Splitting and Nonsplitting, II: A Low$_2$ C.E. Degree Above Which $\mathbf{0}'$ Is Not Splittable

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Abstract

It is shown that there exists a low$_2$ Harrington non-splitting base — that is, a low$_2$ computably enumerable (c.e.) degree $\mathbf{a}$ such that for any c.e. degrees $\mathbf{x}, \mathbf{y}$, if $\mathbf{0}' = \mathbf{x} \vee \mathbf{y}$, then either $\mathbf{0}' = \mathbf{x} \vee \mathbf{a}$ or $\mathbf{0}' = \mathbf{y} \vee \mathbf{a}$. Contrary to prior expectations, the standard Harrington non-splitting construction is incompatible with the low$_2$-ness requirements to be satisfied, and the proof given involves new techniques with potentially wider application.

1 Introduction

We say that a set $A \subseteq \omega$ is *computably enumerable* (c.e.) if there is an algorithm to enumerate the elements of it. For $A, B \subseteq \omega$, we say that A is *Turing reducible to* (or *computable* in) B if there is an algorithm to decide for every $x \in \omega$, whether or not $x \in A$ when given answers to all questions of the form " Is $y \in B$?". We use $A \leq_{\mathrm{T}} B$ to denote that A is Turing reducible to B, and we write $A \equiv_{\mathrm{T}} B$ if $A \leq_{\mathrm{T}} B$ and $B \leq_{\mathrm{T}} A$. A (Turing) *degree* is an equivalence class of A under $\equiv_{\mathrm{T}}$ for some $A \subseteq \omega$. We say that a degree $\mathbf{a}$ is

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computably enumerable (c.e.) if it contains a c.e. set. Post (1944) pioneered the study of the structure of the c.e. degrees. He observed that there is a greatest c.e. degree $\mathbf{0}'$, and asked whether or not there is a c.e. degree other than $\mathbf{0}$ (the least degree) and $\mathbf{0}'$.

Friedberg [1957], and independently Muchnik [1956] answered Post's question affirmatively. Furthermore, Sacks [1963], [1964] showed that:

1.1 Theorem. (Sacks Splitting Theorem, 1963) For any c.e. degree $\mathbf{a} \neq \mathbf{0}$, there exist c.e. degrees $\mathbf{a_0}, \mathbf{a_1}$ such that $\mathbf{a_0}, \mathbf{a_1} < \mathbf{a}$ and $\mathbf{a} = \mathbf{a_0} \vee \mathbf{a_1}$.

1.2 Theorem. (Sacks Density Theorem, 1964) For any c.e. degrees $\mathbf{b} < \mathbf{a}$, there is a c.e. degree $\mathbf{c}$ such that $\mathbf{b} < \mathbf{c} < \mathbf{a}$.

A basic question was then whether or not theorems 1.1 and 1.2 could be combined, which was eventually answered negatively by Lachlan [1975].

1.3 Theorem. (Lachlan Nonsplitting Theorem, 1975) There exist c.e. degrees $\mathbf{b} < \mathbf{a}$ such that for any c.e. degrees $\mathbf{x}, \mathbf{y}$, if $\mathbf{x} \vee \mathbf{y} = \mathbf{a}$, then either $\mathbf{a} \leq \mathbf{x} \vee \mathbf{b}$ or $\mathbf{a} \leq \mathbf{y} \vee \mathbf{b}$.

The proof of this result led to important technical developments in computability theory. On the other hand, the result itself added to a growing impression of structural pathology, unwelcome to the non-specialist. Of course, such results are now viewed as aids to establishing *Turing definability*, and to the re-emergence of useful and attractive mathematical theory at that level. In particular, they seem relevant to the search for natural definitions of a number of classes of Turing degrees which arise directly from easily described sets of reals.

Using the Turing jump, we have a high/low hierarchy for a class of c.e. degrees. Let $\mathbf{x^{(0)}} = \mathbf{x}$, $\mathbf{x^{(n+1)}} = (\mathbf{x^{(n)}})'$, where $\mathbf{y}'$ is the Turing jump of $\mathbf{y}$. We say that a c.e. degree $\mathbf{a}$ is high$_n$ (low$_n$) if $\mathbf{a^{(n)}} = \mathbf{0^{(n+1)}}$ ($\mathbf{a^{(n)}} = \mathbf{0^{(n)}}$), for all $n > 0$. For $n = 1$, we also call an element $\mathbf{a} \in \mathbf{H_1}$, $\mathbf{L_1}$, *high* and *low*, respectively. Then we have:

1.4 Theorem. (Robinson [1971] Low Splitting) For any c.e. degrees $\mathbf{l} < \mathbf{a}$, if $\mathbf{l}$ is low, then there are c.e. degrees $\mathbf{a_0}, \mathbf{a_1}$ such that $\mathbf{l} < \mathbf{a_0}, \mathbf{a_1} < \mathbf{a}$ and $\mathbf{a_0} \vee \mathbf{a_1} = \mathbf{a}$.

This is improved by Arslanov, Cooper and Li [2000]:

1.5 Theorem. (Generalised Low Splitting) For any Δ^0_2 set L, any c.e. set A, if L is of low, and $A \not\leq_{\mathrm{T}} L$, then there exist c.e. sets A_0 and A_1 such that $A_0 \cap A_1 = \emptyset$, $A_0 \cup A_1 = A$, and $A \not\leq_{\mathrm{T}} A_i \oplus L$ for both $i = 0$ and 1.

In addition:

1.6 Theorem. (Harrington [1980] Nonsplitting Theorem) There exists an incomplete c.e. degree $\mathbf{a}$ such that for any c.e. degrees $\mathbf{x}, \mathbf{y}$, if $\mathbf{x} \vee \mathbf{y} = \mathbf{0}'$, then either $\mathbf{0}' \leq \mathbf{x} \vee \mathbf{a}$ or $\mathbf{0}' \leq \mathbf{y} \vee \mathbf{a}$.

It is convenient to introduce the following terminology:

1.7 Definition. (i) Given c.e. degrees $\mathbf{b} < \mathbf{a}$, we say that $(\mathbf{b}, \mathbf{a})$ is a *nonsplitting pair* of c.e. degrees if for any c.e. degrees $\mathbf{x}, \mathbf{y}$, if $\mathbf{x} \vee \mathbf{y} = \mathbf{a}$, then either $\mathbf{a} \leq \mathbf{x} \vee \mathbf{b}$ or $\mathbf{a} \leq \mathbf{y} \vee \mathbf{b}$ holds.

(ii) We say that a c.e. degree is a *Harrington nonsplitting base*, if $\mathbf{a} \neq \mathbf{0}'$ and $(\mathbf{a}, \mathbf{0}')$ is a nonsplitting pair of c.e. degrees.

(iii) We say that a c.e. degree $\mathbf{b}$ is a *Lachlan nonsplitting base* if there is a c.e. degree $\mathbf{a} > \mathbf{b}$ such that $(\mathbf{b}, \mathbf{a})$ is a nonsplitting pair of c.e. degrees.

Then we have:

1.8 Theorem. (Harrington, see Shore and Slaman [1990]) No low$_2$ c.e. degree bounds a nonsplitting pair of c.e. degrees.

And:

1.9 Theorem. (Cooper, Li and Yi [ta]) There exists a nonlow$_2$ c.e. degree which bounds no Lachlan nonsplitting base.

While Cooper and Li [ta] have shown how standard nonsplitting techniques can be adapted to provide a low$_3$ Harrington nonsplitting base, it is clear from the analysis in that paper that the style of proof used there cannot be used to replace low$_3$ with low$_2$. This is contrary to initial impressions, and illustrates the extent to which ones understanding of particular constructions may be relatively shallow, inasmuch as it is necessarily based on a simplified conceptual framework which may turn out to be inappropriate in the light of further investigations. However, in this paper we overcome a number of technical obstacles in proving:

1.10 Theorem. There exists a c.e. degree $\mathbf{a}$ with the following properties:

(1) $\mathbf{a}'' \leq \mathbf{0}''$

(2) for any c.e. degrees $\mathbf{x}$, $\mathbf{y}$, if $\mathbf{0}' \leq \mathbf{x} \vee \mathbf{y}$, then either $\mathbf{0}' \leq \mathbf{x} \vee \mathbf{a}$ or $\mathbf{0}' \leq \mathbf{y} \vee \mathbf{a}$.

The result is, of course, the best possible concerning the nonsplitting of c.e. degrees, by theorem 1.4 or theorem 1.5. Its proof requires a new technique, and the methods developed during the verification of the theorem promise to be as important as the theorem itself. One consequence of the former is that the proof is inevitably longer than that of Cooper and Li [ta].

We now turn to the proof of the main theorem, theorem 1.10. We organise the paper as follows. In section 2, we formulate the conditions of the theorem as requirements, and describe the basic modules for satisfying those requirements; in section 3, we describe the $\mathcal{L}$-strategies working with two $\mathcal{R}$-strategies; in section 4, we describe the general strategies; in section 5, we describe an effective construction to build the tree of strategies, i.e. the priority tree; in section 6, we introduce some definitions which will be useful in the description and the verification of the construction; in section 7, we describe the full construction; and finally in section 8, we verify that the construction satisfies all requirements.

Notation and terminology are standard and generally follow Soare [1987] or P. Odifreddi [1989] (modulo the by now usual translations of 'recursive' into 'computable', etc.). During the course of the construction, notations such as A, Φ are used to denote the current approximations to these objects, and if we want to specially specify the values of objects A, Φ say, to be the values immediately at the end of stage s, then we denote them by $A_s, \Phi[s]$, etc. For a partial computable (p.c.) functional Φ, say, the use function is denoted by the corresponding lower case letter ϕ. The value of the use function of a converging computation is the greatest number which is actually used in the computation. For a p.c. functional which is not built by us, if a computation is not defined, then we define its use function $= -1$. For a p.c. functional which is defined by us, if a computation is not defined, then we regard its use function as ω. During the course of the construction, whenever we define a parameter p, say, as $fresh$, we mean that p is defined as the least natural number which is greater than any number mentioned so far, in particular, if p is defined afresh at stage s, then $p > s$. We use $[t, s]$ to denote the *state* of a parameter immediately before substage t of stage s.

2 Requirements and Methods

2.1 The Requirements

To prove the theorem, we construct c.e. sets A, D to satisfy the following requirements:

$$\mathcal{L}_e : \quad \mathrm{Tot}^A(e) = \Omega(\emptyset''; e)$$
$$\mathcal{R}_{\Phi,X,Y} : \quad D = \Phi^{X,Y} \Rightarrow (\exists \text{ p.c. } \Gamma, \widehat{\Gamma})[K = \Gamma^{X,A} \text{ or } K = \widehat{\Gamma}^{Y,A}]$$

where $\mathrm{Tot}^A = \{e : \Psi_e(A) \text{ is total}\}$, Ω is a p.c. functional built by us, (Φ, Ψ, X, Y) is a typical member of a standard listing of all quadruples of p.c. functionals Φ, Ψ and c.e. sets X, Y, and K is a fixed creative set. We also write $\mathcal{L}_e = \mathcal{L}_\Psi$ with $\Psi = \Psi_e$.

It is easy to verify that Tot^A is Π^A_2-complete, giving $A'' \leq_\mathrm{T} \mathrm{Tot}^A$. So the $\mathcal{L}$-requirements give $A'' \leq_\mathrm{T} \emptyset''$. By the $\mathcal{R}$-requirements, for any c.e. sets X,

Y, if $K \not\leq_{\mathrm{T}} X \oplus A$ and $K \not\leq_{\mathrm{T}} Y \oplus A$, then $D \not\leq_{\mathrm{T}} X \oplus Y$ for some c.e. set D, so that $K \not\leq_{\mathrm{T}} X \oplus Y$. Hence the theorem will follow from the construction which meets all the requirements.

2.2 Satisfying the $\mathcal{L}$-Requirements

To satisfy the $\mathcal{L}$-requirements we have to describe an algorithm, using oracle $\emptyset''$, to decide whether or not $\Psi(A)$ is total, each Ψ.

For an $\mathcal{L}$-requirement $\mathcal{L}_\Psi$ say, we define the "length function" to be $l_\Psi = \max\{x \mid (\forall y < x)[\Psi^A(y)\downarrow]\}$. We say that s is *$\mathcal{L}_\Psi$-expansionary*, if $l_\Psi[s] > l_\Psi[v]$ for all $v < s$.

To satisfy a requirement we will provide an effective procedure or *strategy*. We will arrange all strategies on nodes of a tree, called the *priority tree*. So the name of a strategy will be a string denoting its location on a tree of strategies. Suppose that β is an $\mathcal{L}$-strategy. During the course of the construction, we will build a set C_β to record the computations of β which have already been cleared, which is called a *clearing set* of β. Roughly speaking, the clearing set C_β will record a length l say, for which $\Psi_\beta(A) \upharpoonright (l+1)$ has been cleared of all but a designated finite number of traces for the functionals built by higher priority $\mathcal{R}$-strategies.

For a given $\mathcal{L}$-requirement $\mathcal{L}_\Psi = \mathcal{L}_\beta$ say, we define the "β-believable length function" to be

$$l^+_\Psi = l^+_\beta = \max\{x \in C_\beta \mid (\forall y < x)[\Psi^A(y)\downarrow]\}.$$

We say that s is *believably $\mathcal{L}_\Psi$-expansionary at β*, or is a *β-believable $\mathcal{L}_\Psi$-expansionary* stage, if $l^+_\Psi[s] > l^+_\Psi[v]$ for all $v < s$.

To satisfy the $\mathcal{L}$-requirements we describe a construction with the following properties:

(1) The *true path* TP of the construction is defined (as usual) to be the set of all nodes α such that there are only finitely many stages s at which $\delta_s <_{\mathrm{L}} \alpha$, and that there are infinitely many stages s such that $\alpha \subseteq \delta_s$, where δ_s is the longest node which is visited at stage s.

(2) (Uniqueness) Given an infinite path f through the priority tree T, we have that for every Ψ, there is a unique $\mathcal{L}_\Psi$-strategy $\beta \in f$.

(3) (TP Condition) (i) TP is a total function from the requirements to T, the priority tree.

(ii) $TP \leq_{\mathrm{T}} \emptyset''$.

(4) ($\mathcal{L}$-Strategy Condition) There is a finite set Λ such that for a given $\mathcal{L}_\Psi$-strategy $\beta \in TP$, Ψ^A is total if and only if there is an element $a \in \Lambda$ such that $\beta\hat{\ }\langle a\rangle \in TP$.

Using these properties, we have that $\text{Tot}^A \leq_T \emptyset''$, since we can read out the totality of Ψ^A for every Ψ along the true path TP. Therefore, the key to the satisfaction of the $\mathcal{L}$-requirements is the four properties above.

The basic module for satisfying an $\mathcal{L}$-requirement, $\mathcal{L}_\Psi$ say, is to ensure that Ψ^A is total if and only if there are infinitely many believable $\mathcal{L}_\Psi$-expansionary stages, this being achieved via appropriate regulation of the corresponding clearing set.

However, a given $\mathcal{L}$-strategy will have to deal with the injury from the p.c. functionals built by higher priority $\mathcal{R}$-strategies, as will become clearer below.

2.3 An $\mathcal{R}$-Strategy

Given an $\mathcal{R}$-requirement, $\mathcal{R}_{\Phi,X,Y}$ say, we define the length of agreement function $l_{\Phi,X,Y} = l(D, \Phi^{X,Y})$ as usual. We say (dropping the subscripts) that s is $\mathcal{R}$-*expansionary*, if $l[s] > l[v]$ for all $v < s$. If there are only finitely many $\mathcal{R}$-expansionary stages, then $D \neq \Phi^{X,Y}$. Suppose that there are infinitely many $\mathcal{R}$-expansionary stages.

An $\mathcal{R}$-strategy will initially attempt to build a p.c. functional Γ for which $K = \Gamma^{X,A}$. For the function $\Gamma^{X,A}$, we require that the use function $\gamma^{X,A}$ (or for short, γ) has the following properties:

(i) If $\Gamma^{X,A}(x+1)[s]\downarrow$, then $\Gamma^{X,A}(x)[s]\downarrow$ and $\gamma(x)[s] < \gamma(x+1)[s]$;

(ii) If $\gamma(x)[s]\downarrow$, then $\gamma(x)[s] \notin A_s$;

(iii) If $\gamma(x)[s]\downarrow \neq \gamma(x)[s+1]$, then either $X_s \upharpoonright (\gamma(x)[s]+1) \neq X_{s+1} \upharpoonright (\gamma(x)[s]+1)$, or there is a $y \leq x$ such that $\gamma(y)[s] \in A_{s+1} - A_s$;

(iv) If $\gamma(x)[s]\downarrow \in A_{s+1} - A_s$, then for any $y \geq x$, $\gamma(y)[s+1]\uparrow$;

(v) If $\gamma(x)[s]\downarrow$ and $x \in K_{s+1} - K_s$, then $\gamma(x)[s] \neq \gamma(x)[v]$ for some $v > s$.

We call (i)–(v) γ-*rules.* The γ-rules are essential to the satisfaction of $\mathcal{R}$, in that:

Proposition 2.3.1. If $\Gamma^{X,A}$ is built infinitely often, and for every x we have $\{s \mid \gamma(x)[s] \neq \gamma(x)[s+1]\}$ a finite set, then $K \leq_T X \oplus A$.
Proof. The same as that for the corresponding proposition in subsection 2.3 of Cooper and Li [ta]. □

Hence, if there is a p.c. functional Γ, say, for which $\Gamma^{X,A}$ satisfies the conditions of Proposition 2.3.1, then $K \leq_T X \oplus A$, and $\mathcal{R}$ is satisfied.

However the key to the satisfaction of $\mathcal{R}$ is the building of a sequence $\widehat{\Gamma}^0$, $\widehat{\Gamma}^1$, $\widehat{\Gamma}^2$, $\cdots$, say, of p.c. functionals such that a function $\widehat{\Gamma}^{Y,A}$ $(= \widehat{\Gamma}^j(Y,A)$ say) satisfies the same use-rules as the γ-rules above, and such that if $K \not\leq_T X \oplus A$, then there is a j such that $\widehat{\Gamma}^j(Y, A)$ satisfies the conditions of Proposition 2.3.1, giving $K \leq_{\mathrm{T}} Y \oplus A$, and hence $\mathcal{R}$ again satisfied. So the procedure for satisfying an $\mathcal{R}$-requirement is to show that either there is a p.c. functional Γ say, which satisfies the conditions in Proposition 2.3.1, or there is a p.c. functional $\widehat{\Gamma}$ which satisfies its corresponding conditions.

In the remaining part of this section, we first consider the simple case of just one $\mathcal{R}$-requirement below an $\mathcal{L}$-requirement, and then go on to describe the strategies for satisfying the following requirements with the given priority ranking:

$$\mathcal{R},\ \mathcal{L}_0,\ \mathcal{L}_1,\ \mathcal{L}_2,\ \cdots .$$

2.4 Satisfying $(\mathcal{L}, \mathcal{R})$

Suppose that there are infinitely many (believable) $\mathcal{L}$-expansionary stages. Then we must try to ensure that the building of the p.c. functional Γ for the $\mathcal{R}$-strategy, does not make Ψ^A partial. For this purpose, for the p.c. functional, Γ say, built by the $\mathcal{R}$-strategy, we will define a *boundary*, b say, which is different for different p.c. functionals. Let $\gamma^{(m)}(x)$ be the value of the use function for $\Gamma^{X,A}(x)$ which is defined by the $(m+1)$-th time. To ensure that Ψ^A is total, we define $\gamma^{(m)}(k)$ only if for every $x < b + m + k$, we have that $\Psi^A(x)$ is defined. In so doing, for a fixed x, $\Psi^A(x)$ will be injured by $\gamma^{(m)}(X, A; k)$ only if $x \geq b+m+k$. Note that whenever we define b, we define it afresh. Therefore, there are only finitely many triples (b, m, k) satisfying $x \geq b + m + k$, and $\Psi^A(x)$ will be injured only finitely many times.

