

On the Intensionality of Turing Definitions

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1. *Mathematics and meaning*

“Although it is largely unrecognised, mathematics is, of course, part of our culture, because everything humans do is part of human culture. In a trivial sense mathematical objects and theorems are creations of human minds.

The more profound mystery is why, after being created or discovered (either term is legitimate), they so beautifully fit the outside world.”

— Martin Gardner, *The Night Is Large: Collected Essays 1938–1995*, St. Martin’s Press, New York, 1996, p. 255.

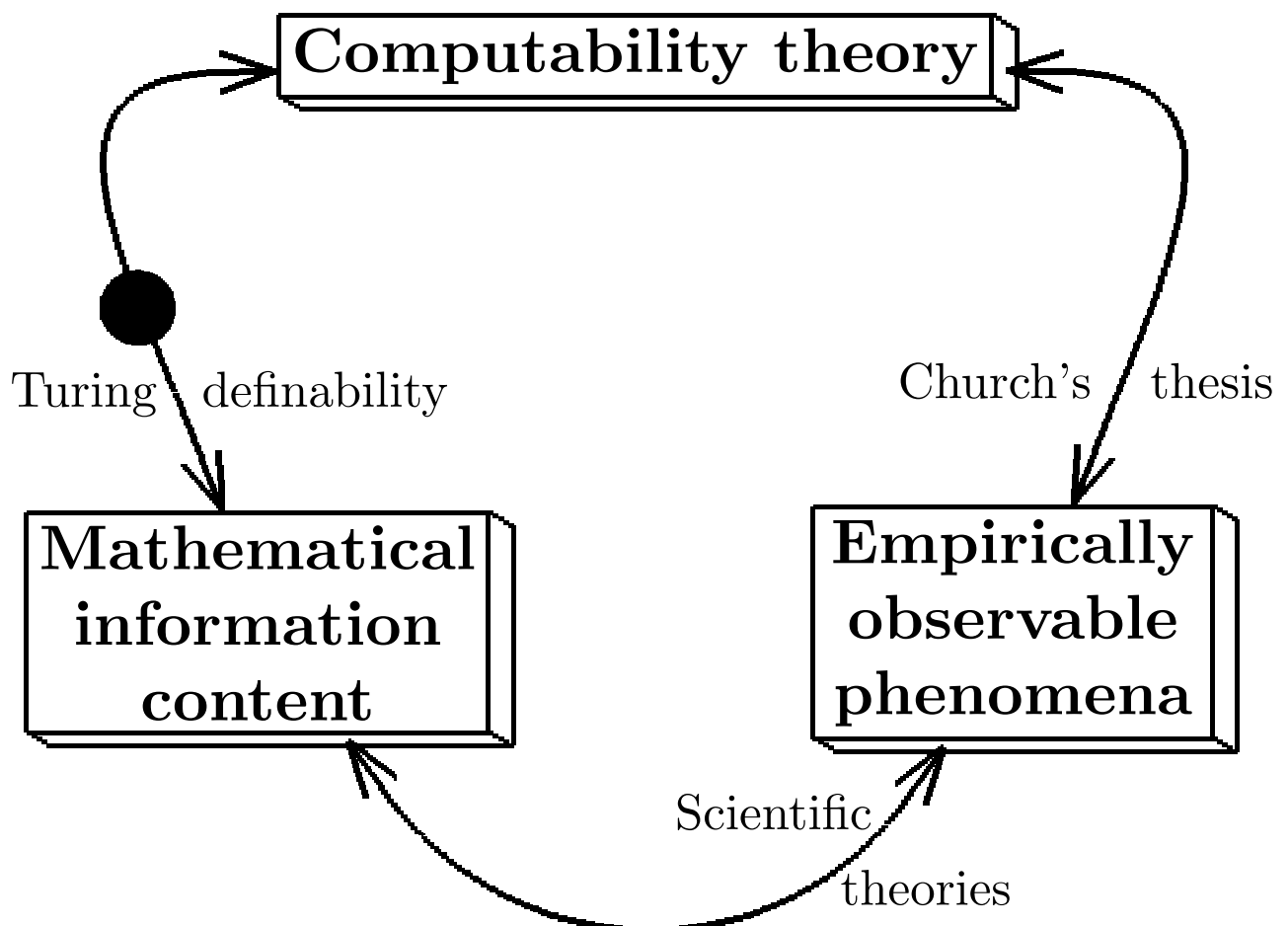
Basic features of this ‘outside world’:

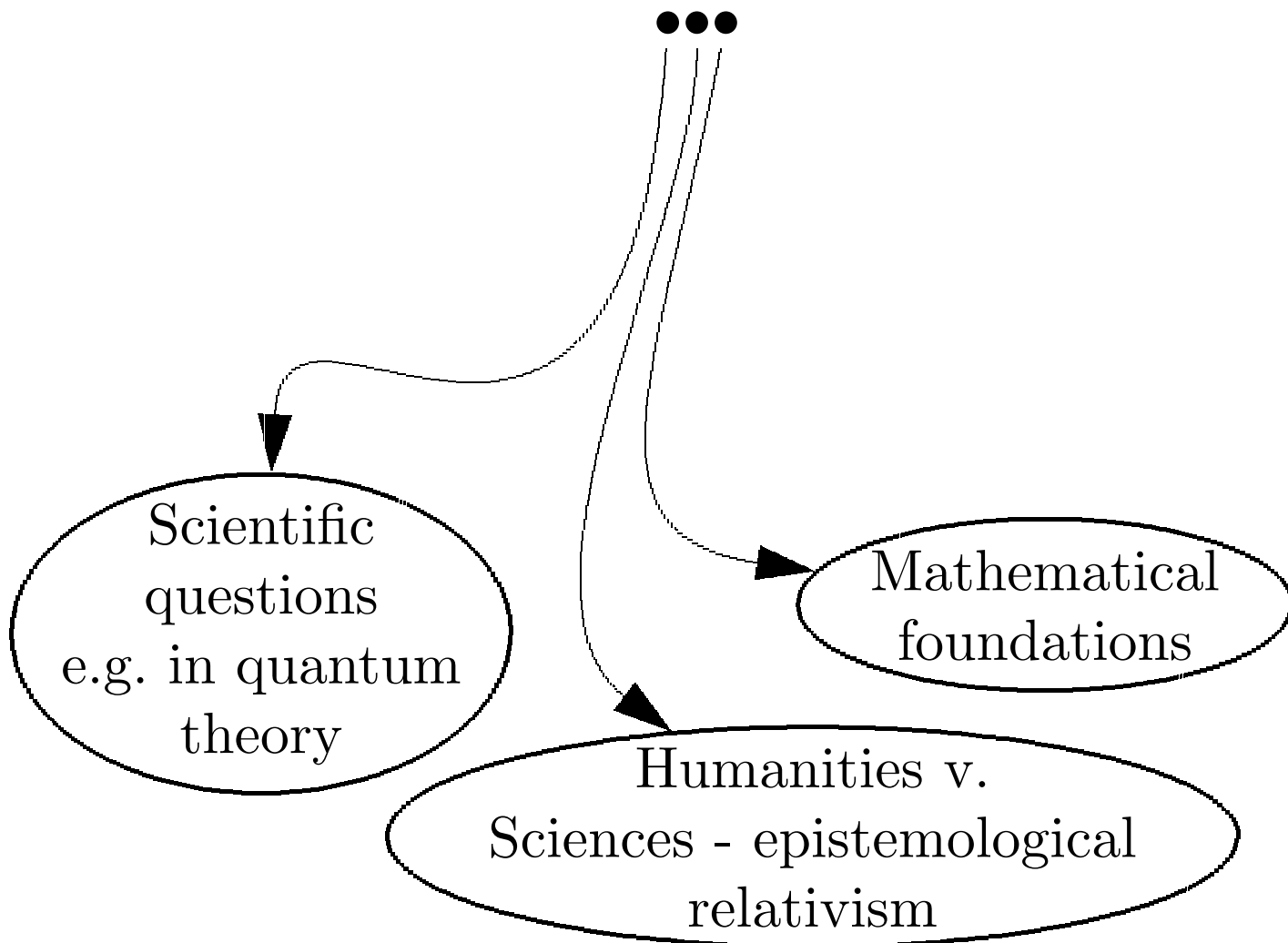
- *Objects* — Easy to interact with, hard to describe.
- *Algorithmic content* — Since Newton and before, familiar component of the universe — computability comes prepackaged, independent of particular language of description.
- *Approximation* — Arising from failure of attempts *in practice* to model universe discretely. Apparent importance of uncompleted infinities and description of physical parameters via reals
- — and parallel inadequacy of epistemological discreteness (i.e., language).
- *Incomputability* as an inconvenient fact of life
- — and — despite apparent (causal) determinism — breakdown of purely local theoretical explanations (e.g., quantum theory, origin of natural laws).

