathbf{x} \not\leq \mathbf{0}'$, in which case the result follows from the definability of the jump, *or* $\mathbf{0} < \mathbf{x} < \mathbf{0}'$, and the result follows via the local definability of the c.e. degrees.

This time we get a Π_5 definition.

Relativising, we get that for each $\mathbf{a}$, the degrees CEA $\mathbf{a}$ are definable in $\boldsymbol{\mathcal{D}}(\geq \mathbf{a})$. But, we can do better, via a relativised local jump-join theorem for 2-CEA operators derived from a d-c.e. set, of the form: If J^2 is a 2-CEA-B operator derived from a special B-d-c.e. set, and $X \leq_T B'$ is not of B-c.e. degree, then one can find an A such that $B \leq_T A \leq_T B''$ and $X \oplus A \equiv_T J^2(A)$.

Using this (again see Rogers [1967], [1967a]) one gets the definability of the relation of "c.e. in", since $\mathbf{x}$ is c.e. in $\mathbf{y}$ if and only if it satisfies

$$(\forall \mathbf{a}, \mathbf{b} \geq \mathbf{y})[\, \mathbf{x} \cup \mathbf{a} \text{ is discretely splittable over } \mathbf{a} \text{ avoiding } \mathbf{b}\,].$$

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