The assumption that Ψ^A is total renders the delayed definition for the p.c. functional Γ harmless, of course. This procedure is important in that it enables us to protect the totality of a p.c. functional from injury arising from the building of p.c. functionals by the strategies which assume the totality of the p.c. functional being protected. Suppose that α is an $\mathcal{L}$-strategy. We temporarily use $0 <_{\mathrm{L}} 1$ to denote infinite and finite actions respectively. The strategies for $\mathcal{R}$-requirements below $\alpha\hat{\ }\langle 1\rangle$ will not make $\Psi_\alpha(A)$ partial, because at the stage at which $\alpha\hat{\ }\langle 0\rangle$ is visited, all p.c. functionals built by strategies $\supseteq \alpha\hat{\ }\langle 1\rangle$ will be set to be totally undefined. So if there are infinitely many α-expansionary stages, then $\Psi_\alpha(A)$ will be total.

2.5 Satisfying $(\mathcal{R}, \mathcal{L}_0)$

Given $\mathcal{L}_0$, if there are infinitely many $\mathcal{L}_0$-expansionary stages, an $\mathcal{L}_0$-strategy will try to preserve the totality of Ψ_0^A while its priority ordering is given to an $\mathcal{R}$-strategy which is building a p.c. functional Γ. An $\mathcal{L}_0$-strategy will work with a fixed *initial value*, i say. Whenever we define the initial value i, we define it afresh. If $K \upharpoonright i$ changes, then any previous action of the $\mathcal{L}_0$-strategy is cancelled, while keeping the initial value unchanged — in which case, we say that the $\mathcal{L}_0$-strategy is *reset*. Clearly, an $\mathcal{L}_0$-strategy is reset only finitely many times.

Suppose that the initial value i is defined. Then an $\mathcal{L}_0$-strategy will define a sequence of *thresholds* $k_0 < k_1 < k_2 < \cdots$ for Γ. The thresholds of the $\mathcal{L}_0$-strategy for Γ are all elements $k > i$, the initial value of the $\mathcal{L}_0$-strategy. Then the $\mathcal{L}_0$-strategy will open a k-*cycle* (or *cycle* k) for every threshold k and proceed via k-cycles in increasing order of k.

A k-cycle of the $\mathcal{L}_0$-strategy will try to build $\gamma(k)$-cleared computations $\Psi_0^A \upharpoonright (x+1)$ and then preserve $\Psi_0^A \upharpoonright (x+1)$ forever for some fixed x. Specifically, cycle k will work with a fixed *witness* x say, and will try to preserve computations $\Psi_0^A \upharpoonright (x+1)$. We ensure that if there are sufficiently many stages at which $\Psi_0^A \upharpoonright (x+1)$ is defined with $\psi_0(x) < \gamma(k)$, then $\Psi_0^A \upharpoonright (x+1)$ is eventually defined at a stage after which it is permanently preserved. Otherwise, the k-cycle of the $\mathcal{L}_0$-strategy builds a p.c. functional $\widehat{\Gamma}^k$ with the intention that $K \leq_{\mathrm{T}} Y \oplus A$, in which case there are infinitely many stages at which we have $\gamma(k) \downarrow \leq \psi_0(x)$, giving $\gamma(k)$ unbounded during the course of the construction, so that Ψ_0^A is partial.

We now examine the cycles of the $\mathcal{L}_0$-strategy. A cycle k will try to preserve the correctness of $\Psi_0^A(l_0)$ for some witness l_0. We ensure that either a computation $\Psi_0^A(l_0)$ is cleared by $\gamma(k)$ and so is preserved, or the k-cycle acts infinitely often. In which case we have that:

- a p.c. functional $\widehat{\Gamma}$ is built,
- there are infinitely many stages at which $\gamma(k) \downarrow \leq \psi_0(l_0)$, and
- $\gamma(k)$ will be unbounded during the course of the construction.

Cycle $k > i$

1. Define a witness l_0 to be the least element $x \notin C_0$, where C_0 is the clearing set of the $\mathcal{L}_0$-strategy.

   When $\gamma(k) \downarrow$, proceed to 2.

2. Wait for a stage at which $\Psi_0^A \upharpoonright (l_0+1)$ is defined.

3. Case (a): If $\psi_0(l_0) < \gamma(k)$, then
   - enumerate l_0 into C_0, and
   - terminate the k-cycle and start the $k+1$-cycle.

   Case (b): Otherwise. If $\psi_0(l_0) \geq \gamma(k)$, proceed to 4 and then 5, as applicable.

4. (Rectify $\widehat{\Gamma}^k$) If there is an x such that $\widehat{\Gamma}^k(Y, A; x) \downarrow = 0 \neq 1 = K(x)$, then let y be the least such x, and enumerate $\widehat{\gamma}^k(y)$ into A.

5. Case (a) (Harrington Honestification): If there is an x such that $\widehat{\Gamma}^k(Y, A; x)\downarrow$, with x a threshold of some strategy with *agitator* d say, (see below) and with $\phi(d)\downarrow\geq \gamma(k)$, enumerate $\gamma(k)$ into A, and return to 2.

   Case (b): Otherwise, proceed to 6.

6. (Build $\widehat{\Gamma}^k$) Let y be the least such number for which $\widehat{\Gamma}^k(Y, A; y)\uparrow$. Then define $\widehat{\Gamma}^k(Y, A; y)\downarrow= K(y)$ with $\widehat{\gamma}^k(y)$ fresh, and return to 2.

We assume that any nontrivial action (via 4 or 6 above) in relation to $\widehat{\Gamma}^k$ is accompanied by an enumeration of $\gamma(k)$ into A, so that the $\widehat{\Gamma}^k$-strategy always works below $\gamma(k)$. This is similar in effect to Harrington capricious destruction.

We notice that honestification is not directly relevant to the outcomes of this simple module, but will be required later in laying down the conditions for a possible *switch* of $\mathcal{R}$-strategy.

We require that the clearing set C_0 has the following property: Given $s' < s$, if l is enumerated into C_0 at stage s', then:

(i) If $A_{s'} \upharpoonright (\psi_0(l)[s']+1) \neq A_s \upharpoonright (\psi_0(l)[s']+1)$, then l is extracted from C_0 at stage s immediately and automatically,

(ii) If l was enumerated into C_0 at stage s' by cycle k of the $\mathcal{L}_0$-strategy and there is a $k' < k$ such that $\gamma(k')$ is enumerated into A at stage s, then l is extracted from C_0 at stage s immediately and automatically.

We now consider:

The Possible Outcomes

For cycle k, the possible infinitary outcomes are as follows:

$$c^k <_{\mathrm{L}} \widehat{c}^k,$$

with the following intuition:

c^k: The p.c. functional $\widehat{\Gamma}^k$ is built infinitely often.

In this case, $\gamma(k) \to \infty$, and there are infinitely many stages at which $\gamma(k)\downarrow\leq \psi_0(l_0)$, so $\psi_0(l_0) \to \infty$, Ψ_0^A is partial. And since, by the module, the clearing set C_0 is presented as a Δ_2^0 set, and is in fact finite, the corresponding believable length function is bounded.

Also, by the construction of $\widehat{\Gamma}$, if $D = \Phi^{X,Y}$, then $\widehat{\Gamma}^k(Y, A) = K$. So $\mathcal{R}$ is satisfied.

$\widehat{c}^k$: $\widehat{\Gamma}^k$ is built only finitely many times, but the k-cycle acts infinitely often.

By the assumption of this case, $\widehat{\Gamma}^k$ is a finite set, so there are only finitely many stages at which step 6 occurs. Then the only possibility is that step 5(a) occurs infinitely many times. Since step 3(a) occurs only finitely many times (otherwise, the k-cycle will enumerate l_0 into C_0 eventually and keep l_0 to be in C_0 permanently), we have that:

(a) there are infinitely many stages at which $\psi_0(l_0) \downarrow \geq \gamma(k)$,

(b) there is a fixed d such that there are infinitely many stages at which $\phi(d) \downarrow \geq \gamma(k)$, and

(c) $\gamma(k)$ will be unbounded during the course of the construction.

Therefore, $\Phi^{X,Y}$ is partial, a global win for $\mathcal{R}$, and Ψ_0^A is partial with corresponding believable length function being again bounded.

For the k-cycles with $k > i$, we always assume the following:

If $i < k_1 < k_2$, and $\gamma(k_1)$ is enumerated into A at stage s, then any previous action of the k_2-cycle of the $\mathcal{L}_0$-strategy is cancelled. In which case, we say that the k_2-cycle of the $\mathcal{L}_0$-strategy is *reset*.

The possible outcomes of the $\mathcal{L}_0$-strategy are related by:

$$-1 <_{\mathrm{L}} c^{i+1} <_{\mathrm{L}} \widehat{c}^{i+1} <_{\mathrm{L}} c^{i+2} <_{\mathrm{L}} \widehat{c}^{i+2} <_{\mathrm{L}} \cdots <_{\mathrm{L}} 2$$

where outcomes c^k and $\widehat{c}^k$ are as above, and outcomes -1 and 2 have the following interpretations:

2: There are only finitely many $\mathcal{L}_0$-expansionary stages.

In this case, Ψ_0^A is partial, there are only finitely many believable $\mathcal{L}_0$-expansionary stages and the $\mathcal{L}_0$-strategy acts only finitely often.

-1: There are infinitely many $\mathcal{L}_0$-expansionary stages, but there is no k-cycle having an infinitary outcome.

In this case, we ensure that the clearing set $C_0 = \omega$, so that Ψ_0^A is total, while the p.c. functional Γ is still correct.

Suppose that α_0 is the $\mathcal{L}_0$-strategy. One of the key points is that the true outcome of α_0 is determined by a sequence of Π_2^0- and Σ_2^0-propositions. We now look at the true outcome of α_0.

If there are infinitely many stages at which some element is enumerated into C_0, then the required properties of the outcome immediately follow.

On the other hand, if there are only finitely many α_0-expansionary stages, the true outcome of α_0 is 2.

While otherwise, there is a fixed k such that the k-cycle of α_0 is reset only finitely many times, and it acts infinitely often. Now let k_0 be the least such k, which is determined by finitely many Σ_2^0-propositions and one Π_2^0-proposition. If $\widehat{\Gamma}^{k_0}$ is built infinitely often then the true outcome of α_0 is c^{k_0}, and otherwise, the true outcome of α_0 is $\widehat{c}^{k_0}$.

Clearly an $\mathcal{L}$-strategy below $\alpha_0\hat{\ }\langle 2\rangle$ is the same as α_0. An $\mathcal{L}$-strategy below $\alpha_0\hat{\ }\langle \widehat{c}^k\rangle$ for some k is just an $\mathcal{L}$-module. The strategies below $\alpha_0\hat{\ }\langle -1\rangle$ and $\alpha_0\hat{\ }\langle c^k\rangle$ for some k will be discussed in 2.6 and 2.7 respectively.

2.6 An $\mathcal{L}$-Strategy Below $\alpha_0\hat{\ }\langle -1\rangle$

Suppose that α_1 is an $\mathcal{L}_1$-strategy below $\alpha_0\hat{\ }\langle -1\rangle$. Then α_1 assumes that:

(a) Γ is built by the $\mathcal{R}$-strategy with $\Gamma^{X,A} = K$, and

(b) Ψ_0^A is preserved to be total by α_0, and $C_0 = \omega$.

α_1 will deal with any injury resulting from the construction of the p.c. functional Γ by the same method as for α_0. It will work with a fixed initial value $i(\alpha_1)$ say. Note that $i(\alpha_1) > i(\alpha_0)$, the initial value of α_0. α_1 will define all elements $k > i(\alpha_1)$ as its thresholds for Γ. Since $i(\alpha_0) < i(\alpha_1)$, if k is a threshold of α_1 for Γ, then it is also a threshold of α_0 for Γ.

For a threshold k of α_1 for Γ, we will open a k-cycle of α_1. Cycle k of α_1 is similar to the k-cycle of α_0, however it will have to deal with the totality of Ψ_0^A which is preserved by α_0. To ensure that Ψ_0^A is total, a k-cycle of α_1 will work with a *boundary*, b say. Whenever we define the boundary b, we define it afresh. Let $\widehat{\Gamma}^k_{\alpha_1}$ be the p.c. functional built by the k-cycle of α_1. At the stage at which we set $\widehat{\Gamma}^k_{\alpha_1}$ to be totally undefined, we redefine the boundary b of the k-cycle of α_1 afresh. First we ensure that, the collection of the hatted p.c. functionals $\widehat{\Gamma}^k_{\alpha_1}$ for all k during the course of the construction will not make Ψ_0^A partial. This is achieved by a *delayed* definition of $\widehat{\Gamma}^k_{\alpha_1}$ using the slow down method from 2.4.

Specifically, steps 1–5 of the k-cycle of α_1 are similar to the corresponding steps of the k-cycle of α_0, but step 6 is replaced by step 6$'$ below.

Slow Down:

6$'$ (Build $\widehat{\Gamma}^k_{\alpha_1}$) – Let y be the least number for which $\widehat{\Gamma}^k_{\alpha_1}(Y, A; y)\uparrow$, let m be number of times that $\widehat{\Gamma}^k_{\alpha}(Y, A; y)$ has been defined, and let $l = y+m+b$, where b is the boundary of the cycle.

– Wait for a stage at which $l \in C_{\alpha_0}$, the clearing set of α_0, and then:

- define $\widehat{\Gamma}^k_{\alpha_1}(Y, A; y)\downarrow = K(y)$ with $\widehat{\gamma}^k_{\alpha_1}(y)$ fresh, and
- return to 2.

This ensures that the hatted p.c. functionals built by various cycles of α_1 will not make Ψ_0^A partial. However, a k-cycle of α_1 may enumerate $\gamma(k)$ into A, which still threatens the totality of Ψ_0^A.

To solve this problem, at the stage at which the k-cycle of α_1 enumerates $\gamma(k)$ into A, we execute the following:

Wake Up Action:

A. For any $k' > k$, the k'-cycle of α_0 is reset — in particular, if l was enumerated into C_{α_0} by the k'-cycle of α_0, then l is extracted from C_{α_0}.

B. Then:
- set $\widehat{\Gamma}^k_{\alpha_0}$ to be totally undefined,
- redefine the boundary of the k-cycle of α_0 afresh.

In this case, we say that the k-cycle of α_0 is *woken up* (by the k-cycle of α_1), or the k-cycle of α_1 *wakes up* the k-cycle of α_0.

The role of the awakened k-cycle of α_0 is: If a strategy below $\alpha_0\hat{\ }\langle -1\rangle$ enumerates some $\gamma(k)$ for some $k < k'$ into A, then the previous action of the k'-cycle of α_0 may have been injured. So we reset the k'-cycle of α_0, and allow the k-cycle of α_0 to begin a new cycle. This ensures that if there is a fixed k such that some strategy below $\alpha_0\hat{\ }\langle -1\rangle$ enumerates $\gamma(k)$ into A infinitely often, then for the least k, k_0 say, α_0 will build its computations $\Psi_0^A \upharpoonright (x+1)$ which are cleared by $\gamma(k_0)$ for larger and larger x, as each such x assumes the role of l_0 in the description of the cycle k. That is to say, α_0 will guess the least k such that $\gamma(k)$ will be unbounded, and allow the k-cycle to clear computations $\Psi_0^A \upharpoonright (x+1)$ for almost every x. Therefore, even if there is a k such that the k-cycle of α_1 acts infinitely often, Ψ_0^A is still kept total, provided that -1 is the true outcome of α_0. On the other hand, if $\Gamma^{X,A}$ is total, then for every threshold k of α_0 for Γ, the k-cycle will preserve a computation $\Psi_0^A(x)$ for some x eventually and permanently. In addition, we note that if the k-cycle of α_1 enumerates $\gamma(k)$ into A, then any x in C_{α_0} which was enumerated by a k'-cycle with $k' > k$ has already been injured, so is extracted from C_{α_0} immediately. The awakening procedure ensures that if α_0 has true outcome -1, then every x will be enumerated by some permanent k-cycle of α_0, i.e. by a cycle which will never be reset.

We say that a k-cycle of α_0 is *active*, if there is no element in C_{α_0} which has been enumerated by the k-cycle of α_0 since the cycle was either reset or woken up for the last time. Clearly α_0 will always allow the least active cycle to act. This ensures that if α_0 has outcome -1, then for every l, there is a k and a stage s such that the k-cycle of α_0 will never be reset after stage s and such that the k-cycle of α_0 enumerates l into C_{α_0}, so that each $\Psi_0^A \upharpoonright (l+1)$ becomes defined and preserved forever.

Note that a k-cycle of α_0 will always define its witness to be the least element which is not in C_{α_0}, so whatever α_1 enumerates, if C_{α_0} is built infinitely often, then $C_{\alpha_0} = \omega$, giving $\Psi^A_{\alpha_0}$ total.

On the other hand, given that Ψ_0^A is preserved to be a total function, the slow down definition of the hatted p.c. functionals in step $6'$ of the cycles for α_1 is not a real problem.

One of the main points is to ensure that for a fixed k, there are only finitely many strategies which can define k as their threshold for Γ. (Otherwise, it is possible that for a fixed k, $\gamma(k)$ is destroyed infinitely many times, but no strategy is responsible for building a hatted p.c. functional $\widehat{\Gamma}^k$.) This is of course ensured by the definition of the initial values and thresholds of the strategies.

2.7 An $\mathcal{L}$-Strategy Below $\alpha_0\hat{}\langle c^k\rangle$ (for some k)

Suppose that α_1 is an $\mathcal{L}_1$-strategy such that $\alpha_0\hat{}\langle c^k\rangle \subseteq \alpha_1$. α_1 will try to preserve the totality of Ψ_1^A. It assumes that:

- an $\mathcal{L}_0$-strategy α_0 is building and rectifying a p.c. functional $\widehat{\Gamma}^k$,
- $\gamma(k) \to \infty$, where k is the least threshold of α_0 for Γ such that the k-cycle of α_0 acts infinitely often, and
- there are infinitely many α_0-expansionary stages.

Given x, we say that $\Psi_1^A(x)\downarrow= y$ is α_1-*believable* if $\psi_1(x) < \gamma(k)$. Then α_1 will use only α_1-believable computations. Note that α_1 assumes that k is the least x such that $\gamma(x)$ is unbounded during the course of the construction. Therefore if α_1 uses only α_1-believable computations, then the building of the p.c. functional Γ will not injure α_1.

However, α_1 will have to deal with the building of the p.c. functional $\widehat{\Gamma}^k$. First α_1 will work with an initial value, $i(\alpha_1)$ say. Then it will define all numbers $m > i(\alpha_1)$ to be its thresholds for $\widehat{\Gamma}^k$. For the thresholds $m > i(\alpha_1)$, we will open m-cycles, and allow them to act in increasing order of m.

As before, we will define a clearing set C_{α_1} to record the numbers x for which $\Psi_1^A \restriction (x+1)$ are defined and preserved.

An m-cycle of α_1 will work with a fixed *agitator* $d(m)$ say, and proceed as follows:

1. Define a witness x_1 to be the least x which is not in C_{α_1}.

2. Define an agitator $d(m)$ say, afresh.

   Suppose that $\widehat{\Gamma}^k(Y, A; m)\downarrow$.