And consequently —

- *Relevance of logical structure* — And in particular — of *mathematical definability* for the modelling of globally determined material phenomena. And specifically —

- *The Turing universe* as an appropriate structure within which to theoretically capture these features.





2. The Turing universe

- Corresponding to the i th Turing machine, Φ_i denotes the i th partial computable (p.c.) functional $2^\omega \rightarrow 2^\omega$.

- A is *Turing reducible* to a B ($A \leq_T B$) iff $A = \Phi_i^B$, some $i \in \omega$ — and A, B are *Turing equivalent* ($A \equiv_T B$) iff $A \leq_T B$ and $B \leq_T A$.

- The *degree of unsolvability* or *Turing degree* of A is defined by

$$\text{deg}(A) = \{X \in 2^\omega \mid A \equiv_T X\}.$$

- \leq is the induced partial ordering on \mathcal{D} (= the set of all degrees), $\mathbf{0}$ = the least degree (consisting of all computable sets of numbers), and \mathcal{D} is the structure $\langle \mathcal{D}, \leq \rangle$.

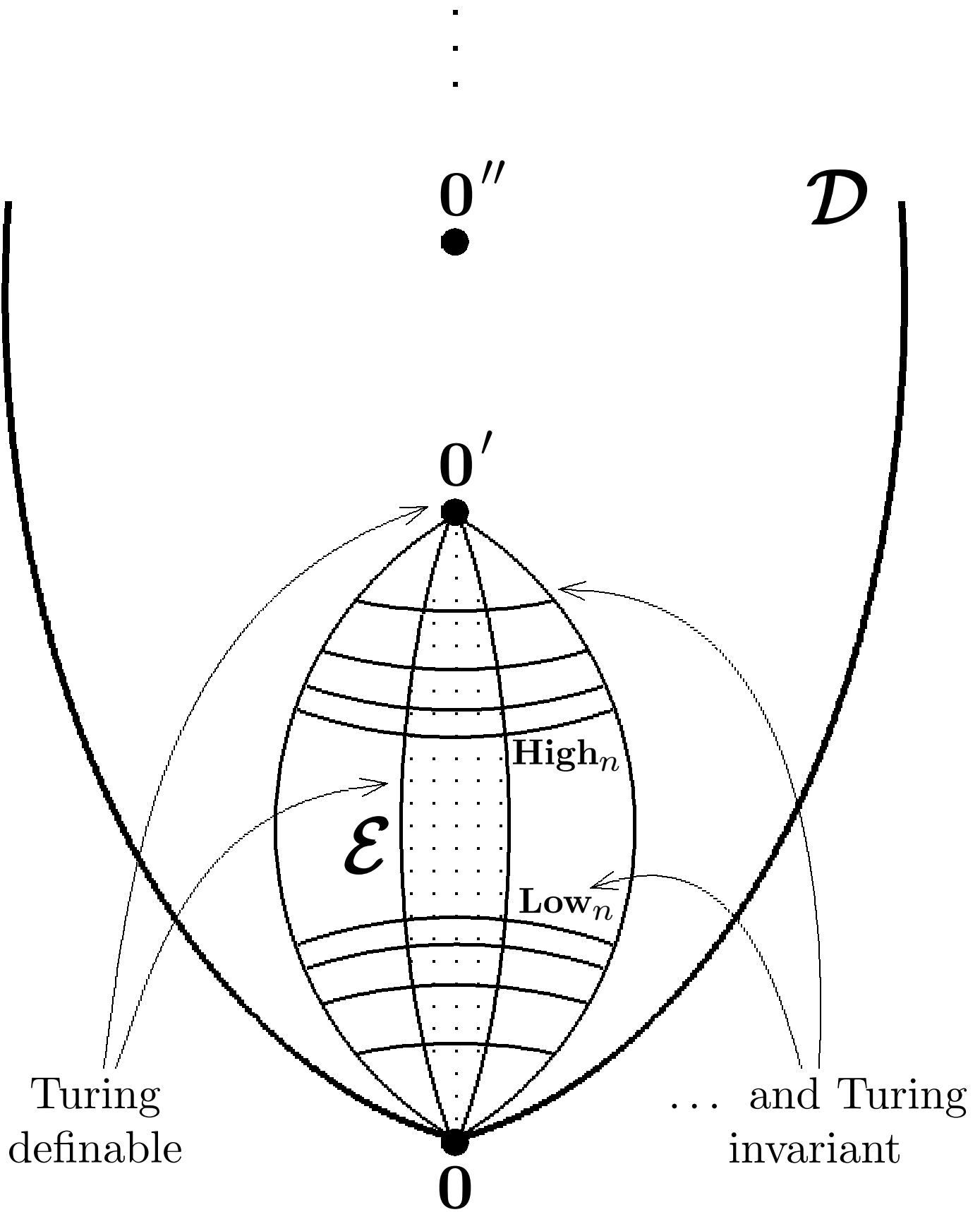
- $W_i^A = \text{dom } \Phi_i^A$ denotes the i th *computably enumerable in A* (A -c.e.) set ($W_i = W_i^\emptyset$ being the i th c.e. set).

- The *jump* — or $n + 1$ th *jump* — of a set A is defined by $A' = A^{(1)} = \{x \mid x \in W_x^A\}$ — or, $A^{(n+1)} = (A^{(n)})'$, respectively.

- The *jump operator* on degrees is defined by $\mathbf{a}' = \deg(A')$, $A \in \mathbf{a}$, where $\mathbf{a} < \mathbf{a}'$, and \mathbf{a}' is the l.u.b. of the degrees of sets c.e. in $A \in \mathbf{a}$.
- And write $\mathbf{a}^{(n+1)} = \deg(A^{(n+1)}) = (\mathbf{a}^{(n)})'$.
- Define the standard ω -*jump* of \mathbf{a} by

$$\mathbf{a}^{(\omega)} = \deg(\bigoplus_{n \in \omega} A^{(n)}), \quad A \in \mathbf{a}.$$
- Write \mathcal{D}' for the structure $\langle \mathcal{D}, \leq, ' \rangle$.
- A relation on \mathcal{D} is *Turing definable* iff it is describable in the first order theory of \mathcal{D} .
- A relation on \mathcal{D} is *Turing invariant* iff it is unchanged by any automorphism $\mathcal{D} \rightarrow \mathcal{D}$.

Note (Rogers, 1965): Given $\langle U, R \rangle$, with just one binary relation R , and $V \subseteq U$, then V is invariant iff V is definable in $\mathcal{L}_{\alpha\beta}$ (with predicate symbol for R), some α, β .



Naturally arising information content
in the Turing universe

I. Via coding techniques:

- Attempt to code information into degree structures via generic techniques — which may originate independently of a given application, and depend, ideally, on a base level of structure common to many different contexts.
- *Origins:* Jockusch and Simpson (1976), Simpson (1977).
- *Further landmarks:* Nerode and Shore (1980), Slaman and Woodin (1986), Nies, Shore and Slaman (1996).

Characteristics of approach I:

- Content from computability theory relatively simple.
- Hence applicable to a broad spectrum of structures — leading to a multiplicity of applications, each ‘minor perturbations’ of others.

- Better understanding of *how much* computability theory is needed for a given definition.
- The promise of ‘easier’ proofs —
- But — may need complex combinatorial structures of implementation, with a resulting distancing from the object being defined.
- Tension between code and object may emerge, particularly at local level — leading to reliance on parameters, or hybrid ‘pull down’ arguments and convergence with approach II:

II. Local theory:

- Curiosity-led investigation of structural, and informational, *content* of Turing relations.
- *Origins*: Kleene and Post (1954).
- *Other landmarks*: Spector, Shoenfield, Yates, Lerman, Lachlan (initial segments); Friedberg, Muchnik (priority); Shoenfield, Sacks (infinite injury); Lachlan, Harrington, Soare (\emptyset''' -priority, tree framework); Martin, Soare, Harrington (information content).

Characteristics of approach II:

- Interest in *content* of computability theory.
- — And in *technical* content.
- Definability emerging from degree-theoretic *practice* — hence definitions which contribute to the *understanding* of relations.
- May deliver definitions of optimal *logical* — but not computability theoretic — complexity.

Note: A definition satisfying the these last two criteria may be said to be *natural*.

- A deficiency of uniformity from relation to relation, and from structure to structure — an impression of *pathology*.
- — And an inevitability of *piecemeal* development of definitions.

Note: It makes no sense to ask which is the *better* approach — but they are aesthetically quite different.

Note: ‘Computably enumerable’ a key test — arises from basic information content — e.g. axiomatisable theories, diophantine equations — all exhibiting sufficient algorithmic closure.

Approach via generic techniques:

- (Slaman and Woodin, 1986) Definability of \mathcal{E} achievable — but additional parameters needed — and argument a hybrid, relying on specific features of Sacks splitting.
- (Shore and Slaman, 1999) Definition of the *one* c.e. degree $\mathbf{0}'$ attained — by superimposing new work within the framework of the 1989 proof on an earlier Slaman and Woodin derivation of the definability of $\mathbf{0}''$. Result —
- Little added to ones *understanding* of the role of the relation of c.e. in — but computability theoretic content relatively simple.

Approach via local theory:

- (1989/90) Local definability of relation of ‘c.e. in’ via theory of d.c.e. degrees and Lachlan nonsplitting techniques. And —
- Global definability via that of Turing jump, needing added ingredient of pseudo-jumps.
- (1999) Further refinement of specific defining properties — now brought within unifying pseudo-jump framework. Result —
- An intuitively informative definition — but involving a \emptyset''' priority argument ...

5. Pseudo-jump operators

DEFINITION 5.1: Say J^n is an n -CEA operator iff there exist $j_0, j_1, \dots, j_{n-1} \in \omega$ such that $J^k(A) \leq_T W_{j_k}^{J^k(A)}$, each $k < n$, $A \subseteq \omega$, and J^n is inductively defined by

$$J^0(A) = A, \quad J^{k+1}(A) = W_{j_k}^{J^k(A)}, \quad (k < n).$$

- If $D = W_i - W_j$, some $i, j \geq 0$, say D is a d -c.e. set (a difference of two c.e. sets).

LEMMA 5.2: If $D = W_i - W_j$ is a d -c.e. set then $A \oplus (W_i^A - W_j^A)$ is a 2-CEA operator.

PROOF (Lachlan; Jockusch and Shore [1984]).

- Make a special choice of the indices j_0, j_1 in definition 5.1 in relation to i, j —

Underlying idea — $W_i - W_j$ is CEA the c.e. set

$$\{\langle x, s \rangle \mid x \in (W_{i,s+1} - W_{i,s})\} \\ \cup \{\langle x, s + 1 \rangle \mid x \in (W_{i,s+1} - W_{i,s}) \cap W_j\}.$$

- Choose j_0, j_1 so that for each set X of numbers

$$W_{j_0}^X = X \oplus [\{\langle x, s \rangle \mid x \in (W_{i,s+1}^X - W_{i,s}^X)\} \\ \cup \{\langle x, s + 1 \rangle \mid x \in (W_{i,s+1}^X - W_{i,s}^X) \cap W_j^X\}],$$

$$W_{j_1}^{X \oplus Y} = \\ X \oplus \{x \mid \exists s [\langle x, s \rangle \in Y \ \& \ \langle x, s \pm 1 \rangle \notin Y]\},$$

and define the operator J^2 by $J^2(A) = W_{j_1}^{J^1(A)}$,
where $J^1(A) = W_{j_0}^A$.

- **Then verify:**

(a) that $J^2(A) = A \oplus (W_i^A - W_j^A)$ and

(b) that J^2 is a 2-CEA operator. □

THE BASIC JUMP-JOIN THEOREM FOR 2-CEA OPERATORS DERIVED FROM A d-c.e. SET: *If J^2 is a 2-CEA operator derived from a d-c.e. set, then if $C \geq_T \emptyset'' \oplus X$ and $X \not\leq_T \emptyset'$, one can find an A such that*

$$X \oplus A \equiv_T C \equiv_T J^2(A).$$

PROOF. An extension of the Posner-Robinson [1981] cupping theorem.

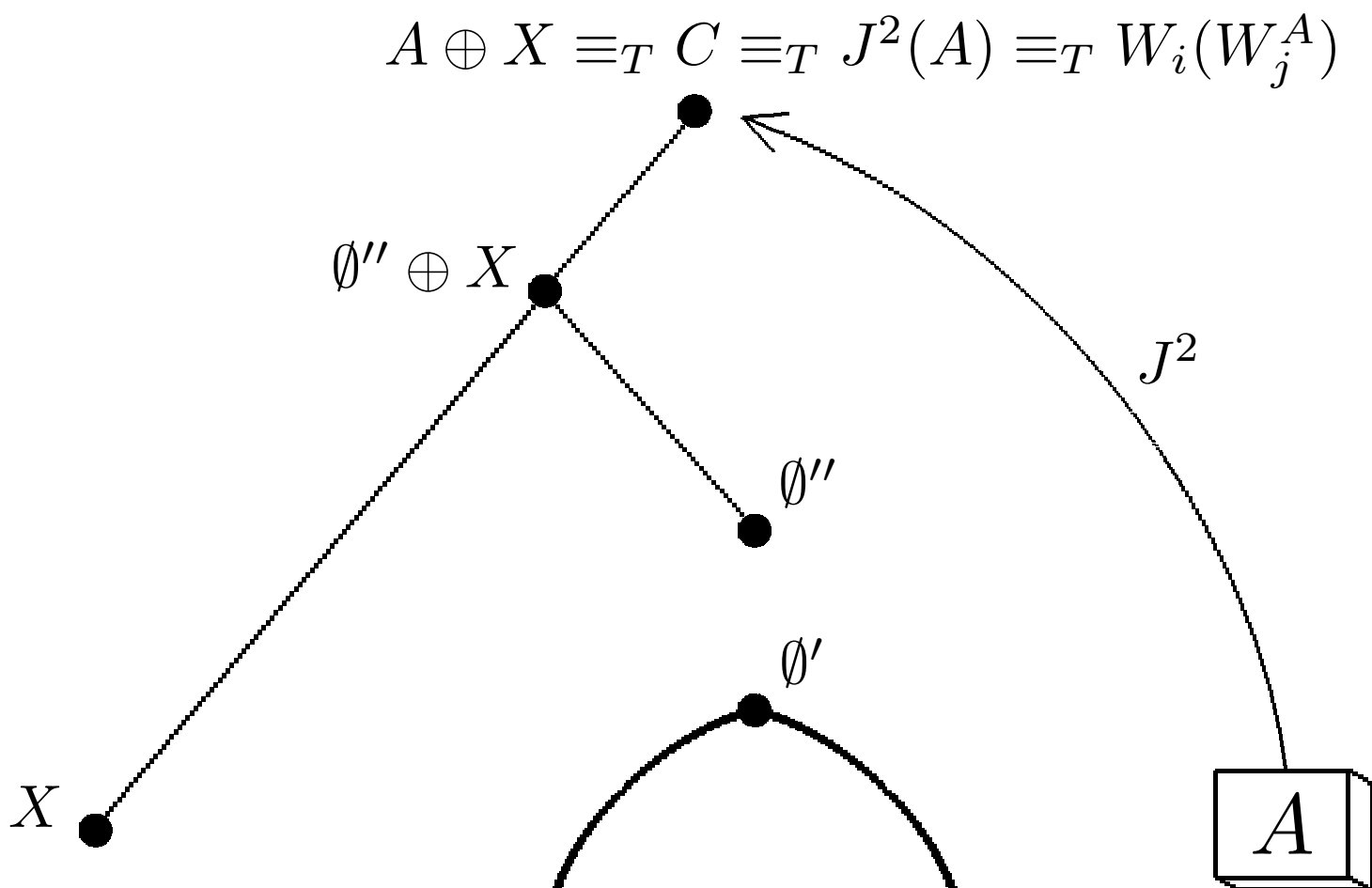
- Assume $(\forall A \subseteq \omega)[J^2(A) = W_i^{W_j^A} = W_i(W_j^A)]$.
- From the proof of Lemma 5.2, can assume that W_i, W_j are given by equations of the form

$$W_j^X = X \oplus [\{\langle x, s \rangle \mid x \in (W_{i',s+1}^X - W_{i',s}^X)\} \\ \cup \{\langle x, s+1 \rangle \mid x \in (W_{i',s+1}^X - W_{i',s}^X) \cap W_{j'}^X\}],$$

$$W_i^{X \oplus Y} = X \oplus \{x \mid \exists s [\langle x, s \rangle \in Y \ \& \ \langle x, s \pm 1 \rangle \notin Y]\},$$

where $J^2(A) = A \oplus (W_{i'}^A - W_{j'}^A)$.

- Without changing the degree of X , can assume that X is \emptyset' -immune (i.e., has no infinite \emptyset' -c.e. subsets).
- Then construct A satisfying the picture:



- To do this, define $A = \bigcup_{n \geq 0} \sigma_n$, where the σ 's $\subset A$ are chosen to —
- Code C into A with help from X , —

- To force certain $\langle x, s \rangle \in W_j^{\sigma'}$, $\sigma' \supset \sigma_n$, —
- And to ensure that, for each $n \in \omega$,

$$n \in J^2(A) \Leftrightarrow n \in W_i(W_j^\sigma \upharpoonright M)[\|\sigma\|],$$

σ corresponding to n , and M depending on the construction. □

Note: The basic jump-join theorem is sufficient for a natural Turing definition of the jump. But a local version is needed to provide that of the relation of ‘computably enumerable in’, which applies to 2-CEA operators derived from dynamically *special* d.c.e. sets —

THE LOCAL JUMP-JOIN THEOREM FOR 2-CEA OPERATORS DERIVED FROM A d-c.e. SET: *If J^2 is a 2-CEA operator derived from a special d-c.e. set, and $X \leq_T \emptyset'$ is not of c.e. degree, then one can find an $A \leq_T \emptyset''$ such that*

$$X \oplus A \equiv_T J^2(A).$$

PROOF.