3. Wait for a stage at which $\Psi_1^A \restriction (x_1+1)$ is defined via α_1-believable computations.

4. If $\psi_1(x_1) < \widehat{\gamma}^k(m)$, then go to step 8.

   Suppose that $\psi_1(x_1) \geq \widehat{\gamma}^k(m)$.

5. (Honestify $\widehat{\Gamma}^k$) If $\widehat{\gamma}^k(m) \leq \phi(d(m))$, enumerate $\widehat{\gamma}^k(m)$ into A, and return to 3.

6. (Create a Hard Link $(\mathcal{R}, \mathcal{L}_1)$) Otherwise:
   - enumerate $d(m)$ into D,
   - create a *hard link* $(\mathcal{R}, \mathcal{L}_1)$.

   At the next $\mathcal{R}$-expansionary stage:

7. (Travel the Hard Link $(\mathcal{R}, \mathcal{L}_1)$) Let s^- be the stage at which the current hard link $(\mathcal{R}, \mathcal{L}_1)$ was created. There are two cases:

   Case 7a. (Successful Travel) If

   $$Y_{s^-} \upharpoonright (\phi(d(m))[s^-] + 1) \neq Y \upharpoonright (\phi(d(m))[s^-] + 1),$$

   then
   - set $\widehat{\gamma}^k(m)$ to be undefined,
   - go to step 8.

   Case 7b. (Unsuccessful Travel) Otherwise:
   - set $\Gamma^{X,A}(k)$ to be undefined,
   - let the $\mathcal{L}_0$-strategy α_0 act.

8. Then:
   - enumerate x_1 into C_{α_1}, and
   - say that x_1 *is enumerated into* C_{α_1} *by the* m*-cycle of* α_1.

We say that an m-cycle of α_1 is *passive at stage* s if the m-cycle of α_1 enumerated some x into C_{α_1} at a stage s^- say, such that $x \in C_{\alpha_1}[v]$ at each stage $v \geq s^-$ since the m-cycle of α_1 was either reset or woken up for the last time. Otherwise the m-cycle of α_1 is *active at stage* s.

α_1 will always allow the least active m-cycle of α_1 to act.

The Possible Outcomes

An m-cycle of α_1 has only one possible infinitary outcome, denote by $\widehat{c}^m$, whereby step 5 of the m-cycle of α_1 acts at infinitely many stages. In this case, we have that both $d(m)$ and x_1 are fixed, but — and this is the key to the success of the $\mathcal{L}_1$-strategy α_1 — there are infinitely many stages at which $\widehat{\gamma}^k(m) \downarrow \leq \psi_1(x_1)$, and at which $\widehat{\gamma}^k(m) \downarrow \leq \phi(d(m))$, and $\widehat{\gamma}^k(m)$ is unbounded during the course of the construction.

So the possible outcomes of the $\mathcal{L}_1$-strategy α_1 are as follows:

$$-1 <_{\mathrm{L}} \widehat{c}^{(i(\alpha_1)+1)} <_{\mathrm{L}} \widehat{c}^{(i(\alpha_1)+2)} <_{\mathrm{L}} \cdots <_{\mathrm{L}} \widehat{c}^{m} <_{\mathrm{L}} \cdots <_{\mathrm{L}} 2,$$

where $\widehat{c}^m$ is the outcome of the m-cycle of α_1 as above, and -1 and 2 have the same meaning as before (see 2.6).

An $\mathcal{L}$-strategy below $\alpha_1\hat{\ }\langle 2\rangle$ is the same as α_1, and an $\mathcal{L}$-strategy below $\alpha_1\hat{\ }\langle \widehat{c}^m\rangle$ for some m is just an $\mathcal{L}$-module. But the $\mathcal{L}$-strategy below $\alpha_1\hat{\ }\langle -1\rangle$ has some new features.

2.8 An $\mathcal{L}$-Strategy Below $\alpha_1\hat{\ }\langle -1\rangle$

Suppose that α_2 is an $\mathcal{L}_2$-strategy below $\alpha_1\hat{\ }\langle -1\rangle$. α_2 is virtually the same as α_1, but it will have to deal with the requested totality of Ψ_1^A, being preserved by α_1. Note that the only possibility, other than via routine rectification, of a trace being enumerated into A during an m-cycle (say) of α_2 is via honestification in step 5.

As before, α_2 will first choose its thresholds for $\widehat{\Gamma}^k$. That is to say, we define an initial value, $i(\alpha_2)$ say, for α_2. Then α_2 will define its thresholds for $\widehat{\Gamma}^k$ as the set of all elements $k > i(\alpha_2)$. $i(\alpha_2)$ will be choosen to be $> i(\alpha_1)$, so a threshold m of α_2 for $\widehat{\Gamma}^k$ is also a threshold of α_1 for $\widehat{\Gamma}^k$. Then an m-cycle of α_2 for $\widehat{\Gamma}^k$ may wake up the same m-cycle of α_1 for $\widehat{\Gamma}^k$. We ensure:

Wake Up Action:

- If the m-cycle of α_2 acts at stage s, then
  - — for any $m' > m$, any previous action of the m'-cycle of α_1 is cancelled, in which case we say that the m'-cycle of α_1 is reset,
  - — we say that the m-cycle of α_1 is *woken up*, or *awakened*.

And obviously, the possible outcomes of α_2 are the same as those of α_1.

Further analysis of outcomes:

We now look in more detail at how the outcomes involve the satisfaction of the requirements. As mentioned above, all $\mathcal{L}$-strategies below α_1 are similar to α_1. This means we mainly need to analyse the satisfaction of the requirements whose strategies are described in 2.5– 2.7.

For α_1:

• First, we note that if $\alpha_1\hat{\ }\langle 2\rangle$ is the true outcome, then Ψ_1^A is partial, with trivially bounded α_1-believable length function.

• If $\widehat{c}^m$ for some m is the true outcome of α_1, let x_1 be the permanent witness of the m-cycle of α_1. Then $\psi_1(x_1)$ will be unbounded during the course of the construction, giving partial Ψ_1^A again, and bounded α_1-believable length function.

And according to the strategy α_1, if $\widehat{c}^m$ is the true outcome of α_1, then the m-cycle of α_1 is reset only finitely many times and it acts infinitely often. Also, there is no strategy below $\alpha_1\hat{\ }\langle\widehat{c}^m\rangle$ persisting in destroying the p.c. functional $\widehat{\Gamma}^k$. And m is the least n such that the n-cycle of α_1 is active permanently. Therefore if $\widehat{c}^m$ is the true outcome of α_1, one has that m is the least n such that $\widehat{\gamma}^k(n)$ is unbounded, and $\Phi^{X,Y}$ is partial.

- Suppose now that α_1 has true outcome -1. We prove the totality of Ψ_1^A by cases:

Case 1. There is an $\mathcal{L}$-strategy α such that $\alpha_1\hat{\ }\langle -1\rangle \subseteq \alpha$ and such that α has true outcome $\widehat{c}^m$ for some m.

We have to prove the following:

(i) the m-cycle of α_1 is reset only finitely many times,

(ii) the m-cycle of α_1 is woken up infinitely often, and

(iii) almost every x is eventually enumerated into C_{α_1} by the m-cycle of α_1.

Clearly (ii) is straightforward by the description of the m-cycle of α.

For (i), first notice that by the choice of α and m, the m-cycle of α is active at almost every stage s, and $\alpha\hat{\ }\langle -1\rangle$ is visited only finitely many times. Therefore there is no $m' < m$ such that there are infinitely many stages at which $\widehat{\gamma}^k(m')$ is enumerated into A by some strategy α' with $\alpha' \not\leq \alpha$. This is because for any γ to the right of $\alpha\hat{\ }\langle\widehat{c}^m\rangle$, γ is initialised infinitely many times, so the initial value of γ will be unbounded during the course of the construction. While there is no strategy below $\alpha\hat{\ }\langle\widehat{c}^m\rangle$ which persists in destroying the p.c. functional $\widehat{\Gamma}^k$. And the strategies below $\alpha\hat{\ }\langle -1\rangle$ act only finitely many times, and by the choice of α and m, for any $m' < m$, the m'-cycle of α acts only finitely many times.

Clearly, both α_1 and α assume that $\widehat{\Gamma}^k$ is correct. Therefore, for any β, if $\alpha_1 \subset \beta \subset \alpha$, either $\beta\hat{\ }\langle -1\rangle \subseteq \alpha$ or $\beta\hat{\ }\langle 2\rangle \subseteq \alpha$. If $\beta\hat{\ }\langle 2\rangle \subseteq \alpha$, then β acts only finitely many times.

Suppose, to the contrary, that there is a strategy β such that $\alpha_1 \subseteq \beta\hat{\ }\langle -1\rangle \subseteq \alpha$ and that there are infinitely many stages at which β enumerates $\widehat{\gamma}^k(m')$ for some $m' < m$ into A. Let γ be the longest such β and let m_0 be the least m' such that γ enumerates $\widehat{\gamma}^k(m')$ infinitely often. By the choice of γ and m_0, the m_0-cycle of γ is woken up only finitely many times. Therefore $\widehat{c}^{m_0}$ is the true outcome of γ, contradicting the choice of γ and α. (i) immediately follows.

Finally, by the definition of C_{α_1} and by the strategy α_1, (iii) follows. Note that if an x is enumerated into C_{α_1} by the m-cycle of α_1 at stage s, then

$\widehat{\Gamma}^k(Y, A; m')$ is undefined for all $m' \geq m$ at stage s. Therefore $\Psi_1^A[s] \upharpoonright (x+1)$ are defined and preserved (where of course one can assume that this occurs after the last stage at which the m-cycle of α_1 is reset). So Ψ_1^A is total and $C_{\alpha_1} = \omega$.

Case 2. Otherwise.

In this case, every m-cycle of α_1 is reset or awakened only finitely many times. For an $m > i(\alpha_1)$, let s_m be the greatest stage at which the m-cycle of α_1 is either reset or awakened. By the assumption that α_1 has true outcome -1, the m-cycle of α_1 will enumerate an element, x say, into C_{α_1} at a stage $v_m > s_m$ say. Note that at stage v_m, for any $m' \geq m$, $\widehat{\Gamma}^k(m')$ is undefined, while by the choice of s_m, $\Psi_1^A[v_m] \upharpoonright (x+1)$ is defined and preserved forever.

Therefore Ψ_1^A is total and clearly, by the definition of C_{α_1}, we have that $C_{\alpha_1} = \omega$.

For α_0:

- It is easy to see that a similar argument proves that if α_0 has true outcome -1, then $\Psi_{\alpha_0}^A$ is total.
- And by the $\mathcal{L}_0$-strategy α_0, it is easy to see that if α_0 has true outcome either 2 or $\widehat{c}^k$ or c^k for some k, then $\Psi_{\alpha_0}^A$ is partial, with bounded correspondingly believable length function. So again, $\mathcal{L}_0$ is satisfied.

Finally, by the definition of the initial values for the $\mathcal{L}$-strategies, we have:

(1) For any fixed k, there are only finitely many strategies which can define k as their threshold for Γ.

(2) For a p.c. functional $\widehat{\Gamma}^k$, for any given m, there are only finitely many strategies which can define m as their threshold for $\widehat{\Gamma}^k$.

This ensures that $\mathcal{R}$ is satisfied by one of the following cases:

Case A. There is an $\mathcal{L}$-strategy α and a number m such that α has true outcome $\widehat{c}^m$.

In this case, there is a fixed y such that $\phi(y)$ will be unbounded during the course of the construction, giving $D \neq \Phi^{X,Y}$.

Case B. Case A does not hold, and there is an $\mathcal{L}$-strategy α and a k such that α has true outcome c^k.

Then we have that $\widehat{\Gamma}^k$ is built infinitely often, giving $\widehat{\Gamma}^k(Y, A)$ total with $\widehat{\Gamma}^k(Y, A) = K$.

Case C. Otherwise.

Noting that we have already assumed the nontrival case in which there are infinitely many $\mathcal{R}$-expansionary stages, this means that $\Gamma^{X,A}$ is total and $\Gamma^{X,A} = K$.

3 Satisfying All $\mathcal{L}$-Requirements With Two $\mathcal{R}$-Requirements

In this section we describe the strategies needed to satisfy the following requirements with the given priority ranking:

$$\mathcal{R}_0,\ \mathcal{L}_0,\ \mathcal{R}_1,\ \mathcal{L}_1,\ \mathcal{L}_2,\ \mathcal{L}_3, \cdots .$$

3.1 Satisfying $(\mathcal{R}_0, \mathcal{R}_1, \mathcal{L}_1)$

An $\mathcal{L}_1$-strategy assumes that an $\mathcal{R}_j$-strategy is building a p.c. functional Γ_j, for $j = 0, 1$. An $\mathcal{L}_1$-strategy will have to deal with the injury arising from the construction of the p.c. functionals Γ_0 and Γ_1.

Suppose that α_1 is an $\mathcal{L}_1$-strategy. α_1 will work with an initial value i_1 say. It defines its thresholds for Γ_j to be the set of all $k > i_1$ for both $j = 0$ and 1.

α_1 will build a clearing set C_{α_1} say. For each threshold k of α_1 for Γ_j, we may need to open a k-cycle of α_1 for Γ_j for both $j = 0$ and 1. In order to achieve this, α_1 is allowed to open cycles of the form $\vec{c} = (k_0, k_1)$ which consist of a cycle k_0 of α_1 for Γ_0, and a cycle k_1 of α_1 for Γ_1 respectively, where k_0, k_1 are thresholds of α_1 for Γ_0, Γ_1 respectively.

Note that a k-cycle of α_1 for Γ_j may be reset or woken up for both $j = 0$ and 1. Before giving the details of the $\vec{c}$-cycles we need to define the following:

(i) We say that the threshold k of α_1 for Γ_1 is *active at stage s* if it has been either reset or woken up since it enumerated an element into C_{α_1} for the last time.

(ii) Given a threshold k of α_1 for Γ_1 and a threshold k_0 of α_1 for Γ_0, we say that the threshold k_0 of α_1 for Γ_0 is *k-active at stage s*, if the $\vec{c} = (k_0, k)$-cycle of α_1 has been either reset or woken up since it enumerated an element into C_{α_1} for the last time.

Then α_1 will open cycles according to the following procedure:

(i) Define k_1 to be the least active threshold of α_1 for Γ_1.

(ii) Define k_0 to be the least k such that threshold k of α_1 for Γ_0 is k_1-active.

(iii) Open the $\vec{c} = (k_0, k_1)$-cycle.

A $\vec{c} = (k_0, k_1)$-cycle of α_1 will consist of two (sub)cycles $[\vec{c}]_0^{k_0}$ and $[\vec{c}]_1^{k_1}$ which deal with the p.c. functionals Γ_0 and Γ_1 respectively.

Cycle $\vec{c} = (k_0, k_1)$ will proceed as follows:

Part I: Cycle $[\vec{c}]_1^{k_1}$

1. Choose a witness x_1 to be the least x which is not in C_{α_1}.

   Suppose that $\gamma_0(k_0) < \gamma_1(k_1) \downarrow$.

2. Wait for a stage at which $\Psi_1^A(x_1) \downarrow$.

3. If $\psi_1(x_1) < \gamma_0(k_0)$, then go to step 17.

4. If $\psi_1(x_1) < \gamma_1(k_1)$, then go to step 10.

5. (Rectify $\widehat{\Gamma}_1^{k_1}$) If there is an x such that $\widehat{\Gamma}_1^{k_1}(Y_1, A; x) \downarrow \neq K(x)$, then let y be the least x, and enumerate $\widehat{\gamma}_1^{k_1}(y)$ into A.

6. (Harrington Honestification) If there is an x such that $\widehat{\Gamma}_1^{k_1}(Y_1, A; x) \downarrow$, that x is a threshold of some threshold for $\widehat{\Gamma}_1^{k_1}$ with an agitator d_1 and such that $\phi_1(d) \downarrow \geq \gamma_1(k_1)$, then enumerate $\gamma_1(k_1)$ into A.

Slow Down: Steps 7–9

7. Let y_1 be the least x such that $\widehat{\Gamma}_1^{k_1}(Y_1, A; x) \uparrow$, let b_1 be the boundary of the k_1-cycle of the $\mathcal{L}_1$-strategy for Γ_1, and let m_1 be the number of times for which $\widehat{\Gamma}_1^{k_1}(y)$ has been defined since b_1 was defined, and let $l_1 = m_1 + b_1 + y_1$.

8. (Build $\widehat{\Gamma}_1^{k_1}$) Wait for a stage at which $\Psi_1^A \upharpoonright (l_1 + 1)$ are all defined, then:

   - define $\widehat{\Gamma}_1^{k_1}(Y_1, A; y_1) \downarrow = K(y_1)$ with $\widehat{\gamma}_1^{k_1}(y_1)$ fresh.

9. Otherwise, then do nothing.

10. If the $\vec{c} = (k_0, k_1)$-cycle of α_1 is $\gamma_1(k_1)$-cleared, then go to step 11.

Part II: Cycle $[\vec{c}]_0^{k_0}$

11. If $\psi_1(x_1) < \gamma_0(k_0)$, then we say that the $\vec{c} = (k_0, k_1)$-cycle of α_1 is $\gamma_0(k_0)$-cleared, and go to step 17.

    Suppose that $\psi_1(x_1) \geq \gamma_0(k_0)$.

12. (Rectify $\widehat{\Gamma}_0^{k_0}$) If there is an x such that $\widehat{\Gamma}_0^{k_0}(Y_0, A; x) \downarrow \neq K(x)$, then let y be the least x, and enumerate $\widehat{\gamma}_0^{k_0}(y)$ into A.

13. (Harrington Honestification) If there is an x such that $\widehat{\Gamma}_0^{k_0}(Y_0, A; x)\downarrow$, that x is a threshold of some strategy for $\widehat{\Gamma}_0^{k_0}$ with agitator d_0, say, and such that $\phi_0(d_0)\downarrow\geq \gamma_0(k_0)$, then enumerate both $\gamma_0(k_0)$ and $\gamma_1(k_0)$ into A.

Slow Down: Steps 14–16

14. Let y_0 be the least x such that $\widehat{\Gamma}_0^{k_0}(Y_0, A; x)\uparrow$, let b_0 be the boundary of the k_0-cycle of the $\mathcal{L}_1$-strategy for Γ_0, let m_0 be the number of times for which $\widehat{\Gamma}_0^{k_0}(y_0)$ has been defined since b_0 was defined for the last time, and let $l_0 = m_0 + b_0 + y_0$.
15. (Build $\widehat{\Gamma}_0^{k_0}$) Wait for a stage at which $\Psi_1^A \upharpoonright (l_0+1)$ are all defined, then define $\widehat{\Gamma}_0^{k_0}(Y_0, A; y_0)\downarrow= K(y_0)$ with $\widehat{\gamma}_0^{k_0}(y_0)$ fresh.
16. Otherwise, then do nothing.
17. If the $\vec{c} = (k_0, k_1)$-cycle of α_1 is $\gamma_0(k_0)$-cleared, then:
    - if $\gamma_0(k_0)$ is defined, then enumerate it into A,
    - enumerate x_1 into C_{α_1},
    - we say that x_1 is enumerated by the $\vec{c} = (k_0, k_1)$-cycle.

Before describing the possible outcomes of the $\mathcal{L}_1$-strategy α_1, we introduce some rules relating to the building of the clearing set C_{α_1}. Intuitively speaking, C_{α_1} is the set of all elements x for which $\Psi_1^A \upharpoonright (x+1)$ has been defined and preserved. Therefore we require that the set C_{α_1} has the following properties.

For $s^- < s$, if x is enumerated into C_{α_1} at stage s^-, then:

(i) $\Psi_1^A \upharpoonright (x+1)$ are defined during stage s^-.

(ii) There is no element $y \leq \psi_1(x)[s^-]$ which is enumerated into A during stage s^-.

(iii) If there is an element $y \leq \psi_1(x)[s^-]$ which is enumerated into A at stage s, then x is extracted from C_{α_1} at stage s automatically.

Suppose that x is enumerated into C_{α_1} at stage s^- by the $\vec{c} = (k_0, k_1)$-cycle.

(iv) If there is a $k < k_1$ such that $\gamma_1(k)$ is enumerated into A at stage s, then we say that the $\vec{c} = (k_0, k_1)$-cycle of α_1 is reset at stage s automatically.

(v) If there is a $k < k_0$ such that $\gamma_0(k)$ is enumerated into A at stage s, we say that the $\vec{c} = (k_0, k_1)$-cycle of α_1 is reset at stage s automatically.

(vi) If the $\vec{c} = (k_0, k_1)$-cycle of α_1 is reset at stage s, then x is extracted from C_{α_1} immediately and automatically.

We say that (i)–(vi) above are the *clearing rules* of α_1. By the clearing rules, if $x \in C_{\alpha_1}$ then $\Psi_1^A \upharpoonright (x+1)$ is defined eventually and permanently.

The key to the satisfaction of $\mathcal{L}_1$ is to ensure the following properties:

(a) C_{α_1} is a Δ_2^0 set,

(b) $C_{\alpha_1} = \omega$ if and only if there are infinitely many stages at which some element x is enumerated into C_{α_1}.

(b) follows from (a) and the strategy. Therefore the point is to ensure (a), which follows from the clearing rules and from the opening of cycles.

By the clearing rules, if $C_{\alpha_1} = \omega$, then $\Psi_{\alpha_1}^A$ is total.

We now look at the outcomes of α_1.

The Possible Outcomes

We use $(0,k)$ to denote the k-cycle of α_1 for Γ_0, and $(1,k)$ to denote the k-cycle of α_1 for Γ_1.

The possible outcomes of $(0,k)$ are:

$$(c,0) <_{\mathrm{L}} (\widehat{c},0)$$

where $(c,0)$ means that a p.c. functional $\widehat{\Gamma}_0^k$ is built infinitely often, $(\widehat{c},0)$ means that $\widehat{\Gamma}_0^k$ is built only finitely many times but the k-cycle of α_1 for Γ_0 acts infinitely many times.

We define the possible outcomes of the k-cycle of α_1 for Γ_1, denote by $(1,k)$, as follows:

$$(0,k) <_{\mathrm{L}} (0,k+1) <_{\mathrm{L}} (0,k+2) <_{\mathrm{L}} \cdots <_{\mathrm{L}} (c,1) <_{\mathrm{L}} (\widehat{c},1).$$

The intuition for these outcomes is:

$(0,k_0)$: the (k_0,k)-cycle of α_1 acts infinitely often.

In this case, note that $(0,k_0)$ has two outcomes $(c,0) <_{\mathrm{L}} (\widehat{c},0)$.

$(c,1)$: $\widehat{\Gamma}_1^k$ is built infinitely many times.

$(\widehat{c},1)$: $\widehat{\Gamma}_1^k$ is built only finitely many times, but the k-cycle of α_1 for Γ_1 acts infinitely often.

We now list the possible outcomes of α_1:

$$-1 <_{\mathrm{L}} (1,1) <_{\mathrm{L}} (1,2) <_{\mathrm{L}} (1,3) <_{\mathrm{L}} \cdots <_{\mathrm{L}} (1,k) <_{\mathrm{L}} \cdots <_{\mathrm{L}} 2,$$

where $(1,k)$ denotes the k-cycle of α_1 for Γ_1, so that the immediate successors of $\alpha_1\hat{\ }\langle(1,k)\rangle$ are the possible outcomes of the k-cycle of α_1 for Γ_1 as described above; 2 means that there are only finitely many α_1-expansionary stages; and

-1 means that there are infinitely many stages at which some element is enumerated into C_{α_1}, in which case $C_{\alpha_1} = \omega$, so that $\Psi^A_{\alpha_1}$ is total.

We now analyse the possible outcomes of α_1. We divide into 6 cases:

Case 1. α_1 has true outcome 2.

In this case, there are only finitely many α_1-expansionary stages. We have:

- Ψ^A_1 is partial,
- α_1 acts only finitely often,
- both Γ_0 and Γ_1 are still correct, and
- so an $\mathcal{L}$-strategy below $\alpha_1\hat{\ }\langle 2\rangle$ will be the same as α_1.

Case 2. α_1 has true outcome $(1,k)\hat{\ }\langle(c,1)\rangle$ for some k.

In this case, the k-cycle of α_1 for Γ_1 is either reset or awakened only finitely many times, and a p.c. functional $\widehat{\Gamma}^k_1$ is built infinitely often. And there is a fixed x such that there are infinitely many stages at which $\gamma_1(k)\downarrow\leq \psi_1(x)$, so that Ψ^A_1 is partial.

An $\mathcal{L}$-strategy below α_1 in this case will be described in 3.2.

Case 3. α_1 has true outcome $(1,k)\hat{\ }\langle(\widehat{c},1)\rangle$ for some k.

In this case, we have that:

- the k-cycle of α_1 for Γ_1 is either reset or woken up only finitely many times, but it acts infinitely often,
- there is a fixed d_1, such that $\phi_1(d_1)$ will be unbounded over the course of the construction,
- there is a fixed x such that $\psi_1(x)$ will be unbounded over the course of the construction.

Therefore $\mathcal{R}_1$ is satisfied, and Ψ^A_1 is partial. The $\mathcal{L}$-strategies below α_1 in this case are the same as that in section 2.

Case 4. α_1 has true outcome $(1,k_1)\hat{\ }\langle(0,k_0)\rangle\hat{\ }\langle(c,0)\rangle$ for some k_0,k_1.

In this case, the (k_0,k_1)-cycle is either reset or awakened only finitely often, but the (k_0,k_1)-cycle builds a p.c. functional $\widehat{\Gamma}^{k_0}_0$ infinitely many times. So we have:

- both $\gamma_0(k_0)$ and $\gamma_1(k_1)$ are unbounded over the course of the construction,
- there is a fixed x such that there are infinitely many stages at which $\gamma_0(k_0)\downarrow\leq\psi_1(x)$ holds.

It follows that Ψ^A_1 is partial.

Note that below this outcome, we will need to introduce a backup strategy for $\mathcal{R}_1$. An $\mathcal{L}$-strategy below this back up strategy for $\mathcal{R}_1$ will be discussed in 3.3.

Case 5. α_1 has true outcome $(1, k_1)\hat{}\langle(0, k_0)\rangle\hat{}\langle(\widehat{c}, 0)\rangle$ for some k_0, k_1.

This means that $\widehat{\Gamma}_0^{k_0}$ is built only finitely many times, but the (k_0, k_1)-cycle acts infinitely often. So we have that:

- there is a fixed d_0 such that $\phi_0(d_0)$ will be unbounded over the course of the construction,
- there is a fixed x such that $\psi_1(x)$ will be unbounded over the course of the construction.

So we have a global win for $\mathcal{R}_0$, and Ψ_1^A is partial. Below this outcome, we must introduce a backup strategy for $\mathcal{R}_1$. The strategies below the backup strategy will be the same as those described in section 2.

Case 6. α_1 has true outcome -1.

In this case, C_{α_1} is built infinitely often. By the definition of C_{α_1} and by the strategy, $C_{\alpha_1} = \omega$, so we have that Ψ_1^A is total.

An $\mathcal{L}$-strategy below $\alpha_1\hat{}\langle -1\rangle$ will be the same as α_1. However it will have to deal with the requested totality of Ψ_1^A by slowing down the definition of p.c. functionals and by waking up cycles of α_1.

The role of the awakening actions is as follows. If a strategy below $\alpha_1\hat{}\langle -1\rangle$ makes a function $\Gamma_j^{X_j,A}$, say, partial, then α_1 will find the least k such that $\Gamma_j^{X_j,A}(k)$ diverges, so that α_1 then builds computations which are cleared by $\gamma_j(k)$. This ensures that even if $\Gamma_j^{X_j,A}$ is partial, the construction of the p.c. functionals Γ_j can never make Ψ_1^A partial.

One of the main things to notice is that although the possible outcomes of α_1 look like an $\omega + \omega + \cdots$ ordering type, the true outcome of such a strategy can be found using an $\emptyset''$-oracle. That is to say, the true path TP is computable from $\emptyset''$, and we can read out the totality of Ψ^A along the true path TP.

As an example, we decide the true outcome of α_1 as follows:

Case A. If there are infinitely many stages at which some elements are enumerated into C_{α_1}, then -1 is the true outcome of α_1.

Case B. Otherwise.

In this case, $\alpha_1\hat{}\langle -1\rangle$ is to the left of the true path TP, so there are only finitely many stages at which some k-cycle of α_1 for Γ_1, or some k-cycle of α_1 for Γ_0, is awakened.

Since case A does not apply, there is a fixed k such that k is a threshold of α_1 for Γ_1 and the threshold k of α_1 for Γ_1 is active permanently. Let k_1 be the least such k.

According to the procedure for the opening of cycles of α_1, there is a stage s_0 say, such that the k_1-cycle of α_1 for Γ_1 acts at every α_1-expansionary stage $> s_0$.

By the choice of k_1 and s_0, there is no k_0 such that k_0 is a threshold of α_1 for Γ_0 and the (k_0, k_1)-cycle enumerates an element into C_{α_1} after stage s_0. Let s_1 be the smallest stage $> s_0$ after which $\alpha_1\hat{}\langle(1, k_1)\rangle$ is never initialised, reset, or woken up. Let k_0 be the least element $> s_1$. We now look at the following subcases:

Subcase B1. There are infinitely many stages at which some node to the left of $\alpha_1\hat{}\langle(1, k_1)\hat{}\langle(c, 1)\rangle$ is visited.

In this case, the (k_0, k_1)-cycle is permanent and acts infinitely often. Therefore for $\beta = \alpha_1\hat{}\langle(1, k_1)\rangle\hat{}\langle(0, k_0)\rangle$, we have:

- if $\widehat{\Gamma}_0^{k_0}$ is built infinitely often, then $\beta\hat{}\langle(c, 0)\rangle$ is on the true path, and
- otherwise, we have $\beta\hat{}\langle(\widehat{c}, 0)\rangle$ on the true path.

Subcase B2. Subcase B1 does not apply, and the k_1-cycle of α_1 for Γ_1 acts infinitely often.

We now have that:

- if $\widehat{\Gamma}_1^{k_1}$ is built infinitely often, then $\alpha_1\hat{}\langle(1, k_1)\rangle\hat{}\langle(c, 1)\rangle$ is on the true path, and
- otherwise, we have $\alpha_1\hat{}\langle(1, k_1)\rangle\hat{}\langle(\widehat{c}, 1)\rangle$ on the true path.

Subcase B3. Otherwise. This means that $\alpha_1\hat{}\langle 2\rangle$ is on the true path.

By the above analysis, a sequence of Π^0_2- and Σ^0_2-propositions determines the true outcome of α_1.

3.2 An $\mathcal{L}$-Strategy Below $\alpha_1\hat{}\langle(1, k_1)\rangle\hat{}\langle(c, 1)\rangle$ (for some k_1)

Suppose that α_2 is an $\mathcal{L}_2$-strategy $\supseteq \alpha_1\hat{}\langle(1, k_1)\rangle\hat{}\langle(c, 1)\rangle$ for some k_1. So α_2 assumes that:

- $\gamma_1(k_1) \to \infty$ during the course of the construction,
- $\widehat{\Gamma}_1^{k_1}$ is built infinitely often by α_1,
- there are infinitely many α_1-expansionary stages, and
- Γ_0 is still correct.

α_2 will have to deal with injury arising from the construction of the p.c. functionals Γ_0 and $\widehat{\Gamma}_1^{k_1}$. α_2 will work with a fixed initial value $i(\alpha_2)$ say, and will use all numbers $> i(\alpha_2)$ as its thresholds for both $\widehat{\Gamma}_1^{k_1}$ and Γ_0.

α_2 will build a clearing set C_{α_2}. It will open cycles $\vec{c} = (k_2, \widehat{k}_2)$, where k_2 is a threshold of α_2 for Γ_0, and $\widehat{k}_2$ is a threshold of α_2 for $\widehat{\Gamma}_1^{k_1}$. For a threshold $\widehat{k}_2$ of α_2 for $\widehat{\Gamma}_1^{k_1}$, we may open a $\widehat{k}_2$-cycle for $\widehat{\Gamma}_1^{k_1}$, and for a threshold k_2 of α_2 for Γ_0, we may open a k_2-cycle of α_2 for Γ_0. Each element of C_{α_2} is enumerated by some $(k_2, \widehat{k}_2)$-cycle for some k_2 and $\widehat{k}_2$.

First, we say:

(i) A threshold $\widehat{k}_2$ of α_2 for $\widehat{\Gamma}_1^{k_1}$ is *active at stage* s, if the $\widehat{k}_2$-cycle of α_2 has been either reset or woken up since it last enumerated an element into C_{α_2}.

(ii) A threshold k_2 of α_2 for Γ_0 is $\widehat{k}_2$-*active at stage* s, if the $(k_2, \widehat{k}_2)$-cycle of α_2 has been either reset or woken up since it last enumerated an element into C_{α_2}.

Cycle $(k_2, \widehat{k}_2)$ of α_2 will deal with the injury arising from the p.c. functionals Γ_0 and $\widehat{\Gamma}_1^{k_1}$. On the other hand, we have to ensure that α_2 will not make Ψ_1^A partial, unless there are infinitely many stages at which $\gamma_1(k_1)$ is enumerated by the k_1-cycle of α_1. Therefore, α_2 will act under the assumption that there are infinitely many α_1-expansionary stages.

Then, to open a $\vec{c} = (k_2, \widehat{k}_2)$-cycle α_2 will:

(i) Let $\widehat{k}_2$ be the least k which is an active threshold of α_2 for $\widehat{\Gamma}_1^{k_1}$.

(ii) Let k_2 be the least k which is $\widehat{k}_2$-active.

(iii) Open a $(k_2, \widehat{k}_2)$-cycle of α_2.

Then this $(k_2, \widehat{k}_2)$-cycle of α_2 will proceed as follows:

1. Define a witness x_2 to be the least element x which is not in C_{α_2}.
2. Define an agitator d_2 of α_2 for $\mathcal{R}_1$.
   Suppose that $\gamma_0(k_2) < \widehat{\gamma}_1^{k_1}(\widehat{k}_2) \downarrow$.
3. Wait for a stage at which $\Psi_2^A \upharpoonright (x_2 + 1)$ is defined with $\psi_2(x_2) < \gamma_1(k_1)$.
4. If $\psi_2(x_2) < \gamma_0(k_2)$, go to step 14.
5. If $\psi_2(x_2) < \widehat{\gamma}_1^{k_1}(\widehat{k}_2)$, we say that $\psi_2(x_2)$ is $\widehat{\gamma}_1^{k_1}(\widehat{k}_2)$-cleared.
   Suppose that $\psi_2(x_2) \geq \gamma_0(k_2)$.
6. (Rectify $\widehat{\Gamma}_0^{k_2}$) If there is an x such that $\widehat{\Gamma}_0^{k_2}(Y_0, A; x) \downarrow \neq K(x)$, then let y be the least such x, and enumerate $\widehat{\gamma}_0^{k_2}(y)$ into A.
7. (Harrington Honestification) If there is an x such that $\widehat{\Gamma}_0^{k_2}(Y_0, A; x) \downarrow$, with x a threshold of some strategy with agitator d, and such that $\phi_1(d) \downarrow \geq \widehat{\gamma}_1^{k_1}(\widehat{k}_2)$, then enumerate both $\gamma_0(k_2)$ and $\widehat{\gamma}_1^{k_1}(\widehat{k}_2)$ into A.

8. (Honestify $\widehat{\Gamma}_1^{k_1}$) If $\widehat{\gamma}_1^{k_1}(\widehat{k}_2) \le \phi_1(d_2)$ and $\psi_2(x_2)$ is not $\widehat{\gamma}_1^{k_2}(\widehat{k}_2)$-cleared, then enumerate $\widehat{\gamma}_1^{k_1}(\widehat{k}_2)$ into A.

Slow Down:

9. Let y be the least x such that $\widehat{\Gamma}_0^{k_2}(Y_0, A; x)\uparrow$, let b be the boundary of the k_2-cycle of α_2 for Γ_0, let m be the number of times $\widehat{\Gamma}_0^{k_2}(y)$ has been defined since b was last defined, and let $l = m + b + y$.

10. (Build $\widehat{\Gamma}_0^{k_2}$) Wait for a stage at which

    - $\Psi_{\alpha_2}^A \upharpoonright (l+1)$ is defined via α_2-believable computations,

    and then define $\widehat{\Gamma}_0^{k_2}(Y_0, A; y)\downarrow = K(y)$ with $\widehat{\gamma}_0^{k_2}(y)$ fresh.

11. If the $(k_2, \widehat{k}_2)$-cycle of α_2 is $\gamma_0(k_2)$-cleared, then:

    - if it is also $\widehat{\gamma}_1^{k_1}(\widehat{k}_2)$-cleared, go to step 14,
    - otherwise, proceed to step 12.

12. (Create a Hard Link $(\mathcal{R}_1, \mathcal{L}_2)$) Then:

    - enumerate d_2 into D,
    - create a hard link $(\mathcal{R}_1, \mathcal{L}_2)$.

13. (Travel the Hard Link $(\mathcal{R}_1, \mathcal{L}_2)$) Travel the hard link $(\mathcal{R}_1, \mathcal{L}_2)$ at the next $\mathcal{R}_1$-expansionary stage. Let s^- be the stage at which the current hard link $(\mathcal{R}_1, \mathcal{L}_2)$ was created. There are two cases:

    Case 13a. $Y_{1,s^-} \upharpoonright (\phi_1(d_2)[s^-]+1) \neq Y_1 \upharpoonright (\phi_1(d_2)[s^-]+1)$. Then:

    - set $\widehat{\gamma}_1^{k_1}(\widehat{k}_2)$ to be undefined,
    - go to step 14.

    Case 13b. Otherwise. Then:

    - set $\gamma_1(k_1)$ to be undefined,
    - say that the $\mathcal{L}_1$-strategy α_1 is $\gamma_1(k_1)$-cleared,
    - and start a backup strategy for the $\mathcal{L}_1$-strategy α_1 to build a p.c. functional $\widehat{\Gamma}_0^{k_0}$ which corresponds to the actions in part II of α_1.

14. Then:

    - we say that α_2 is $\gamma_0(k_2)$- and $\widehat{\gamma}_1^{k_1}(\widehat{k}_2)$-cleared,
    - enumerate x_2 into C_{α_2},
    - we say that x_2 is enumerated into C_{α_2} by the $(k_2, \widehat{k}_2)$-cycle.

The Possible Outcomes

We use $(0,k)$ to denote the k-cycle of α_2 for Γ_0, and $(\widehat{1},\widehat{k})$ to denote the $\widehat{k}$-cycle of α_2 for $\widehat{\Gamma}_1^{k_1}$. The possible outcomes of $(0,k)$ are

$$(c,0) <_{\mathrm{L}} (\widehat{c},0),$$

where $(c,0)$ means that $\widehat{\Gamma}_0^k$ is built infinitely often, and $(\widehat{c},0)$ means that $\widehat{\Gamma}_0^k$ is built only finitely many times, but that there are infinitely many stages at which $\gamma_0(k)$ is enumerated into A by honestification via step 7.