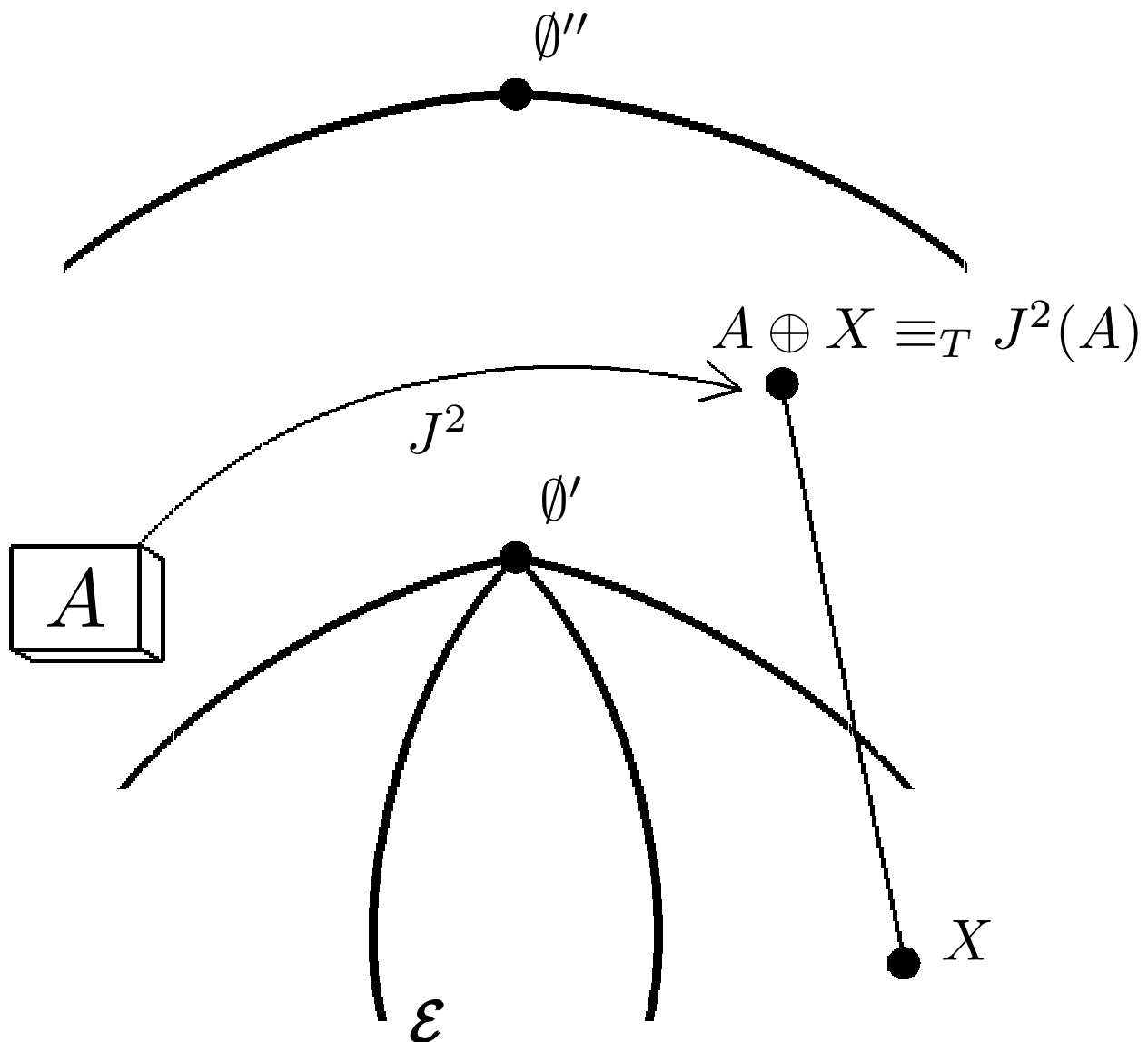
- Let $J^2(A) = W_{j_1}^{W_{j_0}^A} = W_{j_1}(W_{j_0}^A)$, all $A \subseteq \omega$.
- Again — can assume

$$W_{j_0}^X = X \oplus [\{\langle x, s \rangle \mid x \in (W_{i,s+1}^X - W_{i,s}^X)\} \\ \cup \{\langle x, s+1 \rangle \mid x \in (W_{i,s+1}^X - W_{i,s}^X) \cap W_j^X\}],$$

$$W_{j_1}^{X \oplus Y} = X \oplus \{x \mid \exists s [\langle x, s \rangle \in Y \ \& \ \langle x, s \pm 1 \rangle \notin Y]\},$$

where $J^2(A) = A \oplus (W_i^A - W_j^A)$.

- **In fact** — can assume J^2 to be defined by $J^2(A) = W_i^A - W_j^A$ — and then construct a set A satisfying the picture:



- Again — define $A = \cup_{n \geq 0} \sigma_n$, with —
- The σ 's $\subset A$ chosen to code X into A — and to force appropriate membership relations between numbers n and $J^2(A)$.

●● **This time** — σ_n ($= \lim_s \sigma_n^s$) is chosen relative to computably enumerable sets $S_{i,n}$, $S_{j,n}$ via an infinite injury priority argument — with the true path, represented by $\{\sigma_n\}_{n \geq 0}$, retrievable from $A \oplus X$ and from $J^2(A)$.

Definition of $S_{i,n}$, $S_{j,n}$:

● Let

$$S_{i,n} = \{\sigma \supseteq \sigma_n \mid n \in W_i^\sigma\},$$

$$S_{j,n} = \{\sigma \in S_{i,n} \mid n \in W_j^\sigma\}.$$

●● Notice — $S_{i,n}$, $S_{j,n}$ are c.e. sets of strings.

The forcing relation:

●● Say σ, τ *force* $n \in J^2(A)$ iff $\sigma \in S_{i,n}$, and for no $\tau' \approx \tau$ is $\tau' \in S_{j,n}$.

●● And say σ, τ *force* $n \notin J^2(A)$ iff $\tau \approx$ some $\tau' \in S_{i,n} \Rightarrow \tau \in S_{j,n}$.

The overall requirements:

- $\{\sigma_n\}_{n \geq 0}$ is retrievable from $A \oplus X$.
 - $\{\sigma_n\}_{n \geq 0}$ is retrievable from $J^2(A)$.
 - $\sigma_{n+1}(|\sigma_{n+1}| - 1) = X(n)$
- So $A \oplus X$ retrievable from $\{\sigma_n\}_{n \geq 0}$.

The prioritised requirements:

$$\mathcal{R}_n : (\exists \tau \subset A)[\sigma_{n+1}, \tau \Vdash n \in J^2(A) \text{ or } \sigma_{n+1}, \tau \Vdash n \notin J^2(A)].$$

Ensuring — The forcing relation does its job and — if σ_{n+1} is sufficiently informative about how it does it — $J^2(A)$ is retrievable from $\{\sigma_n\}_{n \geq 0}$.

Aim:

- To *select* least π , and then $\sigma \supseteq \sigma_n \hat{0}^{\langle \pi \rangle} \hat{1}$, $\tau \supseteq \sigma \hat{X}(n)$, for which *either*
 - (a) π is to the right of X , $\sigma \supset \sigma_n \hat{0}^{\langle \pi \rangle} \hat{1}$, and σ, τ force $n \in J^2(A)$ at stage $s+1$,
or
 - (b) π is to the left of X , $\sigma = \sigma_n \hat{0}^{\langle \pi \rangle} \hat{1}$, and σ, τ force $n \notin J^2(A)$ at stage $s+1$.
- To *define* $\sigma_{n+1}^{s+1} = \sigma \hat{X}(n)$, and choose $A^{s+1} \supset \tau_{n+1}^{s+1} = \tau$.
- And to maintain satisfaction of \mathcal{R}_n at later stages $t+1$ via π, τ still satisfying (a) or (b), with $\tau \subset A^{t+1}$.

That is, want to get —

LEMMA 7.1: There exist π, τ permanently satisfying either (a) or (b).

●● First notice that if no such π, τ exist permanently satisfying clause (a) —

● Then for each π to the right of X — each $\sigma \supseteq \sigma_n \wedge 0^{\langle \pi \rangle} \wedge 1$, $\tau \supseteq \sigma \wedge X(n)$ — one *either* has σ permanently $\notin S_{i,n}$, or $\tau \approx$ some $\tau' \in S_{j,n}$.