The possible outcomes of $(\widehat{1},\widehat{k})$ are defined by:

$$(0,\widehat{k}+1) \ <_{\mathrm{L}} \ (0,\widehat{k}+2) \ <_{\mathrm{L}} \ \cdots \ <_{\mathrm{L}} \ (\widehat{c},1)$$

where $(0,k)$ stands for the k-cycle of α_2 for Γ_0, $(\widehat{c},1)$ means that there are only finitely many stages at which some k-cycle of α_2 for Γ_0 acts, but that there are infinitely many stages at which $\widehat{\gamma}_1^{k_1}(\widehat{k})$ is enumerated into A by honestification via step 8.

Note that the immediate successors of $(0,k)$ are $(c,0) <_{\mathrm{L}} (\widehat{c},0)$.

Now we define the possible outcomes of α_2 as follows:

$$-1 \ <_{\mathrm{L}} \ (\widehat{1},1) \ <_{\mathrm{L}} \ (\widehat{1},2) \ <_{\mathrm{L}} \ \cdots \ <_{\mathrm{L}} \ 2.$$

The interpretation of the outcomes is the same as that in 3.1.

As before, if there are infinitely many stages at which we enumerate some element into C_{α_2}, then:

- C_{α_2} is Δ_2^0,
- -1 is the true outcome of α_2, and
- $C_{\alpha_2} = \omega$ and $\Psi_{\alpha_2}^A$ is total.

Otherwise, if there are only finitely many α_2-expansionary stages, then 2 is the true outcome of α_2.

Otherwise, there is a fixed $\widehat{k}$ such that $\widehat{k}$ is a threshold of α_2 for $\widehat{\Gamma}_1^{k_1}$ and it is active at almost every stage. Let $\widehat{k}_2$ be the least such $\widehat{k}$. Now if there are infinitely many stages at which some nodes to the left of $\alpha_2\hat{\ }\langle(\widehat{1},\widehat{k}_2)\rangle\hat{\ }\langle(\widehat{c},1)\rangle$ act, then there is a fixed k such that k is a threshold of α_2 for Γ_0, and such that k is $\widehat{k}_2$-active at almost every stage. Now if $\widehat{\Gamma}_0^{k_2}$ is built infinitely often, then $(\widehat{1},\widehat{k}_2)\hat{\ }\langle(0,k_2)\rangle\hat{\ }\langle(c,0)\rangle$ is the true outcome of α_2, or else $(\widehat{1},\widehat{k}_2))\hat{\ }\langle(0,k_2)\rangle\hat{\ }\langle(\widehat{c},0)\rangle$ is the true outcome of α_2 .

Otherwise, the true outcome of α_2 is $(\widehat{1},\widehat{k}_2)\hat{\ }\langle(\widehat{c},1)\rangle$.

Thus the true outcome of α_2 is determined by a sequence of Π_2^0- and Σ_2^0-propositions.

An $\mathcal{L}$-strategy below $\alpha_2\hat{}\langle 2\rangle$ is exactly the same as α_2. An $\mathcal{L}$-strategy below $\alpha_2\hat{}\langle -1\rangle$ is the same as α_2, but will have to deal with the requested totality of Ψ_2^A just as before.

The strategies below $\alpha_2\hat{}\langle(\widehat{1},\widehat{k})\rangle\hat{}\langle(\widehat{c},1)\rangle$ for a given $\widehat{k}$ are the same as those in section 2.

Below $\beta = \alpha_2\hat{}\langle(\widehat{1},\widehat{k})\rangle\hat{}\langle(0,k)\rangle\hat{}\langle(\widehat{c},0)\rangle$, some $\widehat{k},k$, we must introduce a backup strategy, $\widehat{\tau}_1$ say, for $\mathcal{R}_1$, and the strategies below $\widehat{\tau}_1$ are the same as that in section 2.

Suppose that α_2 has true outcome $o = (\widehat{1},\widehat{k}_2)\hat{}\langle(0,k_2)\rangle\hat{}\langle(c,0)\rangle$ for some $\widehat{k}_2,k_2$. Then below $\alpha_2\hat{}\langle o\rangle$, we need to introduce a backup strategy, for $\mathcal{R}_1$.

The backup strategy for $\mathcal{R}_1$, $\mathcal{R}_1^c$ say, is similar to the $\mathcal{R}_1$-strategy, and will build a p.c. functional Γ_1^c. The strategies below the $\mathcal{R}_1^c$-strategy will be discussed in 3.3.

3.3 An $\mathcal{L}$-Strategy Below $\alpha_2\hat{}\langle(\widehat{1},\widehat{k}_2)\rangle\hat{}\langle(0,k_2)\rangle\hat{}\langle(c,0)\rangle$ (for some $\widehat{k}_2,k_2$)

Suppose that α_3 is an $\mathcal{L}_3$-strategy below $\alpha_2\hat{}\langle(\widehat{1},\widehat{k}_2)\rangle\hat{}\langle(0,k_2)\rangle\hat{}\langle(c,0)\rangle$, some $\widehat{k}_2,k_2$. α_3 assumes:

- an $\mathcal{L}_2$-strategy α_2 is building a p.c. functional $\widehat{\Gamma}_0^{k_2}$,
- a backup strategy of $\mathcal{R}_1$ is building a p.c. functional Γ_1^c, say, which is constructed according to the $\mathcal{R}_1$-strategy, and
- there are infinitely many α_j-expansionary stages for both $j = 1$ and 2.

α_3 will have to deal with the injury from the building of Γ_1^c and $\widehat{\Gamma}_0^{k_2}$. It will work with a fixed initial value, $i(\alpha_3)$ say. It defines its thresholds for Γ_1^c as the set of all numbers $k > i(\alpha_3)$, and defines its thresholds for $\widehat{\Gamma}_0^{k_2}$ to be the set of all numbers $k > i(\alpha_3)$. α_3 may open cycles $(\widehat{k}_3,k_3)$, where k_3 is a threshold of α_3 for Γ_1^c and $\widehat{k}_3$ is a threshold of α_3 for $\widehat{\Gamma}_0^{k_2}$.

We define the notions of an active threshold of α_3 for Γ_1^c, and of a k-active threshold of α_3 for $\widehat{\Gamma}_0^{k_2}$, just as before. Then α_3 will open a $(\widehat{k}_3,k_3)$-cycle as follows:

(i) Define k_3 to be the least k with k an active threshold of α_3 for Γ_1^c.

(ii) Define $\widehat{k}_3$ to be the least $\widehat{k}$ with $\widehat{k}$ a k_3-active threshold of α_3 for $\widehat{\Gamma}_0^{k_2}$, and define an agitator d_3 for the threshold $\widehat{k}_3$.

(iii) Open a $(\widehat{k}_3,k_3)$-cycle.

This $(\widehat{k}_3,k_3)$-cycle will proceed as follows:

1. Define a witness, x_3 say, to be the least element x not in C_{α_3}.

   Suppose that $\widehat{\gamma}_0^{k_2}(\widehat{k}_3) < \gamma_1^c(k_3) \downarrow$.

2. Wait for a stage at which $\Psi_3^A \upharpoonright (x_3+1)$ is defined via α_3-believable computations.

3. If $\psi_3(x_3) < \widehat{\gamma}_0(\widehat{k}_3)$, then go to step 14.

4. If $\psi_3(x_3) < \gamma_1^c(k_3^c)$, then go to step 10.

5. (Rectify $\widehat{\Gamma}^c$) If there is an x such that $\widehat{\Gamma}_1^{c,k_3}(x) \downarrow \neq K(x)$, then let y be the least such x, and enumerate $\widehat{\gamma}_1^{c,k_3}(y)$ into A.

6. (Harrington Honestification) If there is an x such that $\widehat{\Gamma}_1^{c,k_3}(x) \downarrow$, with x a threshold for $\widehat{\Gamma}_1^c$ with agitator d, such that $\phi_3(d) \geq \gamma_1^c(k_3^c)$, then enumerate $\gamma_1^c(k_3)$ into A.

Slow Down:

7. Let y be the least x such that $\widehat{\Gamma}_1^{c,k_3}(x) \uparrow$, let b be the boundary of the k_3^c-cycle for Γ_1^c, let m be the number of times for which $\widehat{\Gamma}_1^c(y)$ has been defined, and let $l = m + b + y$.

8. (Build $\widehat{\Gamma}_1^{c,k_3}$) Wait for a stage at which $\Psi_3^A \upharpoonright (l+1)$ is defined via α_3-believable computations. Then define $\widehat{\Gamma}_1^{c,k_3}(y) \downarrow = K(y)$ with $\widehat{\gamma}_1^{c,k_3}(y)$ fresh.

9. Otherwise, then do nothing.

10. If $\Psi_3^A \upharpoonright (x_3+1)$ are $\gamma_1^c(k_3)$-cleared, then go on to step 11.

11. (Honestify $\widehat{\Gamma}_0^{k_2}$) If $\phi_0(d_3) \geq \widehat{\gamma}_0^{k_2}(\widehat{k}_3)$, then enumerate both $\widehat{\gamma}_0^{k_2}(\widehat{k}_3)$ and $\gamma_1^c(k_3)$ into A.

12. (Create a Hard Link $(\mathcal{R}_0, \mathcal{L}_3)$) Otherwise. Then:
    - enumerate d_3 into D, and
    - create a hard link $(\mathcal{R}_0, \mathcal{L}_3)$.

13. (Travel the Hard Link $(\mathcal{R}_0, \mathcal{L}_3)$) Travel the hard link $(\mathcal{R}_0, \mathcal{L}_3)$ at the next $\mathcal{R}_0$-expansionary stage. Let s^- be the stage at which the current hard link $(\mathcal{R}_0, \mathcal{L}_3)$ was created. Then there are two cases.

    Case 13a. $Y_{0,s^-} \upharpoonright (\phi_0(d_3)[s^-]+1) \neq Y_0 \upharpoonright (\phi_0(d_3)[s^-]+1)$. Then
    - set $\widehat{\gamma}_0^{k_2}(\widehat{k}_3)$ to be undefined,

- we say that $\Psi_3^A \upharpoonright (x_3+1)$ is $\widehat{\gamma}_0^{k_2}(\widehat{k}_3)$-cleared, and go to step 14.

Case 13b. Otherwise. Then:

- set $\gamma_0(k_2)$ to be undefined,
- we say that the $\mathcal{L}_2$-strategy α_2 is $\gamma_0(k_2)$-cleared.

[Note: At this point, the $\widehat{k}_2$-cycle of α_2 has passed honestification, and $\Psi_2^A(x_2)$ is currently defined. So we have:

- if $\widehat{\gamma}_1^{k_1}(\widehat{k}_2)$ is not defined, then $\Psi_2^A(x_2)\downarrow$ has been cleared by both $\widehat{\gamma}_1^{k_1}(\widehat{k}_2)$ and $\gamma_0(k_2)$,
- and otherwise, we can create a hard link $(\mathcal{R}_1, \mathcal{L}_2)$, which is necessary for α_2 to utilise the honestification of α_2.]

14. Then:
    - enumerate x_3 into C_{α_3}, and
    - stop.

The Possible Outcomes

We use $(\widehat{0}, \widehat{k})$ to denote the $\widehat{k}$-cycle of α_3 for $\widehat{\Gamma}_0^{k_2}$, and $(1, k)$ to denote the k-cycle of α_3 for Γ_1^c.

The only possible outcome of $(\widehat{0}, \widehat{k})$ is $(\widehat{c}, 0)$. The possible outcomes of $(1, k)$ are given by:

$$(\widehat{0}, k+1) <_{\mathrm{L}} (\widehat{0}, k+2) <_{\mathrm{L}} \cdots <_{\mathrm{L}} (c, 1) <_{\mathrm{L}} (\widehat{c}, 1).$$

And for the possible outcomes of α_3 we have:

$$-1 <_{\mathrm{L}} (1,1) <_{\mathrm{L}} (1,2) <_{\mathrm{L}} (1,3) <_{\mathrm{L}} \cdots <_{\mathrm{L}} 2.$$

As before, a sequence of Π_2^0- and Σ_2^0-propositions completely determines the true outcome of α_3.

An $\mathcal{L}$-strategy below $\alpha_3\hat{\ }\langle -1\rangle$ or $\alpha_3\hat{\ }\langle 2\rangle$ is the same as α_3. The strategies below $\alpha_3\hat{\ }\langle(1,k_3)\rangle\hat{\ }\langle(\widehat{c},1)\rangle$ or $\alpha_3\hat{\ }\langle(1,k_3)\rangle\hat{\ }\langle(\widehat{0},\widehat{k}_3)\rangle\hat{\ }\langle(\widehat{c},0)\rangle$ for some k_3, $\widehat{k}_3$ are the same as that in section 2. In 3.4, we will describe an $\mathcal{L}$-strategy below $\alpha_3\hat{\ }\langle(1,k_3)\rangle\hat{\ }\langle(c,1)\rangle$ for some k_3.

3.4 An $\mathcal{L}$-Strategy Below $\alpha_3\hat{\ }\langle(1,k_3)\rangle\hat{\ }\langle(c,1)\rangle$ (some k_3)

Suppose that α_4 is an $\mathcal{L}_4$-strategy below $\alpha_3\hat{\ }\langle(1,k_3)\rangle\hat{\ }\langle(c,1)\rangle$ for some k_3. Then α_4 assumes that:

- a p.c. functional $\widehat{\Gamma}_1^{c,k_3}$ is built by α_3,

- a p.c. functional $\widehat{\Gamma}_0^{k_2}$ is built by α_2.

α_4 will have to deal with injury arising from the construction of $\widehat{\Gamma}_1^{c,k_3}$ and $\widehat{\Gamma}_0^{k_2}$. It will work with a fixed initial value $i(\alpha_4)$, and may define its thresholds for $\widehat{\Gamma}_1^{c,k_3}$ to be all numbers $k > i(\alpha_4)$, and define its thresholds for $\widehat{\Gamma}_0^{k_2}$ to be numbers $k > i(\alpha_4)$. We define the notion of 'active threshold' just as before.

Then α_4 will open a $(\widehat{k}_4, \widehat{k}_4^c)$-cycle as follows:

(i) Define $\widehat{k}_4^c$ to be the least active threshold of α_4 for $\widehat{\Gamma}_1^{c,k_3}$.

(ii) Define $\widehat{k}_4$ to be the least $\widehat{k}_4^c$-active threshold of α_4 for $\widehat{\Gamma}_0^{k_2}$.

(iii) Open a $(\widehat{k}_4, \widehat{k}_4^c)$-cycle.

This $(\widehat{k}_4, \widehat{k}_4^c)$-cycle will proceed as follows:

1. Define a witness x_4 to be the least x which is not in C_{α_4}.
2. Define an agitator d_4^c to be fresh.
3. Define an agitator d_4 afresh.
   Suppose that $\widehat{\gamma}_0^{k_2}(\widehat{k}_4) < \widehat{\gamma}_1^{c,k_3}(\widehat{k}_4^c)\downarrow$.
4. Wait for a stage at which $\Psi_4^A \upharpoonright (x_4+1)$ is defined via α_4-believable computations.
5. If $\psi_4(x_4) < \widehat{\gamma}_0^{k_2}(\widehat{k}_4)$, then go to step 14.
6. If $\psi_4(x_4) < \widehat{\gamma}_1^c(\widehat{k}_4^c)$, then go to step 11.
7. (Honestify $\widehat{\Gamma}_1^{c,k_3}$) Otherwise. Then:
   - If $\widehat{\gamma}_1^{c,k_3}(\widehat{k}_4^c) \le \phi_1^c(d_4^c)$, enumerate $\widehat{\gamma}_1^{c,k_3}(\widehat{k}_4^c)$ into A,
   - If $\widehat{\gamma}_0^{k_2}(\widehat{k}_4) \le \phi_4(d_4)$, enumerate both $\widehat{\gamma}_0^{k_2}(\widehat{k}_4)$ and $\widehat{\gamma}_1^c(\widehat{k}_4^c)$ into A.
8. (Create Hard Links) Otherwise. Then:
   - enumerate d_4^c, d_4 into D,
   - create a hard link $(\mathcal{R}_1^c, \mathcal{L}_4)$, and
   - create a hard link $(\mathcal{R}_0, \mathcal{L}_4)$.

9. (Travel the Hard Link $(\mathcal{R}_0, \mathcal{L}_4)$) Travel the hard link $(\mathcal{R}_0, \mathcal{L}_4)$ at the next $\mathcal{R}_0$-expansionary stage. Let s^- be the stage at which the current hard link $(\mathcal{R}_0, \mathcal{L}_4)$ was created. There are two cases:

   Case 9a. $Y_{0,s^-} \upharpoonright (\phi_0(d_4)+1) \neq Y_0 \upharpoonright (\phi_0(d_4)+1)$. Then:

   - set $\widehat{\gamma}_0^{k_2}(\widehat{k}_4)$ to be undefined.

   Case 9b. Otherwise. Then set $\gamma_0(k_2)$ to be undefined.

10. (Travel the Hard Link $(\mathcal{R}_1^c, \mathcal{L}_4)$) Travel the hard link $(\mathcal{R}_1^c, \mathcal{L}_4)$ at the next $\mathcal{R}_1^c$-expansionary stage. Let s^- be the stage at which the current hard link $(\mathcal{R}_1^c, \mathcal{L}_4)$ was created. There are two cases.

    Case 10a. $Y_{1,s^-} \upharpoonright (\phi_1^c(d_4^c)[s^-]+1) \neq Y_1 \upharpoonright (\phi_1^c(d_4^c)[s^-]+1)$. Then:

    - set $\widehat{\gamma}_1^{c,k_3}(\widehat{k}_4^c)$ to be undefined,
    - go to step 14.

    Case 10b. Otherwise. Then:

    - set $\gamma_1^c(k_3)$ to be undefined.

    Suppose that $\psi_4(x_4) < \widehat{\gamma}_1^c(\widehat{k}_4^c)$.

11. (Honestify $\widehat{\Gamma}_0^{k_2}$) If $\widehat{\gamma}_0^{k_2}(\widehat{k}_4) \leq \phi_0(d_4)$, then enumerate both $\widehat{\gamma}_0^{k_2}(\widehat{k}_4)$ and $\widehat{\gamma}_1^{c,k_3}(\widehat{k}_4^c)$ into A.

12. (Create Hard Links) Otherwise. Then:

    - enumerate d_4 into D,
    - create a hard link $(\mathcal{R}_0, \mathcal{L}_4)$.

13. (Travel the Hard Link $(\mathcal{R}_0, \mathcal{L}_4)$) Travel the hard link $(\mathcal{R}_0, \mathcal{L}_4)$ at the next $\mathcal{R}_0$-expansionary stage. Let s^- be the stage at which the current hard link $(\mathcal{R}_0, \mathcal{L}_4)$ was created. Then there are two cases:

    Case 13a. $Y_{0,s^-} \upharpoonright (\phi_0(d_4)[s^-]+1) \neq Y_0 \upharpoonright (\phi_0(d_4)[s^-]+1)$. Then:

    - set $\widehat{\gamma}_0^{k_2}(\widehat{k}_4)$ to be undefined,
    - we say that $\Psi_4^A \upharpoonright (x_4+1)$ are $\widehat{\gamma}_0^{k_2}(\widehat{k}_4)$-cleared, and go to step 14.

    Case 13b. Otherwise. Then:

    - set $\gamma_0(k_2)$ to be undefined.

14. Then:

    - enumerate x_4 into C_{α_4},
    - we say that x_4 is enumerated by the $(\widehat{k}_4, \widehat{k}_4^c)$-cycle, and stop.

The Possible Outcomes

Denote by $(\widehat{0},\widehat{k}_4)$ the $\widehat{k}_4$-cycle of α_4 for $\widehat{\Gamma}_0^{k_2}$, and by $(\widehat{1},\widehat{k}_4^c)$ the $\widehat{k}_4^c$-cycle of α_4 for $\widehat{\Gamma}_1^{c,k_3}$.