●● And if no such π, τ exist permanently satisfying clause (b) —

● Then for each π to the left of X one has for each $\tau \supseteq \sigma_n \wedge 0^{\langle \pi \rangle} \wedge 1 \wedge X(n)$ that $\tau \approx$ some $\tau' \in S_{i,n}$, but τ permanently $\notin S_{j,n}$.

●● **There are two cases** to consider:

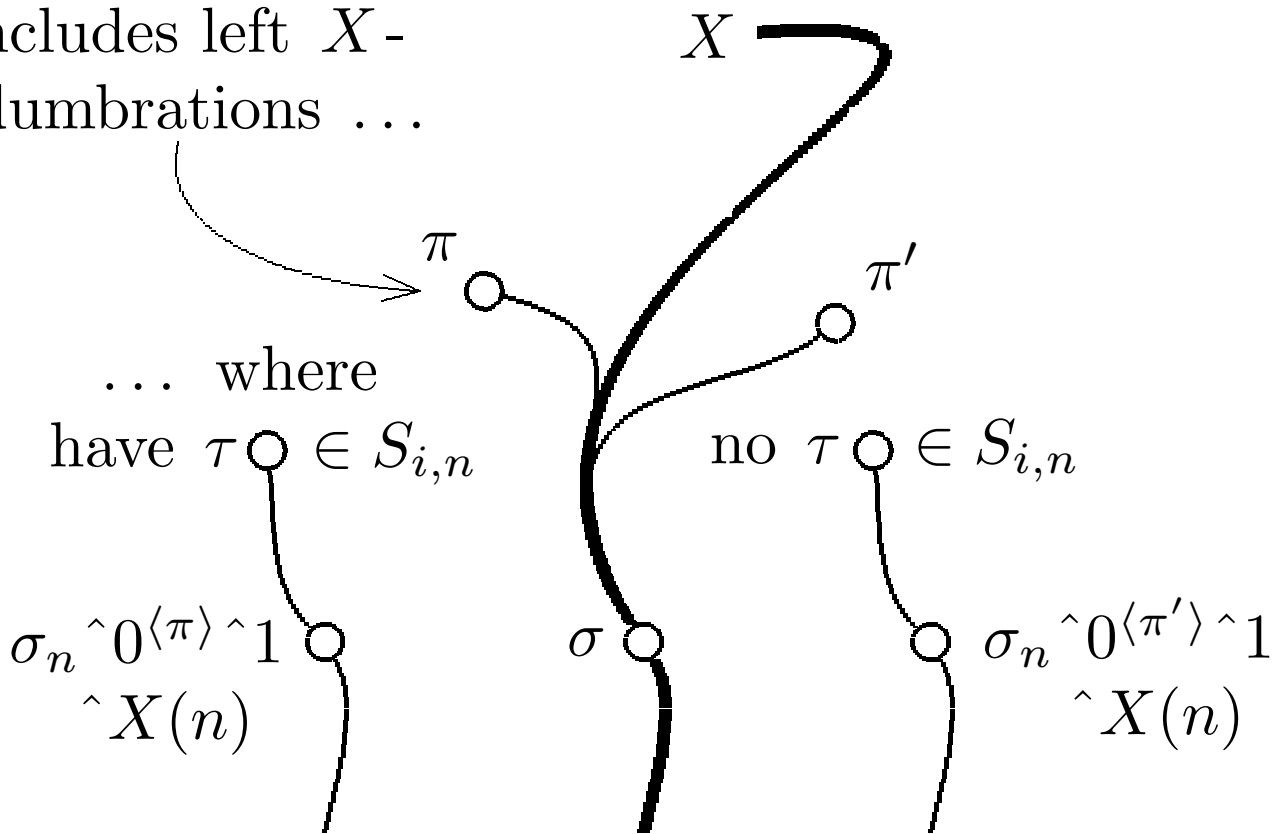
I. There exists some $\sigma \subset X$ — for which, for each $\pi \supset \sigma$ to the right of X , there does not exist a $\tau \supseteq \sigma_n \wedge 0^{\langle \pi \rangle} \wedge 1 \wedge X(n)$ with $\tau \in S_{i,n}$.

II. Otherwise.

•• **Assuming I applies** —

Call a τ , for which there is some $\pi \supset \tau$, for which $\pi \supset \sigma$ — and for which there exists some $\tau' \in S_{i,n}$ with $\tau' \supseteq \sigma_n \hat{0}^{\langle \pi \rangle} \hat{1} \hat{X}(n)$ — a *left X-adumbration*.

includes left X-adumbrations ...



Then notice —

- The set of all left X -adumbrations is c.e., and
- There exist only finitely many strings minimally to the right of a given τ .

•• **So** — verifying that for $\tau \supset \sigma$ have :

$\tau \subset X \Leftrightarrow \tau$ is a left X -adumbration,
and no $\tau' \supset \sigma$ minimally to
the right of τ is a left X -
adumbration,

and hence

$X \equiv_T$ the set of all left X -adumbrations

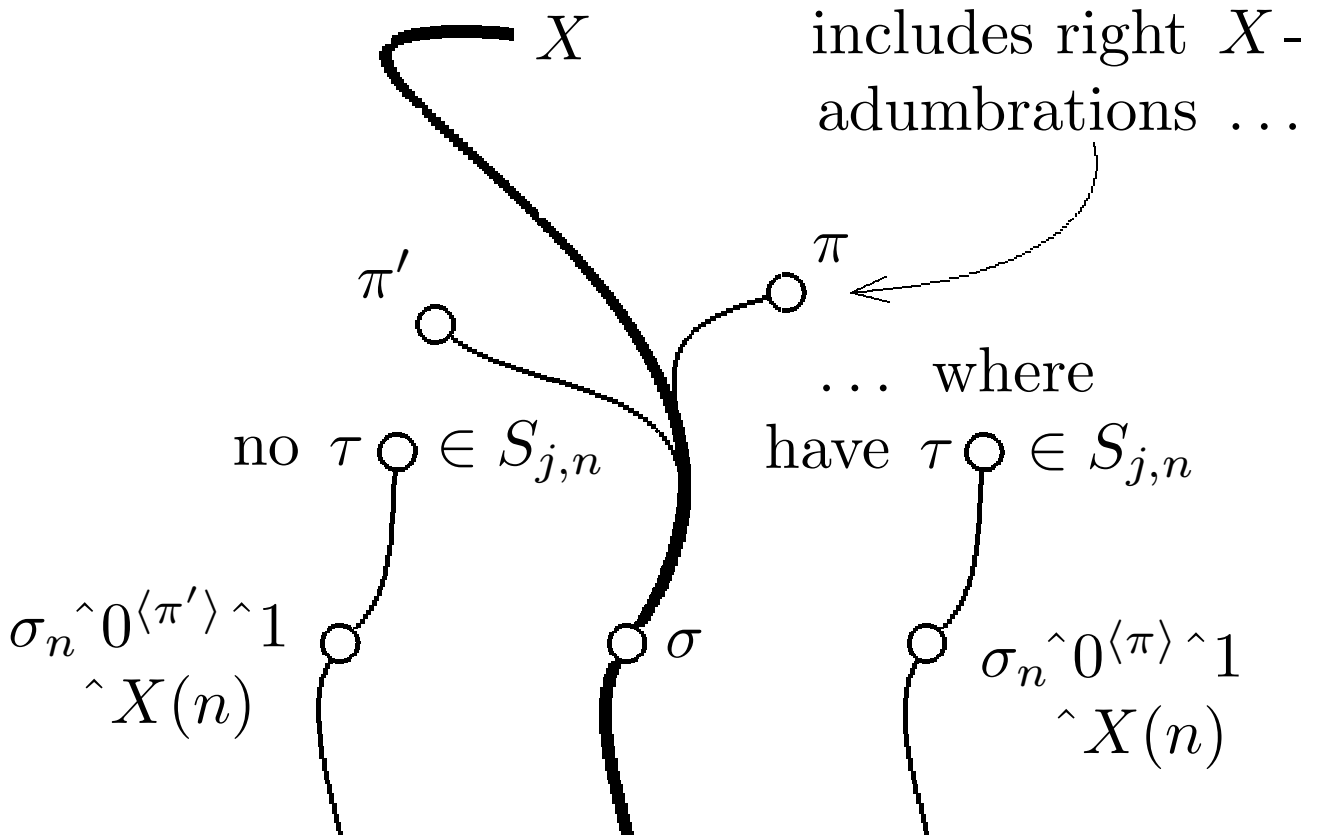
— get a contradiction to X not being of c.e. degree.