The unique outcome of $(\widehat{0},\widehat{k})$ is $(\widehat{c},0)$. The possible outcomes of $(\widehat{1},\widehat{k}^c)$ are as follows:

$$(\widehat{0},\widehat{k}^c+1) <_{\mathrm{L}} (\widehat{0},\widehat{k}^c+2) <_{\mathrm{L}} \cdots <_{\mathrm{L}} (\widehat{c},1).$$

And the possible outcomes of α_4 are given by:

$$-1 <_{\mathrm{L}} (\widehat{1},1) <_{\mathrm{L}} (\widehat{1},2) <_{\mathrm{L}} \cdots <_{\mathrm{L}} 2.$$

Now an $\mathcal{L}$-strategy below $\alpha_4\hat{\ }\langle 2\rangle$ or $\alpha_4\hat{\ }\langle -1\rangle$ is the same as α_4. The strategies below α_4 relating to other cases are the same as the corresponding ones in section 2.

This completes the description of the $\mathcal{L}$-strategies below two $\mathcal{R}$-strategies.

4 The General Strategies

In this section, we describe the strategies in the general case. We first introduce some notation.

Given a node ξ:

(i) We always use ξ^- to denote the longest strategy $\xi' \subset \xi$.

(ii) We define an initial value $i(\xi)$ to be the greatest stage at which ξ was initialised.

(iii) If k is a threshold of ξ for Γ (some Γ, unhatted, or hatted p.c. functional), then we define a k-cycle of ξ for Γ, and define a *boundary* for the cycle to be the greatest stage at which the cycle was either reset or woken up.

Now we describe the strategies.

4.1 A General $\mathcal{R}$-Strategy

An $\mathcal{R}$-strategy, τ say, will build a p.c. functional Γ to show that $\Gamma^{X,A} = K$ some given X. The $\mathcal{R}$-strategy τ is the same as an $\mathcal{R}$-module, but it will have to deal with all $\mathcal{L}$-strategies α such that $\alpha\hat{\ }\langle -1\rangle \subseteq \tau$.

Then an $\mathcal{R}$-strategy τ will proceed as follows:

1. (Rectify Γ) If there is an x such that $\Gamma^{X,A}(x)\downarrow = 1 \neq 0 = K(x)$, then let y be the least such x, enumerate $\gamma(y)$ into A, and close the current stage.

Slow Down: Steps 2–4

2. Then:
   - let y be the least x such that $\Gamma^{X,A}(x)\uparrow$,
   - let m be the number of times for which $\Gamma^{X,A}(y)$ has been defined since τ was last initialised,
   - let $l = y + m + i(\tau)$, where $i(\tau)$ is the initial value of τ.
3. (Build Γ) If we have:

   (3a) $l(D, \Phi^{X,Y}) > y$, and

   (3b) for every α with $\alpha\hat{\ }\langle -1\rangle \subseteq \tau$, we have that $l \in C_\alpha$.

   Then:

   - define $\Gamma^{X,A}(y)\downarrow = K(y)$ with $\gamma(y)$ fresh.
4. Otherwise, do nothing.

To satisfy $\mathcal{R}$, we must guarantee the following:

(a) If there are only finitely many τ-expansionary stages, then $D \neq \Phi^{X,Y}$.

(b) While if this is not the case, then there are infinitely many stages at which step 3 occurs.

By the strategy above, this ensures that if $\Gamma^{X,A}$ is total, then $\Gamma^{X,A} = K$.

(c) If Γ is built infinitely often, but $\Gamma^{X,A}$ is partial, then either $\Phi^{X,Y}$ is partial or there is a fixed k such that $\widehat{\Gamma}^k$ is built infinitely often with $\widehat{\Gamma}^k(Y, A) = K$.

Clearly any construction with these properties succeeds in satisfying the $\mathcal{R}$-requirements.

4.2 A General $\mathcal{L}$-Strategy

Given an $\mathcal{L}$-strategy α, say, we say that $\mathcal{R}_{\Phi,X,Y}$ is 1-*active* at α if α assumes some p.c. functional $\widehat{\Gamma}$ being built to satisfy $\widehat{\Gamma}^{Y,A} = K$, which has not been destroyed. And that $\mathcal{R}_{\Phi,X,Y}$ is 0-*active* at α, if α assumes that some p.c. functional Γ is being built to give $\Gamma^{X,A} = K$, which has again not been destroyed. We then say that $\mathcal{R}_{\Phi,X,Y}$ is *active* at α if $\mathcal{R}_{\Phi,X,Y}$ is either 1- or 0-active at α.

Assume a list of all $\mathcal{R}$-requirements which are active at α:

$$\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_n.$$

Let Γ_1^*, Γ_2^*, $\cdots$, Γ_n^* be the p.c. functionals constructed to satisfy $\mathcal{R}_1$, $\mathcal{R}_2$, $\cdots$, $\mathcal{R}_n$ respectively. Then α will have to deal with the injury from the p.c. functionals Γ_1^*, Γ_2^*, $\cdots$, Γ_n^*.

α will work with an initial value, f say, and will define thresholds k_j^* for Γ_j^* to be greater than f, for all $j \in \{1, 2, \cdots, n\}$. For every j, $1 \leq j \leq n$, and for each corresponding k_j^*, we may open the k_j^*-cycle. For every j, if $1 \leq j < n$, then the k_{j+1}^*-cycle will be comprised of the k_j^*-cycle for Γ_j^* for every $k_j^* > k_{j+1}^*$. Suppose that $k_n^*, \cdots, k_1^*$ are all defined. Then we will open the $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$-cycle.

As before, α will build a Δ_2^0 set C_α to record the computations $\Psi_\alpha^A(x)$ which have already been preserved. We first look at the actions of a $\vec{c}$-cycle.

Cycle $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$ will proceed as follows:

1. Define a witness x to be the least element which is not in C_α.

2. If there is an i such that $\mathcal{R}_i$ is 1-active at α, but d_i is not defined, then for the greatest such i define d_i afresh.

   Suppose that $\gamma_1^*(k_1^*) < \gamma_2^*(k_2^*) < \cdots < \gamma_n^*(k_n^*)$.

3. Wait for a stage at which $\Psi^A(x)\downarrow$.

4. If $\psi(x) < \gamma_1^*(k_1^*)$, then go to step 15.

5. If there is no i such that $\mathcal{R}_i$ is 0-active at α and $\gamma_i^*(k_i^*) \leq \psi(x)$, then go to step 12. Otherwise, let j be the maximal such i and proceed to step 6j.

6j. If there is an $i > j$ such that $\mathcal{R}_i$ is 1-active at α, and $\gamma_i^*(k_i^*) \leq \phi_i(d_i)$, then let i_0 be the least such i. And enumerate $\gamma_i^*(k_i^*)$ into A for all i with $i_0 \leq i \leq n$.

7j. (Rectify $\widehat{\Gamma}_j^{k_j^*}$) Otherwise. Then:

- if there is an x such that $\widehat{\Gamma}_j^{k_j^*}(Y_j, A; x)\downarrow \neq K(x)$, let y be the least such x, and enumerate $\widehat{\gamma}_j^{k_j^*}(y)$ into A.

8j. (Harrington Honestification) If there is an x such that x is a threshold of the strategy for $\widehat{\Gamma}_j^{k_j^*}$ with agitator d and $\widehat{\gamma}_j^{k_j^*}(x) \leq \phi_j(d)$, then enumerate $\gamma_j^*(k_j^*)$ into A.

Slow Down: Steps 9j–11j

9j. Let y be the least x such that $\widehat{\Gamma}_j^{k_j^*}(Y_j, A; x) \uparrow$, let m be the number of times $\widehat{\Gamma}_j^{k_j^*}(Y_j, A; y)$ has been defined since the k_j^*-cycle of α for Γ_j^* was last either reset or woken up, let b be the boundary of the k_j^*-cycle of α for Γ_j^*, and let $l = y + m + b$.

10j. (Build $\widehat{\Gamma}_j^{k_j^*}$) Say for some β with $\beta\hat{}\langle -1\rangle \subseteq \alpha$ we have that $l \in C_\beta$. Then:

- define $\widehat{\Gamma}_j^{k_j^*}(Y_j, A; y) \downarrow = K(y)$ with $\widehat{\gamma}_j^{k_j^*}(y)$ fresh.

11j. Otherwise, then do nothing.

12. If there is an i such that $\mathcal{R}_i$ is 1-active at α, and $\gamma_i^*(k_i^*) \leq \phi_i(d_i)$, then let j be the greatest such i, and enumerate $\gamma_i^*(k_i^*)$ into A for all i with $j \leq i \leq n$.

13. (Create Hard Links) Otherwise, if there is an i such that $\mathcal{R}_i$ is 1-active at α, and $\gamma_i^*(k_i^*) \leq \psi_\alpha(x)$, then let j be the maximal such i. And for every $i \leq j$, if $\mathcal{R}_i$ is 1-active at α, then:

- enumerate d_i into D, and
- create a hard link $(\mathcal{R}_i, \mathcal{L})$.

14. (Travel a Hard Link) Say there is an i such that a hard link $(\mathcal{R}_i, \mathcal{L})$ was created and has neither been cancelled nor been travelled. Then let j be the least i, and let s^- be the stage at which the current hard link $(\mathcal{R}_j, \mathcal{L})$ was created. There are two cases:

Case 14a. $Y_{j,s^-} \upharpoonright (\phi_j(d_j)[s^-] + 1) \neq Y_j \upharpoonright (\phi_j(d_j)[s^-] + 1)$. Then:

- set $\gamma_j^*(k_j^*)$ to be undefined.

Case 14b. Otherwise. Then set $\gamma_j(k)$ to be undefined, where k is the threshold of the strategy which is destroying the p.c. functional Γ_j.

15. Then:

- enumerate x into C_α, and
- say that x is enumerated by the $\vec{c}$-cycle.

The Possible Outcomes

If $\mathcal{R}_j$ is i-active at α, then let $i_j = i$. We use (i_j, k, j) to denote the k-cycle of α for Γ_j^*.

The possible outcomes of the k_{j+1}-cycle of α for Γ_{j+1}^* can be listed:

$$(i_j, 0, j) <_{\mathrm{L}} (i_j, 1, j) <_{\mathrm{L}} \cdots <_{\mathrm{L}} (c, j+1) <_{\mathrm{L}} (\widehat{c}, j+1),$$

where the interpretation of $(c, j+1)$ is that the p.c. functional $\widehat{\Gamma}^{k_{j+1}}_{j+1}$ is built infinitely often. So if $i_{j+1} = 1$, we can drop the outcome $(c, j+1)$.

The possible outcomes of α are given by:

$$-1 <_{\mathrm{L}} (i_n, 0, n) <_{\mathrm{L}} (i_n, 1, n) <_{\mathrm{L}} \cdots <_{\mathrm{L}} 2,$$

where the immediate successors of (i_n, k, n) are the possible outcomes of the k-cycle of α for Γ^*_n described as above.

Hence the possible outcomes of α form a subtree of height $n+1$.

We now look at the opening of the $\vec{c}$-cycles.

Opening Cycles $\vec{c}$

We say that the x enumerated into C_α in step 15 of the $\vec{c} = (k^*_1, k^*_2, \cdots, k^*_n)$-cycle is a *fruit* of the string $\alpha\hat{}\langle(i_n, k^*_n, n)\rangle\hat{}\cdots\hat{}\langle(i_1, k^*_1, 1)\rangle$. And we locate x at node $\beta_j = \alpha\hat{}\langle(i_n, k^*_n, n)\rangle\hat{}\cdots\hat{}\langle(i_j, k^*_j, j)\rangle$ (say), and say that x is a *fruit* of β_j for all j with $1 \leq j \leq n$.

Let $\beta_j = \alpha\hat{}\langle(i_n, k^*_n, n)\rangle\hat{}\cdots\hat{}\langle(i_j, k^*_j, j)\rangle$ for every j with $1 \leq j \leq n$.

The definition of C_α ensures that:

- if x is a fruit of β_j and there is a $k < k^*_j$ such that $\gamma^*_j(k)$ is enumerated into A, then x is extracted from C_α immediately and automatically,
- if the computation $\Psi^A_\alpha(x)\downarrow$ has been injured since x was enumerated into C_α for the last time, then x is extracted from C_α immediately and automatially,
- if x is not in C_α, then x is no longer a fruit of any string.

With this notation, α will open a $\vec{c} = (k^*_1, k^*_2, \cdots, k^*_n)$-cycle as follows:

(1) We say that a threshold k of α for Γ^*_n is *active*, if the k-cycle of α for Γ^*_n has been either reset or woken up since $\alpha\hat{}\langle(i_n, k, n)\rangle$ last produced a fruit.

Let k^*_n be the least active threshold of α for Γ^*_n, and let $\beta_n = \alpha\hat{}\langle(i_n, k^*_n, n)\rangle$.

Suppose that $k^*_n, \cdots, k^*_{j+1}$ are all defined but k^*_j has not been defined.

(2) We say that a threshold k of α for Γ^*_j is $(k^*_{j+1}, \cdots, k^*_n)$-*active*, if $k > k^*_{j+1}$, and the k-cycle of α for Γ^*_j has been either reset or awakened since $\beta_{j+1}\hat{}\langle(i_j, k, j)\rangle$ last produced a fruit.

Let k^*_j be the least $(k^*_{j+1}, \cdots, k^*_n)$-active threshold of α for Γ^*_j, and let $\beta_j = \beta_{j+1}\hat{}\langle(i_j, k^*_j, j)\rangle$.

(3) Suppose that k^*_1 is defined. Then open a $\vec{c} = (k^*_1, k^*_2, \cdots, k^*_n)$-cycle.

We now look at the satisfaction of $\mathcal{L}$.

By the definition of witnesses in step 1 of the $\vec{c}$-cycles, if C_α is a Δ_2^0 set then $C_\alpha = \omega$. By the definition of C_α, if $C_\alpha = \omega$ then Ψ_α^A is total.

Note that the slowing down of the definition of the p.c. functionals ensures that the p.c. functionals built by strategies below $\alpha\hat{}\langle -1\rangle$ will not make Ψ_α^A partial. That is to say, for a fixed x, $\Psi_\alpha^A(x)$ will eventually remain uninjured by strategies below $\alpha\hat{}\langle -1\rangle$.

We say that a $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$-cycle is *reset* if there exist j, k for which $k < k_j^*$ and $\gamma_j^*(k)$ is enumerated into A.

By the definition of the $\vec{c}$-cycles, there are infinitely many stages at which we open a $\vec{c}$-cycle which is never reset, and which consequently is called a *permanent cycle*. By the definition of C_α, if x was enumerated by a $\vec{c}$-cycle, then at the stage at which the $\vec{c}$-cycle is reset, x is extracted from C_α immediately and automatically. So, following $\Psi_\alpha^A(x)$ becoming free from injury by strategies below $\alpha\hat{}\langle -1\rangle$, a given x enumerated by a permanent cycle will be retained in C_α forever.

Therefore, if -1 is the true outcome of α, then Ψ_α^A is total.

Otherwise, the wake up method ensures that if there are infinitely many α-expansionary stages, then there is a permanent $\vec{c}$-cycle, a fixed witness x of the $\vec{c}$-cycle such that there is a fixed j for which $\gamma_j^*(k_k^*)$ is unbounded during the course of the construction, and for which there are infinitely many stages at which $\gamma_j^*(k_j^*) \downarrow \leq \psi_\alpha(x)$, so that Ψ_α^A is partial. So in any case, if -1 is not the true outcome of α, then Ψ_α^A is partial.

However, we have to ensure that the true outcome is computable from $\emptyset''$.

Determining the True Outcome

The true outcome of α will be determined by one of the following cases:

Case 1. There are infinitely many stages at which C_α is built.

In this case, the true outcome of α is -1. Since C_α is Δ_2^0, we have that $C_\alpha = \omega$ holds (as above), so that Ψ_α^A is total.

Case 2. There are only finitely many α-expansionary stages.

In this case, the true outcome of α is 2, and Ψ_α^A is obviously partial.

Case 3. Otherwise, there is a permanent $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$-cycle say, which acts infinitely often.

By the algorithm for opening cycles, $\vec{c}$ is determined by a Σ_2^0-proposition. By the description of the $\vec{c}$-cycle, the true outcome of cycle $\vec{c}$ is determined by a sequence of Σ_2^0- and Π_2^0-propositions.

We conclude that the true outcome of α is determined by a fixed sequence of Σ_2^0- and Π_2^0-propositions.

5 The Priority Tree

In this section, we describe the *priority tree* T.

5.1 Definition. (i) Define the *priority ranking of the requirements* as follows:

$$\mathcal{R}_0 < \mathcal{L}_0 < \mathcal{R}_1 < \mathcal{L}_1 < \mathcal{R}_2 < \mathcal{L}_2 < \cdots .$$

(ii) The *possible outcomes* of an $\mathcal{R}$-strategy are $0 <_{\mathrm{L}} 1$ to denote infinite and finite actions respectively.

(iii) The *possible outcomes* of an $\mathcal{L}$-strategy are those of the subtree described in 4.2.

We now consider the satisfaction along a given strategy.

5.2 Definition. Given a strategy ξ:

(i) We say that $\mathcal{L}_\Psi$ is *satisfied at* ξ if there is an $\mathcal{L}_\Psi$-strategy α such that $\alpha \subset \xi$.

(ii) We say that $\mathcal{R}_{\Phi,X,Y}$ is *satisfied at* ξ if either there is an $\mathcal{R}_{\Phi,X,Y}$-strategy τ such that $\tau\hat{\ }\langle 1\rangle \subseteq \xi$, or there is a node β such that $\beta\hat{\ }\langle(i,k,e)\rangle\hat{\ }\langle(\widehat{c},e)\rangle \subseteq \xi$ for some $i \leq 1$, and some $k \in \omega$.

(iii) We say that $\mathcal{R}_{\Phi,X,Y}$ is 1-*active at* ξ if $\mathcal{R}_{\Phi,X,Y}$ is not satisfied at ξ, but there is a node β such that both (a) and (b) below hold:

(a) $\beta\hat{\ }\langle(0,k,e)\rangle\hat{\ }\langle(c,e)\rangle \subseteq \xi$ for some $k \in \omega$, and

(b) there exist no i', k' such that

$$\beta\hat{\ }\langle(0,k,e)\rangle\hat{\ }\langle(c,e)\rangle \subseteq \gamma\hat{\ }\langle(i',k',e')\rangle\hat{\ }\langle(a,e')\rangle \subseteq \xi,$$

any γ, any $a \in \{c, \widehat{c}\}$, and any $e' < e$.

(iv) We say that $\mathcal{R}_{\Phi,X,Y}$ is 0-*active at* ξ if $\mathcal{R}_{\Phi,X,Y}$ is not satisfied, and is not 1-active at ξ, but there is an $\mathcal{R}_{\Phi,X,Y}$-strategy τ such that both (a) and (b) below hold:

(a) $\tau\hat{\ }\langle 0\rangle \subseteq \xi$, and

(b) there exist no i', k' such that $\tau\hat{\ }\langle 0\rangle \subseteq \gamma\hat{\ }\langle(i,'k',e')\rangle\hat{\ }\langle(a,e')\rangle \subseteq \xi$, any γ, any $a \in \{c, \widehat{c}\}$, and any $e' < e$.

(v) We say $\mathcal{R}_{\Phi,X,Y}$ is *active at* ξ if $\mathcal{R}_{\Phi,X,Y}$ is either 0- or 1-active at ξ.

We now describe the priority tree T.

5.3 Definition. (i) Define the root node λ to be an $\mathcal{R}_0$-strategy.

(ii) The immediate successors of a strategy are the possible outcomes of that strategy.

(iii) A strategy ξ will work on the highest priority requirement which is neither satisfied, nor active at ξ.

(iv) We define the index of a strategy to be the index of the requirement on which the strategy works.

(v) The priority tree T will be built as a c.e. set of all strategies which appear during the course of the construction. At the stage at which a strategy appears in the construction for the first time, we enumerate it into T.

[Remark: Since certain possible outcomes of an $\mathcal{L}$-strategy form a subtree, not every node is a strategy.]

We now look at the structure of the priority tree T.

5.4 Proposition. (Finite Injury Along Any Infinite Path Proposition) Given an infinite path P through T, we have that, for any requirement $\mathcal{X}$, there exist ξ_0 and $i \leq 1$ such that either (i) or (ii) below holds:

(i) $\mathcal{X}$ is satisfied at ξ for all ξ with $\xi_0 \subseteq \xi \in P$,

(ii) $\mathcal{X}$ is i-active at ξ for all ξ with $\xi_0 \subseteq \xi \in P$.

Proof. By induction on the priority ranking of the requirements.

Say $\mathcal{X} = \mathcal{R}_0$.

If $\lambda\hat{\ }\langle 1\rangle \in P$, then by definition 5.2 (ii), for $\xi_0 = \lambda\hat{\ }\langle 1\rangle$ $\mathcal{R}_0$ is satisfied at ξ for all ξ with $\xi_0 \subseteq \xi \in P$.

If there is a node β such that $\beta\hat{\ }\langle(\widehat{c}, 0)\rangle \in P$, then by definition 5.2 (ii), for $\xi_0 = \beta\hat{\ }\langle(\widehat{c}, 0)\rangle$ $\mathcal{R}_0$ is satisfied at ξ for all ξ with $\xi_0 \subseteq \xi \in P$.

Otherwise, if there is a node β such that $\beta\hat{\ }\langle(c, 0)\rangle \in P$, then by definition 5.2 (iii) and by definition 5.3, for $\xi_0 = \beta\hat{\ }\langle(c, 0)\rangle$, $\mathcal{R}_0$ is 1-active at ξ for all ξ with $\xi_0 \subseteq \xi \in P$. This is because otherwise, by definition 5.2 (iv) and definition 5.3, for $\xi_0 = \lambda\hat{\ }\langle 0\rangle$ $\mathcal{R}_0$ is 0-active at ξ for all ξ with $\xi_0 \subseteq \xi \in P$.

Suppose by induction that the proposition holds for all requirements $\mathcal{Y} < \mathcal{X}$. Let ξ_0 be the shortest node such that for every $\mathcal{Y} < \mathcal{X}$, either $\mathcal{Y}$ is satisfied at ξ for all ξ with $\xi_0 \subseteq \xi \in P$, or $\mathcal{Y}$ is i-active at ξ for all ξ with $\xi_0 \subseteq \xi \in P$ for some $i \leq 1$.

By definition 5.3, there is an $\mathcal{X}$-strategy $\subseteq \xi_0$. Let α be the longest $\mathcal{X}$-strategy $\subseteq \xi_0$. We consider two cases:

Case 1. $\mathcal{X} = \mathcal{R}_{\Phi,X,Y}$ for some e.

By the choice of α, the same argument as that for $\mathcal{R}_0$ supplies a proof for $\mathcal{X}$.

Case 2. $\mathcal{X} = \mathcal{L}_\Psi$ for some e.

Let ξ_0 be the shortest strategy β such that $\alpha \subset \beta \in P$. Then by definition 5.2 (i), $\mathcal{L}_\Psi$ is satisfied at all ξ with $\xi_0 \subseteq \xi \in P$.

The proposition follows. $\square$

We need to further examine the structure of the priority tree.

5.5 Definition. Given a node ξ:

(i) We say that ξ is a 1-*node*, if $\xi = \beta\hat{\ }\langle(c, e)\rangle$, some β, some $e \in \omega$ — in which case, we define $E(\xi) = e$.

(ii) We say that ξ is a -1-node, if $\xi = \beta\hat{\ }\langle(\widehat{c}, e)\rangle$, some β, some $e \in \omega$ — in which case, we define $E(\xi) = e$.

(iii) We say that ξ is a 0-node, if $\xi = \tau\hat{\ }\langle 0\rangle$ for some $\mathcal{R}_{\Phi,X,Y}$-strategy τ — in which case we define $E(\xi) = e$.

(iv) We say that $\alpha \subset \xi$ is 1-*active at* ξ, if α is a 1-node, and there is no ± 1-node β such that $\alpha \subseteq \beta \subseteq \xi$ and $E(\beta) < E(\alpha)$.

(v) We say that $\alpha \subset \xi$ is 0-*active at* ξ, if α is a 0-node and there is no ± 1-node β such that $\alpha \subset \beta \subseteq \xi$ and $E(\beta) \leq E(\alpha)$.

(v) We say that $\alpha \subset \xi$ is *active at* ξ, if α is i-active at ξ for some $i \leq 1$.

We now fix some notation relative to a given $\mathcal{L}$-strategy.

5.6 Definition. Given an $\mathcal{L}$-strategy α:

(i) Suppose that $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_n$ are all nodes β which are active at α.

(ii) For every $j \in \{1, 2, \cdots, n\}$, let τ_j be the longest $\mathcal{R}_{\Phi,X,Y}$-strategy $\subset \beta_j$ for $e = E(\beta_j)$.

(iii) If β_j is 0-active at α, then we also use Γ_{β_j} to denote the p.c. functional built by τ_j, in which case, we we set $\Gamma^*_{\beta_j} = \Gamma_{\beta_j}$.

If β_j is 1-active at α, then we use $\widehat{\Gamma}_{\beta_j}$ to denote the hatted p.c. functional built by β_j^- to destroy the p.c. functional Γ_{τ_j}, where β_j^- is the longest strategy $\subset \beta_j$ — in which case, we set $\Gamma^*_{\beta_j} = \widehat{\Gamma}_{\beta_j}$.

(iv) Define $n(\alpha) = n$, and $e_j = e(\tau_j)$ for all j with $1 \leq j \leq n$.

(v) If β_j is i-active at α, then define $i_j = i$.

(vi) Suppose that $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$ is the current cycle of α. Then we use δ_j to denote

$$\alpha\hat{\ }\langle(i_n, k_n^*, e_n)\rangle\hat{\ }\cdots\hat{\ }\langle(i_j, k_j^*, e_j)\rangle$$

for all $j \in \{1, 2, \cdots, n\}$.

(vii) If β_j is 1-active at α, then let α_j be the longest $\mathcal{L}$-strategy $\subset \beta_j$. We say that (τ_j, α_j) is a *loop* of α.

(viii) Suppose that (τ_j, α_j) is a loop of α, and that $\beta_{j,1} \subset \beta_{j,2} \subset \cdots \subset \beta_{j,n_j}$ are all β which are active at α_j. Let $e_{j,i} = E(\beta_{j,i})$, and $\tau_{j,i}$ be the longest $\mathcal{R}_{e_{j,i}}$-strategy $\subset \beta_{j,i}$, and let l_j be the number such that $\tau_{j,l_j} = \tau_j$.

Then we have:

5.7 Proposition. Given an $\mathcal{L}_e$-strategy α, assume the notation in definition 5.6. Then:

(i) if $e_j = e(\tau_j)$, for all $j \in \{1, 2, \cdots, n\}$, then $e_1 < e_2 < \cdots < e_n \leq e$.

(ii) $\tau_1 \subset \beta_1 \subset \tau_2 \subset \beta_2 \subset \cdots \subset \tau_n \subset \beta_n$.

Proof. Immediate from definitions 5.3 and 5.6. □

6 Parameters and Definitions

The parameters and definitions introduced in this section will be useful in the description and verification of the construction.

Suppose that α is an $\mathcal{L}$-strategy, that $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$ is the current cycle of α, and that $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_n$ are all β which are active at α. We define:

6.1 Definition. (i) If β_j is 0-active at α, then:

- we say that $\delta_j = \alpha\hat{\ }\langle(i_n, k_n^*, e_n)\rangle\hat{\ }\cdots\hat{\ }\langle(0, k_j^*, e_j)\rangle$ is a 0-*extension* of cycle $\vec{c}$ of α,
- we use γ^{δ_j} to denote the value of $\gamma_{\tau_j}(k_j^*)$,
- we use $k(\tau_j, \alpha, \vec{c})$ to denote k_j^*.

(ii) Define $E^0(\alpha, \vec{c})$ to be the set of all 0-extensions of cycle $\vec{c}$ of α.

(iii) If β_j is 1-active at α, then:

- we say that $\delta_j = \alpha\hat{\ }\langle(i_n, k_n^*, e_n)\rangle\hat{\ }\cdots\hat{\ }\langle(1, k_j^*, e_j)\rangle$ is a 1-*extension* of cycle $\vec{c}$ of α,
- we use γ^{δ_j} to denote the value of $\widehat{\gamma}_{\beta_j}(k_j^*)$,
- we use $\widehat{k}(\tau_j, \alpha, \vec{c})$ to denote k_j^*,
- we use $d(\tau_j, \alpha, \vec{c})$ to denote the agitator of cycle k_j^* of cycle $\vec{c}$ of α for $\widehat{\Gamma}_{\beta_j}$.

(iv) Define $E^1(\alpha, \vec{c})$ to be the set of all 1-extensions of cycle $\vec{c}$ of α.

(v) Let $E(\alpha, \vec{c}) = E^0(\alpha, \vec{c}) \cup E^1(\alpha, \vec{c})$. We say that an element of $E(\alpha, \vec{c})$ is an *extension* of the cycle $\vec{c}$ of α.

Given an $\mathcal{L}$-strategy α and a cycle $\vec{c}$ of α, α will ensure that for any $\delta_1, \delta_2 \in E(\alpha, \vec{c})$, if $\delta_1 \subset \delta_2$, then $\gamma^{\delta_1} > \gamma^{\delta_2}$. This will be ensured by the application of *well-ordering* during the construction.

6.2 Parameters. Given a strategy ξ:

(i) If $\xi = \tau$ is an $\mathcal{R}$-strategy, then the following parameters are associated with τ:

(a) $b(\tau)$: a *boundary* of τ,

(b) Γ_τ: the p.c. functional built by τ,

(c) $d(\tau, \alpha, \vec{c})$: the agitator of cycle $\vec{c}$ of α for τ,

(d) a hard link or a soft link (τ, α) may be created and travelled for some $\mathcal{L}$-strategy α.

(ii) If $\xi = \alpha$ is an $\mathcal{L}$-strategy, then assume definition 5.6. We have the following parameters for α:

(a) $f(\alpha)$: an *initial value* of α,

(b) C_α: the Δ^0_2 set built by α,

(c) the parameters in definition 6.1,

(d) a hard link or a soft link (τ, α) for some $\mathcal{R}$-strategy τ may be created and travelled,

(e) a witness $x(\vec{c}, \alpha)$ may be defined for cycle $\vec{c}$ of α, for some $\vec{c}$,

(f) a *boundary* $b(k^*_j, \vec{c}, \alpha)$ may be defined for cycle k^*_j of cycle $\vec{c}$ of α for $\Gamma^*_{\beta_j}$,

(g) $p(\vec{c}, \alpha)$: a parameter which is defined for cycle $\vec{c}$ of α.

6.3 Definition. Given an $\mathcal{L}$-strategy α:

(i) Define $U(\alpha) = \min\{\gamma^\beta \mid \beta\hat{\ }\langle(a, e)\rangle \subseteq \alpha$ for some $a \in \{c, \widehat{c}\}$, and some $e \in \omega$.

(ii) We say that $\Psi^A_\alpha(y) \downarrow= z$ is *α-believable*, if $\psi_\alpha(y) < U(\alpha)$.

(iii) Let $l(\alpha) = \max\{x | (\forall y < x)[\Psi^A_\alpha(y) \downarrow$ via α-believable computations$]\}$.

(iv) We say that s is *α-expansionary* if α is visited at stage s, and $l(\alpha)[s] > l(\alpha)[v]$ for all stages $v < s$ at which α was visited.

6.4 Definition. Given an $\mathcal{R}_{\Phi,X,Y}$-strategy τ:

(i) Define $l(\tau) = \max\{x | (\forall y < x)[\Phi(X, Y; y) \downarrow= D(y)]\}$.

(ii) We say that s is τ-expansionary, if:

(a) $l(\tau) > l(\tau)[v]$ for all $v < s$ at which τ was visited, and
(b) for any α and $\vec{c}$, if $d(\tau, \alpha, \vec{c}) \downarrow$, then $l(\tau) > d(\tau, \alpha, \vec{c})$.

6.5 Definition. Given an $\mathcal{L}$-strategy α, and a stage s, let $\vec{c} = (k^*_1, k^*_2, \cdots, k^*_n)$ be the current cycle of α, and let $x = x(\vec{c}, \alpha)$ be the witness of cycle $\vec{c}$ of α. Then:

(i) We say that cycle $\vec{c}$ of α is *preservable at stage s*, if $\Psi^A_\alpha(x)$ is currently defined via an α-believable computation. In which case we define $p(\vec{c}, \alpha)$ to be the least stage at which the current computation of $\Psi^A_\alpha(x)$ was established and after which there is no element $y \leq$ the current value $\psi_\alpha(x)$ of the use function which has been enumerated into A.

(ii) If $p(\vec{c}, \alpha) \downarrow = v$, and there is an element which is less than $\psi_\alpha(x)[v]$ and which is enumerated into A at stage s, then $p(\vec{c}, \alpha)$ is set to be undefined immediately and automatically.

(iii) Given $\beta \in E(\vec{c}, \alpha)$, we say that cycle $\vec{c}$ of α is *β-clear at stage s*, if:

(a) $p(\vec{c}, \alpha) \downarrow = v$ for some v,

(b) $\gamma^\beta > \psi_\alpha(x)[v]$.

(iv) We say that cycle $\vec{c}$ of α is *satisfied at stage s*, if x is in C_α.

Assume the notation from definition 5.6. Then a cycle $\vec{c}$ of α may require *τ_j-honestification* for some j.

6.6 Definition. Given an $\mathcal{L}$-strategy α and a stage s, let $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$ be the current cycle of α. Let b be the boundary of cycle $\vec{c}$ of α which is defined as the greatest stage at which cycle $\vec{c}$ of α is either defined, or reset or woken up. Let n_j be the number of times the cycle $\vec{c}$ of α has received τ_j-honestification since the cycle was last either reset or woken up.

Then we say that cycle $\vec{c}$ of α *requires τ_j-honestification at stage s*, if:

(i) For any $\mathcal{R}$-strategy τ, if $\tau_j \subset \tau \subset \alpha$, then $p_\tau > b + n_j$, where p_τ is the number of times that Γ_τ has been defined since τ was last initialised.

(ii) For any β, if $\tau_j \subset \beta\hat{\ }\langle -1\rangle \subseteq \alpha$, then $l = b + n_j \in C_\beta$.

(iii) For any β, if $\tau_j \subset \beta\hat{\ }\langle (c, e)\rangle \subseteq \alpha$, then let $p(\beta)$ be the number of times that $\widehat{\Gamma}_\beta$ has been defined since β was either last initialised or reset. Then we have that $p(\beta) > l = b + n_j$.

(iv) For every β, if $\tau_j \subset \beta \subset \gamma\hat{\ }\langle (\widehat{c}, e)\rangle \subseteq \alpha$ for some γ and some e, let $p(\beta)$ be the number of times that β has received honestification since β was last either reset or initialised. Then we have that $p(\beta) > b + n_j$.

(v) One of the following cases occurs:

Case 1. β_j is 1-active at α, in which case one of the subcases (1a)–(1b) below holds:

(1a) $\widehat{\gamma}_{\beta_j}(k_j^*) \downarrow \leq \phi_{\tau_j}(d(\tau_j, \alpha, \vec{c}))$.

Then let $\delta_i = \alpha\hat{\ }\langle (i_n, k_n^*, e_n)\rangle\hat{\ }\cdots\hat{\ }\langle (i_i, k_i^*, e_i)\rangle$ for all $i \in \{1, 2, \cdots, n\}$.

(1b) $\gamma^{\delta_{j-1}} > \gamma^{\delta_j}$.

Case 2. β_j is 0-active at α, in which case one of the subcases (2a)–(2b) below holds:

(2a) There is a strategy with agitator d for $\widehat{\Gamma}_{\delta_j}$ such that $\phi_{\tau_j}(d) \geq \gamma_{\tau_j}(k_j^*)$,

(2b) The same as (1b) in case 1.

Action. (Receiving τ_j-Honestification) If the cycle $\vec{c}$ of α requires τ_j-honestification, then let cycle $\vec{c}$ of α receive such honestification as follows:

- For any i, if $\delta_i \in E(\vec{c}, \alpha)$, $i \geq j$ and $\gamma^{\delta_i} \downarrow$, then enumerate γ^{δ_i} into A.

During the course of the construction, we may implement the action of *capricious destruction* for a cycle $\vec{c}$ of α, relative to an $\mathcal{L}$-strategy α.

6.7 Definition. Given an $\mathcal{L}$-strategy α, let $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$ be the current cycle of α. We say that $\vec{c}$ *requires* τ_j*-destruction at stage* s, if:

(i) β_j is 0-active at α,

(ii) $p(\vec{c}, \alpha) \downarrow$,

(iii) For any $i > j$, if β_i is 0-active at α, then $\vec{c}$ is δ_i-clear at stage s (where δ_i is as in definition 6.6),

(iv) $\vec{c}$ is not δ_j-clear at stage s,

(v) For any i, if $j < i \leq n$, and β_i is 1-active at α, then $\vec{c}$ does not require τ_i-honestification at stage s.

Action. (Receiving τ_j-Destruction) If cycle $\vec{c}$ of α requires τ_j-destruction at stage s, then let $\vec{c}$ receive τ_j-destruction as follows:

1. (Rectify $\widehat{\Gamma}_{\delta_j}$) If there is an x such that $\widehat{\Gamma}_{\delta_j}^{Y_{\tau_j}, A}(x) \downarrow \neq K(x)$, then let y be the least such x, and enumerate $\widehat{\gamma}_{\delta_j}(y)$ into A.

2. (Harrington Honestification) If there is a strategy with agitator d for $\widehat{\Gamma}_{\delta_j}$ such that $\gamma_{\tau_j}(k_j^*) \leq \phi_{\tau_j}(d)$, then enumerate $\gamma_{\tau_j}(k_j^*)$ into A.

Slow Down: Steps 3–5

3. Then:
   - Let y be the least x such that $\widehat{\Gamma}_{\delta_j}^{Y_{\tau_j}, A}(x) \uparrow$,
   - Let m be the number of times that $\widehat{\Gamma}_{\delta_j}^{Y_{\tau_j}, A}(y)$ has been defined since cycle k_j^* of α for Γ_{τ_j} was last either reset or awakened,
   - Let b be the boundary of cycle k_j^* of α for Γ_{τ_j}, which is defined as the greatest stage at which cycle k_j^* of α for Γ_{τ_j} was either reset or woken up, or opened,
   - Let $l = y + m + b$.

4. (Build $\widehat{\Gamma}_{\delta_j}$) If:

   (4a) For every β, if $\beta\hat{\ }\langle -1 \rangle \subseteq \alpha$, then $l \in C_\beta$, and

   (4b) $\Psi_\alpha^A \upharpoonright (l+1)$ are all defined via α-believable computations, then:

   - let $\widehat{\Gamma}_{\delta_j}^{Y_{\tau_j}, A}(y) \downarrow = K(y)$ with $\widehat{\gamma}_{\delta_j}(y)$ fresh.