•• **And assuming case II applies —**

- Get infinitely many strings $\sigma \subset X$, with some $\pi \supset \sigma$ to the right of X , for which $\sigma_n \hat{0}^{\langle \pi \rangle} \hat{1} \hat{X}(n) \subset$ some $\tau \in S_{i,n}$ —

- So can modify the definition from case I —

Call a τ — for which there is some $\pi \supset \tau$ for which there exists a $\tau' \in S_{j,n}$ with $\tau' \supset \sigma_n \hat{0}^{\langle \pi' \rangle} \hat{1} \hat{X}(n)$ — a *right X-adumbration*.



And note —

- The set of all right X -adumbrations is c.e., and —

- For each τ have :

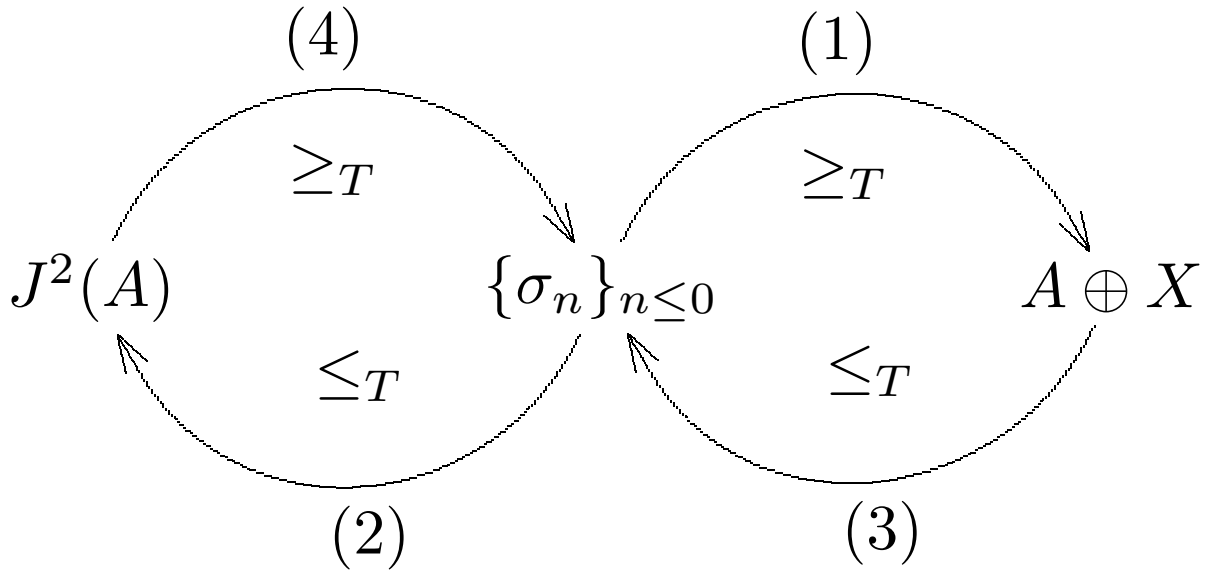
$\tau \subset X \Leftrightarrow \tau$ is a right X -adumbration,
and no τ' minimally to the
left of τ is a right X -adumbration,

and hence

$X \equiv_T$ the set of all right X -adumbrations

— again contradicting the assumption that X is not of c.e. degree.

For background requirements — need to verify following Turing reductions:



(1) $A \upharpoonright (|\sigma_{n+1}| - 1)$ and $X(n)$ will be computable from σ_{n+1} —

- Using $\sigma_{n+1} \approx A$ and $X(n) = \sigma_{n+1}(|\sigma_{n+1}| - 1)$.

(2) Can decide $n \in J^2(A)$ using σ_n and σ_{n+1} —

- Since the π for which \mathcal{R}_n is permanently satisfied will be retrievable from σ_n and A , and

$$\left\| \left\| \begin{array}{l} n \in W_i^A - W_j^A \Leftrightarrow \\ \sigma_{n+1} \upharpoonright (|\sigma_{n+1}| - 2) \supset \sigma_n \hat{0}^{\langle \pi \rangle} \hat{1}. \end{array} \right. \right\|$$

(3) *With careful choice of τ_{n+1}^{s+1} can determine σ_{n+1} from σ_n using A and X —*

- Since to retrieve a *true* stage $s + 1$ — that is a stage $s + 1$ at which one defines $\sigma_{n+1}^{s+1} = \sigma_{n+1}$ — one can use σ_n and A to find π for lemma 7.1 —

- And then check whether π is to the left of X or not to see which of cases (a) or (b) apply at stage $s + 1$ —

- If π is to the left of X — so (b) applies — one has $\sigma_{n+1} = \sigma_{n+1}^{s+1} = \sigma_n \hat{0}^{\langle \pi \rangle} \hat{1} \hat{X}(n)$.

- If π is to the right of X — so (a) applies — one can follow through the module until one finds $\sigma_n \hat{0}^{\langle \pi \rangle} \hat{1} \subset \sigma \subset A$ with $\sigma \in S_{i,n}$ in 1(a), with help from A —

- Then $\sigma_{n+1} = \sigma_{n+1}^{s+1} = \sigma \hat{X}(n)$.

- **So** — from (2) and (3) one gets that:

Given σ_n — can decide $n \in J^2(A)$ using A and X .

(4) Can determine σ_{n+1} from σ_n using $J^2(A)$ —

- Since — using $A \leq_T J^2(A)$ — to retrieve a true stage one can find π from σ_n and A —
- Then verify whether $n \in J^2(A)$ so as to see if one ends up permanently in case (a) or (b) —
- And as for (3) — if (b) applies — one has $\sigma_{n+1} = \sigma_n \hat{0}^{\langle \pi \rangle} \hat{1} \hat{X}(n)$.
- And if (a) applies — one can follow through the module until one finds $\sigma_n \hat{0}^{\langle \pi \rangle} \hat{1} \subset \sigma \subset A$ with $\sigma \in S_{i,n}$ in 1(a), with help from A —
- In which case $\sigma_{n+1} = \sigma \hat{X}(n)$.
- **So** — from (1) and (4) one gets that:

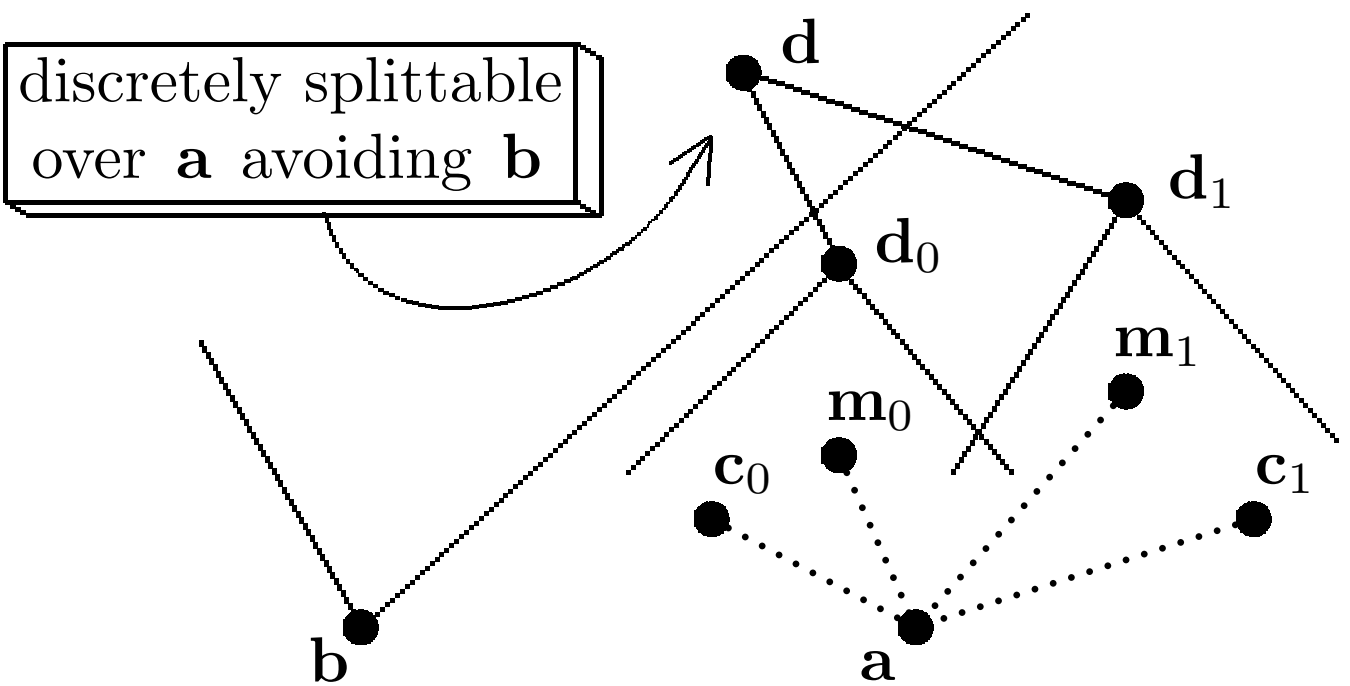
Given σ_n — can compute $A \upharpoonright (\sigma_{n+1} - 1)$ and $X(n)$ using $J^2(A)$.

8. *A particular 2-CEA operator*

DEFINITION 8.1: • Given a, b, d , say d is splittable over a avoiding b iff $a, b \leq d$, $b \not\leq a$ imply there exist $d_0, d_1 < d$ for which $a < d_0, d_1$, $b \not\leq d_0$ and d_1 , and $d = d_0 \cup d_1$.

• Further, d is discretely splittable over a avoiding b iff each such d_i has the Cooper-Epstein property — that is, for each $c_i < d_i$, a has a minimal cover $m_i \leq d_i$ but not $\leq c_i$.

• d is relatively unsplittable iff not discretely splittable over a avoiding b , some a, b .



Then notice: By the relativised Sacks Splitting Theorem (see Soare [1987], p.124) — and Cooper & Epstein [1987], Theorem 1 — can show (see later) there is no relatively unsplittable c.e. degree.

THEOREM 8.2: *There exists a relatively unsplittable d-c.e. degree.*

- *That is, there is a d-c.e. set $D = W_i - W_j$ and sets $A, B \leq_T D$ such that $\text{deg}(D)$ is not discretely splittable over $\text{deg}(A)$ avoiding $\text{deg}(B)$.*

Some consequences —

- One can now get a 2-CEA operator J such that for each C one has $J(C) = C \oplus (W_i^C - W_j^C)$ and $\text{deg}(J(C))$ is relatively unsplittable —

- Since $\text{deg}(J(C))$ is not discretely splittable over \mathbf{a} avoiding \mathbf{b} , some $\mathbf{a}, \mathbf{b} \geq \text{deg}(C)$.
- Then, using J with the basic Jump-Join Theorem for 2-CEA Operators derived from a d-c.e. set one gets:

THEOREM 8.3 (DEFINABILITY OF $\mathbf{0}'$):
 $\mathbf{0}'$ is Π_6 Turing definable as the largest degree satisfying

$$(\forall \mathbf{a}, \mathbf{b})[\mathbf{x} \cup \mathbf{a} \text{ is discretely splittable over } \mathbf{a} \text{ avoiding } \mathbf{b}]. \quad (\dagger)$$

- Since firstly can verify the following —

|| BASIC FACT: Each c.e. degree — including $\mathbf{0}'$ — satisfies (\dagger) . ||

- Assume given \mathbf{x} c.e. — and \mathbf{a}, \mathbf{b} with $\mathbf{b} \not\leq \mathbf{a}$, $\mathbf{b} \leq \mathbf{x} \cup \mathbf{a}$ and $\mathbf{x} \cup \mathbf{a} \not\leq \mathbf{a}$ —

- And let $\mathbf{b}_1 = \mathbf{a} \cup \mathbf{b}$ — so $\mathbf{a} < \mathbf{b}_1 \leq \mathbf{x} \cup \mathbf{a}$.
- Then since $\mathbf{x} \cup \mathbf{a}$ is c.e. in \mathbf{a} —
- The *Sacks splitting theorem* — relativised to $\mathcal{D}(\geq \mathbf{a})$ — gives a splitting $\mathbf{d}_0, \mathbf{d}_1$ of $\mathbf{x} \cup \mathbf{a}$ low over \mathbf{a} avoiding \mathbf{b}_1 — and hence avoiding \mathbf{b} .
- And then — the relativised Theorem 1 of Cooper and Epstein [1987] — gives minimal covers $\mathbf{m}_0, \mathbf{m}_1$ of \mathbf{a} with each $\mathbf{m}_i \leq \mathbf{d}_i$ and $\mathbf{m}_i \not\leq_T \mathbf{c}_i$ for any given $\mathbf{c}_i < \mathbf{d}_i$. \square

•• **On the other hand** —

- Say $X \not\leq_T \emptyset'$.
- Then if $D \geq_T \emptyset'' \oplus X$ — the basic Jump-Join Theorem gives an A_1 for which $X \oplus A_1 \equiv_T D \equiv_T J(A_1)$ — with $\deg(X \oplus A_1)$ relatively unsplittable over some $\mathbf{a} \geq \deg(A_1)$ avoiding some \mathbf{b} —
- Immediately giving \mathbf{a}, \mathbf{b} with $\deg(X) \cup \mathbf{a}$ not discretely splittable over \mathbf{a} avoiding \mathbf{b} . \square

- **Relativising get** — For each \mathbf{a} , \mathbf{a}' is definable in $\mathcal{D}(\geq \mathbf{a})$, giving:

THEOREM 8.4 (DEFINABILITY OF THE JUMP): *The Turing jump is first order definable in \mathcal{D} .*

- So — answering Rogers [1967] — get:

COROLLARY 8.5: *The jump is invariant.*

Useful fact — One can now eliminate the jump from many pre-existing results for the structure \mathcal{D}' — often greatly improving those previously available for \mathcal{D} .

- Such *definability* results include those of:

- The set of arithmetical degrees — building on Jockusch and Soare [1970]. (Obtained previously by Jockusch and Shore [1984].)
- All jump classes High_n and Low_n , $n \geq 0$.
- Any relation on $\mathcal{D} (\geq \mathbf{0}^{(3)})$ definable in second-order arithmetic — from Shore [1982].

- Concerning *homogeneity* one gets:

- If $\mathcal{D} (\geq \mathbf{a}) \cong \mathcal{D} (\geq \mathbf{b})$ then $\mathbf{a}^{(3)} = \mathbf{b}^{(3)}$ — using Richter [1979].
- If $\mathcal{D} \equiv \mathcal{D} (\geq \mathbf{a})$ then $\mathbf{a}^{(3)} = \mathbf{0}^{(3)}$ — from Shore [1981].

Note — In particular, get $\mathcal{D} \not\equiv \mathcal{D}(\geq \mathbf{0}')$ — but proof does not give a *natural* elementary difference between \mathcal{D} and $\mathcal{D}(\geq \mathbf{0}')$ —

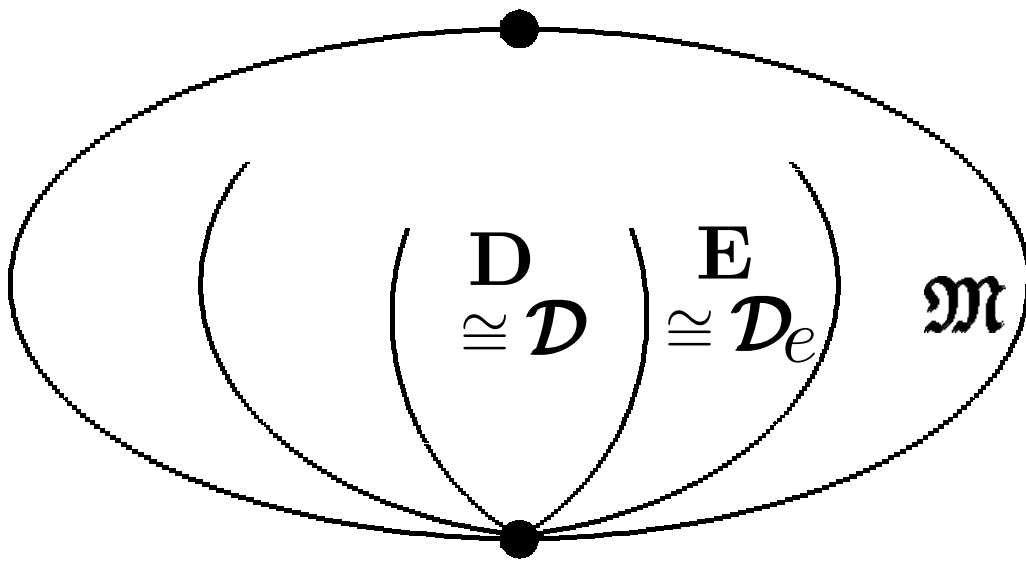
General observation: Properties arising from degree theoretic codings typically entail eight or more alternations of quantifier.

- Concerning *automorphisms* — one gets:

• If ψ is an automorphism of \mathcal{D} — then $\psi(\mathbf{a}) = \mathbf{a}$ for all $\mathbf{a} \geq \mathbf{0}^{(3)}$ — using the result of Epstein [1979] and Richter [1979] for \mathcal{D}' .

Important note — Many of these results have since been obtained directly using coding methods — and where feasible improved by a factor of one — see Nies, Shore & Slaman [1996].

- Additionally — There is a natural embedding of the Turing degrees into the *Medvedev lattice* \mathfrak{M} — as the *degrees of solvability* \mathbf{D} , and —
- (Dyment [1976]) \mathbf{D} is definable in \mathfrak{M} .



- So — answering Rogers [1967, p.288] —

•• The jump operation on the degrees of solvability is definable in \mathfrak{M} .

- **Finally** — Let J be the 2-CEA operator given by theorem 8.2 — where for each C one has $J(C) = C \oplus (W_i^C - W_j^C)$ and $\deg(J(C))$ is relatively unsplittable —

- Then — Using J with the *local* Jump-Join Theorem for 2-CEA Operators derived from a special d-c.e. set one immediately gets:

THEOREM 8.6 (LOCAL DEFINABILITY OF THE COMPUTABLY ENUMERABLE DEGREES): $\mathbf{x} \leq \mathbf{0}'$ is *computably enumerable iff it satisfies*

$$(\forall \mathbf{a}, \mathbf{b} \leq \mathbf{0}'') [\mathbf{x} \cup \mathbf{a} \text{ is discretely splittable over } \mathbf{a} \text{ avoiding } \mathbf{b}]. \quad (\dagger\dagger)$$

- Since every c.e. \mathbf{x} satisfies (\dagger) above — so must also satisfy $(\dagger\dagger)$ — and —

- If $\mathbf{x} \leq \mathbf{0}'$ is *not* c.e., can use the local jump-join theorem as in the proof of the definability of $\mathbf{0}'$ to get \mathbf{a}, \mathbf{b} negating $(\dagger\dagger)$. □

- **Hence** —

THEOREM 8.7 — DEFINABILITY OF THE C.E. DEGREES:

\mathbf{x} is computably enumerable iff it satisfies

*$(\forall \mathbf{a}, \mathbf{b})[\mathbf{x} \cup \mathbf{a}$ is discretely
splittable over \mathbf{a} avoiding \mathbf{b}].*

- Since again — Every c.e. \mathbf{x} satisfies (\dagger).
- And if \mathbf{x} is not c.e. —
- *Either $\mathbf{x} \not\leq \mathbf{0}'$* — in which case result follows from theorem 8.3 —
- *Or $\mathbf{0} < \mathbf{x} < \mathbf{0}'$* — and result follows via theorem 8.6. \square

Note — This time get a Π_5 definition. $\|$

Relativising — For each \mathbf{a} , the degrees CEA \mathbf{a} are definable in $\mathcal{D}(\geq \mathbf{a})$.

•• But — can do better:

THE RELATIVISED LOCAL JUMP-JOIN THEOREM FOR 2-CEA OPERATORS DERIVED FROM A d-c.e. SET: *If J^2 is a 2-CEA- B operator got from a special B -d-c.e. set, and $X \leq_T B'$ is not of B -c.e. degree, then one can find an A such that $B \leq_T A \leq_T B''$ and*

$$X \oplus A \equiv_T J^2(A).$$

• So — again see Rogers [1967] — get:

THEOREM 8.8 — DEFINABILITY OF THE RELATION OF “C.E. IN”:

\mathbf{x} is c.e. in \mathbf{y} iff it satisfies

$(\forall \mathbf{a}, \mathbf{b} \geq \mathbf{y})[\mathbf{x} \cup \mathbf{a}$ is discretely
splittable over \mathbf{a} avoiding $\mathbf{b}]$.

9. The requirements for theorem 8.2

NEED TO SHOW — *There exists a d.c.e. degree \mathbf{d} which is not discretely splittable over \mathbf{a} avoiding \mathbf{b} , some $\mathbf{a}, \mathbf{b} < \mathbf{d}$.*

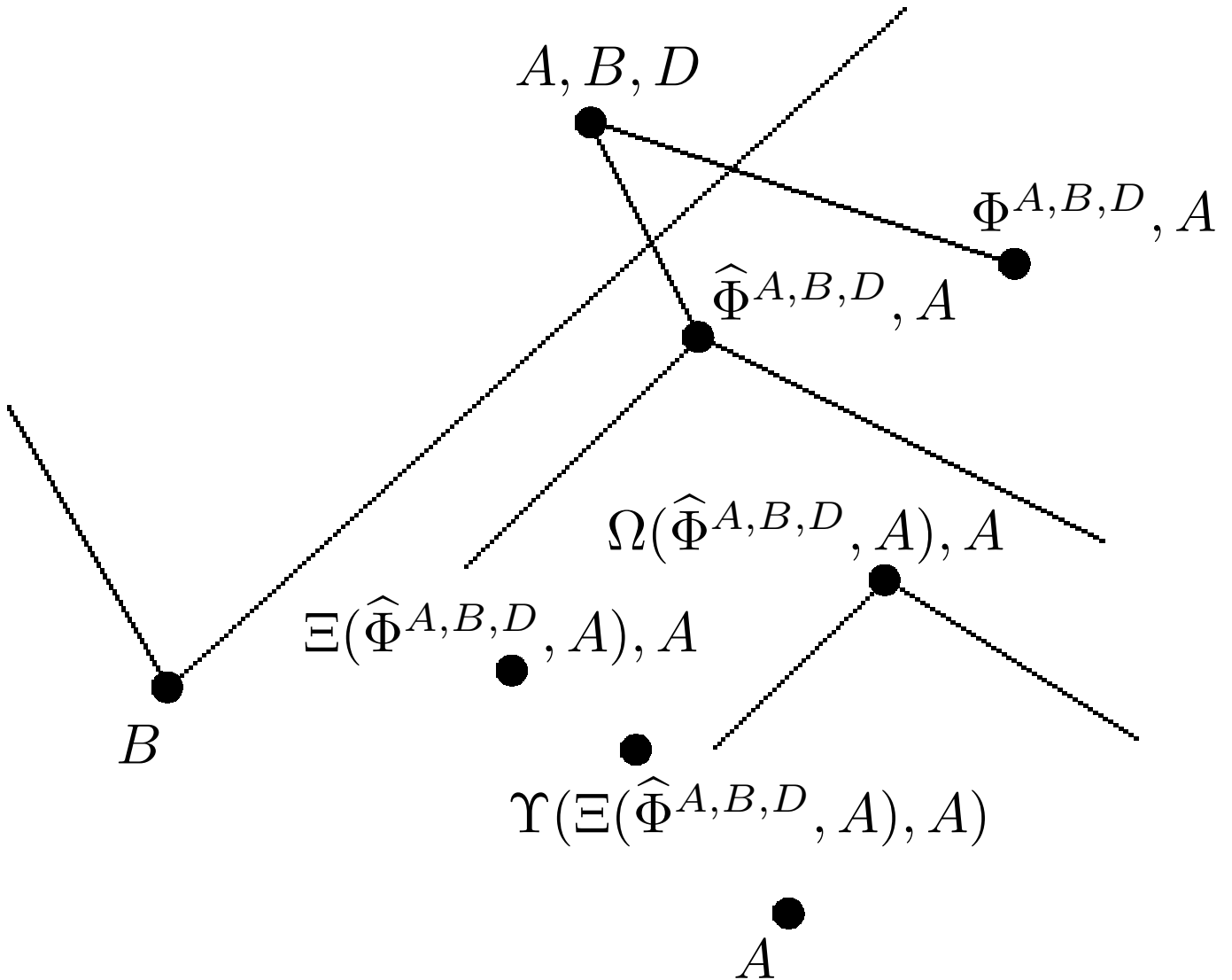
THE REQUIREMENTS:

$$\begin{aligned}
 \mathcal{P}_\Theta &: B \neq \Theta^A \\
 \mathcal{Q}_{\Psi, \Phi, \hat{\Phi}, \Xi, W, \bar{W}} &: D = \Psi(\Phi^{A, B, D}, \hat{\Phi}^{A, B, D}, A) \\
 \Rightarrow B &= \Gamma^{\Phi^{A, B, D}, A} \vee B = \Lambda^{\hat{\Phi}^{A, B, D}, A} \\
 &\vee \Omega(\hat{\Phi}^{A, B, D}, A) \neq \hat{\Theta}^A \\
 \&\mathcal{E} [\Xi^{\hat{\Phi}^{A, B, D}, A} = \Delta^{\Omega(\hat{\Phi}^{A, B, D}, A), A} \\
 &\vee \Upsilon(\Xi^{\hat{\Phi}^{A, B, D}, A}, A) \Vdash W^A \\
 &\vee \bar{\Upsilon}(\Xi^{\hat{\Phi}^{A, B, D}, A}, A) \Vdash \bar{W}^A]
 \end{aligned}$$

Where —

- If $X \Vdash W^A$ for each A -c.e. W^A , then $X \oplus A$ is 1-generic over A .

Corresponding picture:



Note — The strategy for two \mathcal{P} -requirements below one \mathcal{Q} -requirement only involves the Γ and Λ strategies —

In fact — one needs six requirements to fully activate all four modular levels —