5. (Delayed Action) Otherwise, do nothing.

During the course of the construction, we may create a hard link (τ, α) for certain $\mathcal{R}$-, $\mathcal{L}$-strategies τ, α, respectively.

6.8 Definition. Assume $\alpha, \vec{c}$ to be the same as in definition 6.7. We say that cycle $\vec{c}$ of α *requires a switch of strategy at stage* s, if: (i) $p(\vec{c}, \alpha) \downarrow= v$, say.

(ii) For every $\delta \in E^0(\vec{c}, \alpha)$, cycle $\vec{c}$ of α is δ-clear at stage s.

(iii) There is no j such that β_j is 1-active at α, and cycle $\vec{c}$ of α requires τ_j-honestification.

(iv) There is a j such that β_j is 1-active at α, and cycle $\vec{c}$ of α is not δ_j-clear at stage s.

(v) For every j, if $1 \leq j \leq n$ and β_j is 1-active at α, then $d(\tau_j, \alpha, \vec{c})$ is defined.

(vi) For every j, if β_j is 1-active at α, then let $p(\tau_j)$ be the number of times for which Γ_{τ_j} has been defined since τ_j was initialised for the last time, let $b(\alpha, k_j^*)$ be the boundary of cycle k_j^* of α for $\widehat{\Gamma}_{\beta_j}$, and let $q(\alpha, j)$ be the number of times for which a hard link (τ_j, α) has been created since cycle k_j^* of α for $\widehat{\Gamma}_{\beta_j}$ was either reset or woken up for the last time, then we have that

$$p(\tau_j) > q(\alpha, j) + b(\alpha, k_j^*) + k_j^*.$$

(vii) For every j, if β_j is 1-active at α, then for $l = q(\alpha, j) + b(\alpha, k_j^*) + k_j^*$, where $q(\alpha, j), b(\alpha, k_j^*)$ are defined in (vi) above, we have:

(a) for every $\mathcal{R}$-strategy τ, if $\tau_j \subset \tau \subset \tau\hat{\ }\langle 0 \rangle \subseteq \alpha$, then $p(\tau) > l$,

(b) for every β, if $\tau_j \subset \beta\hat{\ }\langle -1 \rangle \subseteq \alpha$, then $l \in C_\beta$,

(c) for every β, if $\tau_j \subset \beta\hat{\ }\langle (c, e) \rangle \subseteq \alpha$ for some e, then let $p(\beta)$ be the number of times for which $\widehat{\Gamma}_\beta$ has been defined since β was either initialised or reset for the last time, we have that $p(\beta) > l$,

(d) for every β, if $\tau_j \subset \beta \subset \gamma\hat{\ }\langle (\widehat{c}, e) \rangle \subseteq \alpha$ for some γ, some e, let $p(\beta)$ be the number of times for which β has received honestification since β was either reset or initialised for the last time, we have that $p(\beta) > l$.

Action. (Implementing a Switch of Strategy) If $\vec{c}$ requires a switch of strategy at stage s, then let $\vec{c}$ implement it as follows.

- For every j, in decreasing order of j, if β_j is 1-active at α and cycle $\vec{c}$ is not δ_j-clear, then enumerate $d(\tau_j, \alpha, \vec{c})$ into D, and create a hard link (τ_j, α).

6.9 Definition. Given an $\mathcal{L}$-strategy α, let $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$ be the current cycle of α, and let $x = x(\vec{c}, \alpha)$ be the witness of $\vec{c}$. Then:

(i) If x is enumerated into C_α at stage s, we say that x is *a fruit of* $\delta_j = \alpha\hat{\ }\langle(i_n, k_n^*, e_n)\rangle\hat{\ }\cdots\hat{\ }\langle(i_j, k_j^*, e_j)\rangle$ for every $j \in \{1, 2, \cdots, n\}$.

(ii) If x is extracted from C_α at stage s, then x is no longer a fruit of δ at the end of stage s, for any δ.

If the current cycle $\vec{c}$ of α is not defined, then α will open a cycle $\vec{c}$ as follows:

6.10 Definition. (i) We say that a threshold k of α for $\Gamma^*_{\beta_n}$ is *active*, if:

(a) $k > f(\alpha)$ (the initial value of α),

(b) There is a greatest stage s^- at which either the k-cycle of α for $\Gamma^*_{\beta_n}$ was opened or reset, or $\alpha\hat{\ }\langle(i_n, k, e_n)\rangle$ reached fruition, and

(c) the k-cycle of α for $\Gamma^*_{\beta_n}$ has been awakened subsequent to stage s^-.

(ii) Define k_n^* to be the least active threshold of α for $\Gamma^*_{\beta_n}$, and let $\delta_n = \alpha\hat{\ }\langle(i_n, k_n^*, e_n)\rangle$.

Suppose that $k_n^*, \cdots, k_{j+1}^*$ are all defined, but k_j^* is not defined.

(iii) We say that a threshold k of α for $\Gamma^*_{\beta_j}$ is $(k_{j+1}^*, \cdots, k_n^*)$-*active*, if:

(a) $k > k_{j+1}^*$,

(b) There is a greatest stage s^- at which either the k-cycle of α for $\Gamma^*_{\beta_j}$ was reset or $\delta_{j+1}\hat{\ }\langle(i_j, k, e_j)\rangle$ reached fruition, and

(c) the k-cycle of α for $\Gamma^*_{\beta_j}$ has been awakened since stage s^-.

(iv) Then:

- Define k_j^* to be the least $(k_{j+1}^*, \cdots, k_n^*)$-active threshold of α for $\Gamma^*_{\beta_j}$,
- and let $\delta_j = \delta_{j+1}\hat{\ }\langle(i_j, k_j^*, e_j)\rangle$.

(v) Suppose that $k_1^*, k_2^*, \cdots, k_n^*$ are all defined.

Then define $\vec{c}(\alpha) = \vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$, and open a cycle $\vec{c}$ of α.

7 The Construction

We describe the construction using the priority tree from section 5, and the definitions from section 6. We first introduce some special notation used during the construction.

7.1 The Use Rules

For any p.c. functional Γ, say, which is built by us, the use function γ will satisfy the γ–rules in 2.2.

During the course of the construction, we may initialise or reset a strategy.

7.2 Initialisation

We require that:

(i) If an $\mathcal{R}$-strategy τ is initialised, then:

- any parameter which is associated with τ is cancelled,
- the boundary $b(\tau)$ is defined afresh.

(ii) If an $\mathcal{L}$-strategy α is initialised, then:

- any previous action of α is cancelled,
- the initial value $f(\alpha)$ is defined afresh.

7.3 Reset

Given an $\mathcal{L}$-strategy α, we define:

(i) If k is a threshold of α for Γ^* for some Γ^*, and there is a $k' < k$ such that $\gamma^*(k')$ is enumerated into A, then the k-cycle of α for Γ^* is *reset*. This means that every previous action of the k-cycle of α for Γ^* is cancelled.

(ii) Given a cycle $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$ of α, if there is a j such that cycle k_j^* of α for $\Gamma^*_{\beta_j}$ is reset, then any previous action of cycle $\vec{c}$ of α is cancelled, in which case, we say that $\vec{c}$ is *reset*.

7.4 Waking Up Rules

Suppose that $\alpha_1 \subset \alpha_2$ are $\mathcal{L}$-strategies such that $\alpha_1\hat{\ }\langle -1\rangle \subseteq \alpha_2$, and that Γ^* is a p.c. functional.

If k is a threshold of α_j for Γ^* for both $j = 1$ and 2, and α_2 enumerates $\gamma^*(k)$ into A at a stage, s say, then we say that the k-cycle of α_1 for Γ^* is *woken up at stage s*.

7.5 Background Activity

We list some actions which will be executed automatically:

(i) The actions of resetting and waking up will proceed automatically.

(ii) If $\xi_1 < \xi_2$ and ξ_1 is initialised, then ξ_2 is simultaneously initialised.

(iii) If a link (α, β) (either hard or soft) is travelled, then it is removed immediately afterwards.

(iv) The construction of the priority tree T will proceed automatically.

(v) If ξ is initialised and $\gamma^\xi\downarrow$, then enumerate γ^ξ into A.

We now describe:

7.6 The Full Construction

Stage $s = 0$. Set $T = D = A = \emptyset$.

Stage $s > 0$. We say that a node ξ is *visited at stage* s, if it is *eligible to act at a substage of stage* s, where this eligibility of ξ is inductively described during the description of the substages below. We start by allowing the root node λ to be eligible to act at substage 0 of stage s.

Substage t. Let ξ be eligible to act at substage t of stage s. If $t = s$, then initialise any δ with $\xi <_{\mathrm{L}} \delta$ and go to stage $s + 1$. Otherwise, there are 2 cases:

Case 1. $\xi = \tau$ is an $\mathcal{R}_{\Phi,X,Y}$-strategy. Implement the following:

Program τ

1. If s is not τ-expansionary, then let $\tau\hat{\ }\langle 1\rangle$ be eligible to act next.

   Suppose that s is τ-expansionary.

2. If there is a hard link (τ, α) which was created and which has neither been cancelled nor travelled, then let α_0 be the $<$-least α, and let α_0 be eligible to act next.

3. If there is a soft link (τ, α) which was created, and which has neither been cancelled nor travelled, then let α_0 be the $<$-least such α, and let α_0 be eligible to act next.

4. (Rectify Γ) If there is an x such that $\Gamma_\tau^{X,A}(x) \downarrow \neq K(x)$, then let y be the least such x. Enumerate $\gamma_\tau(y)$ into A, initialise any δ with $\tau\hat{\ }\langle 0\rangle <_{\mathrm{L}} \delta$, and go to stage $s + 1$.

Slow Down: Steps 5–7

5. Then:
   - Let y be the least x such that $\Gamma_\tau^{X_\tau,A}(x) \uparrow$,
   - Let m be the number of times that $\Gamma_\tau^{X_\tau,A}(y)$ has been defined since τ was last initialised, and
   - Let b be the boundary of τ, and let $l = y + m + b$.

6. (Build Γ) If for every $\mathcal{L}$-strategy α with $\alpha\hat{\ }\langle -1\rangle \subseteq \tau$ we have that $l \in C_\alpha$, then:
   - Define $\Gamma_\tau^{X_\tau,A}(y) \downarrow= K(y)$ with $\gamma_\tau(y)$ fresh, and
   - Let $\tau\hat{\ }\langle 0\rangle$ be eligible to act next.

7. Otherwise. Then initialise any ξ with $\tau\hat{}\langle 0\rangle <_{\mathrm{L}} \xi$, and go to stage $s+1$.

Case 2. $\xi = \alpha$ is an $\mathcal{L}_e$-strategy for some e. Implement the following:

Program α

1. (Travel a Hard Link) If there is an i such that a hard link (τ_i, α) has been created and has not been cancelled and not been travelled, then let j be the least such i, and let s^- be the stage at which the current hard link (τ_j, α) was created. Suppose that $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$ is the current cycle of α. There are two cases:

   Case 1a. (Successful Travel)

   $$Y_{\tau_j, s^-} \upharpoonright (\phi_{\tau_j}(d(\tau_j, \alpha, \vec{c}))[s^-] + 1) \neq Y_{\tau_j} \upharpoonright (\phi_{\tau_j}(d(\tau_j, \alpha, \vec{c}))[s^-] + 1).$$

   Then

   - Set $\widehat{\Gamma}_{\beta_j}^{Y_{\tau_j}, A}(k_j^*)$ to be undefined,
   - If there is no hard link (τ_i, α) currently existing, then go to step 12,
   - Otherwise, initialise any ξ with $\alpha\hat{}\langle -1\rangle <_{\mathrm{L}} \xi$, and go to stage $s+1$.

   Case 1b. (Unsuccessful Travel) Otherwise:

   - Set $\Gamma_{\tau_j}^{X_{\tau_j}, A}(k_j)$, where k_j is the threshold of β_j for destroying Γ_{τ_j},
   - For any $i > j$, if a hard link (τ_i, α) exists, then cancel the hard link (τ_i, α), and
   - Let α_j be the longest $\mathcal{L}$-strategy $\subset \beta_j$, let $\vec{c}_j$ be the current cycle of α_j, set $\alpha \leftarrow \alpha_j$ and let α go to step 5.

2. (Travel a Soft Link) If there is a soft link (τ_i, α) in existence, which has neither been cancelled nor travelled, then let j be the least such i, and travel the soft link (τ_j, α) by allowing $\delta_j\hat{}\langle(\widehat{c}, e_j)\rangle$ to be eligible to act next.

3. If either

   (3a) $n(\alpha) = 0$ and s is not α-expansionary, or

   (3b) $n(\alpha) \neq 0$, $\vec{c}(\alpha) \downarrow = \vec{c}$, $x(\vec{c}, \alpha) \downarrow = x$, but $\Psi_\alpha^A(x)$ is not defined via an α-believable computation, or

   (3c) $n(\alpha) \neq 0$, $\vec{c}(\alpha) \uparrow$, and s is not α-expansionary, then:

   - Let $\alpha\hat{}\langle 2\rangle$ be eligible to act next.

4. • If $n(\alpha) = 0$, then go to step 12.

   • If $n(\alpha) \neq 0$ and $\vec{c}(\alpha) \uparrow$, then go to step 11.

   Suppose that $\vec{c}(\alpha) \downarrow = \vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$. We say that the cycle $\vec{c}$ of α *requires β_j-attention at stage s*, if $\Psi_\alpha^A(x(\vec{c}, \alpha)) \downarrow$ via an α-believable computation and one of the following cases occurs:

   Case 1. For every $j \in \{1, 2, \cdots, n\}$, $\psi_\alpha(x(\vec{c}, \alpha)) < \gamma_j^*(k_j^*)$.

   In this case, we say that cycle $\vec{c}$ of α is *ready at stage s*.

   Case 2. There are $i < j$ such that both $\gamma_i^*(k_i^*)$ and $\gamma_j^*(k_j^*)$ are defined, and such that $\gamma_i^*(k_i^*) \geq \gamma_j^*(k_j^*)$.

   In this case, we say that cycle $\vec{c}$ of α *requires well-ordering at stage s*.

   Case 3. Cycle $\vec{c}$ of α requires a switch of strategy at stage s.

   We say that cycle $\vec{c}$ of α *requires τ_j-attention at stage s*, if either the cycle $\vec{c}$ of α requires τ_j-destruction at stage s, or the cycle $\vec{c}$ of α requires τ_j-honestification at stage s.

   Case 4. There is a j such that the cycle $\vec{c}$ of α requires τ_j-attention at stage s.

5. If $\vec{c}$ is ready at stage s, then go to step 12.

6. If the cycle $\vec{c}$ of α requires well-ordering at stage s, then:

   • Let j be the greatest number for which there is an $i < j$ such that both $\gamma_i^*(k_i^*)$ and $\gamma_j^*(k_j^*)$ are defined, and such that $\gamma_i^*(k_i^*) \geq \gamma_j^*(k_j^*)$, and

   • For every $i \geq j$, if $\gamma_i^*(k_i^*)$ is defined, enumerate $\gamma_i^*(k_i^*)$ into A, and

   • Initialise all ξ with $\delta_j <_{\mathrm{L}} \xi$, and go to stage $s + 1$.

7. (Create Hard Links) If the cycle $\vec{c}$ of α requires a switch of strategy at stage s, then:

   • Let j be the greatest i such that β_i is 1-active at α, and $\psi_\alpha(x(\vec{c}, \alpha)) \geq \gamma_i^*(k_i^*)$, and

   • For every $i \leq j$, in decreasing order of i, if β_i is 1-active at α, then enumerate d_i into D, and create a hard link (τ_i, α), and

   • Initalise any strategy ξ with $\alpha\hat{\ }\langle -1 \rangle <_{\mathrm{L}} \xi$, and go to stage $s + 1$.

8. (Create a Soft Link) If there is an i such that cycle $\vec{c}$ of α requires τ_i-attention at stage s, then let j be the greatest such i, and let $\vec{c}$ require τ_j-attention via the following cases:

Case 8a. $\vec{c}$ requires τ_j-honestification at stage s. Then:

- For any $i \geq j$, if $\gamma_i^*(k_i^*)$ is defined, then enumerate it into A, and
- Create a soft link (τ_j, α), initialise any ξ with $\delta_j <_{\mathrm{L}} \xi$, and go to stage $s+1$.

Case 8b. Otherwise. Then go on to the next step.

9. Suppose that the cycle $\vec{c}$ of α requies τ_j-destruction at stage s. Then let it receive τ_j-destruction as follows:

Step 9a. (Rectify $\widehat{\Gamma}_{\delta_j}$) If there is an x such that $\widehat{\Gamma}_{\delta_j}^{Y_{\tau_j},A}(x) \downarrow \neq K(x)$, then:

- Let y be the least such x, and enumerate both $\widehat{\gamma}_{\delta_j}(y)$ and $\gamma_{\tau_j}(k_j^*)$ into A, and
- Initialise any ξ with $\delta_j\hat{\ }\langle(c, e_j)\rangle <_{\mathrm{L}} \xi$, and go to stage $s+1$.

Step 9b. (Harrington Honestification) If there is a strategy γ and a cycle $\vec{c}'$ of γ such that $\delta_j\hat{\ }\langle(c, e_j)\rangle$ is 1-active at γ, $d(\tau_j, \gamma, \vec{c}') \downarrow = d$ (say), and such that $\gamma_{\tau_j}(k_j^*) \leq \phi_{\tau_j}(d)$, or there is an x such that $\gamma_{\tau_j}(k_j^*) \leq \widehat{\gamma}_{\delta_j}(x)$, then:

- For every $i \geq j$, if $\gamma_{\tau_i}(k_i^*)$ is defined, then enumerate it into A, and
- Create a soft link (τ_j, α), initialise any ξ with $\delta_j <_{\mathrm{L}} \xi$, and go to stage $s+1$.

Slow Down: Steps 9c–9e

Step 9c. Then:

- Let y be the least x such that $\widehat{\Gamma}_{\delta_j}^{Y_{\tau_j},A}(x) \uparrow$,
- Let b be the boundary of cycle k_j^* of α for Γ_{τ_j},
- Let m be the number of times that $\widehat{\Gamma}_{\delta_j}^{Y_{\tau_j},A}(y)$ has been defined since the cycle k_j^* was last either reset or woken up, and
- Let $l = y + b + m$.

Step 9d. (Build $\widehat{\Gamma}_{\delta_j}$) If for every β with $\beta\hat{\ }\langle -1\rangle \subseteq \alpha$ we have that $l \in C_\beta$, then:

- Define $\widehat{\Gamma}_{\delta_j}^{Y_{\tau_j},A}(y) \downarrow = K(y)$ with $\widehat{\gamma}_{\delta_j}(y)$ fresh, and
- Let $\delta_j\hat{\ }\langle(c, e_j)\rangle$ be eligible to act next.

Step 9e. (Delayed Action) Otherwise. Then initialise any ξ for which $\delta_j <_{\mathrm{L}} \xi$, and go to stage $s+1$.

10. Otherwise. Then initialise all ξ with $\alpha\hat{}\langle 2\rangle <_{\mathrm{L}} \xi$, and go to stage $s+1$.

11. (Opening Cycles) If $\vec{c}(\alpha)$ is not defined, then:
    - Define $\vec{c}(\alpha) = \vec{c}$ according to definition 6.10,
    - Define a witness $x(\vec{c}, \alpha)$ to be the least x which is not in C_α, and
    - Initialise any ξ with $\alpha\hat{}\langle 2\rangle \leq \xi$, and go to stage $s+1$.

12. Then:
    - If $n(\alpha) \neq 0$, enumerate $x(\vec{c}, \alpha)$ into C_α,
    - If $n(\alpha) = 0$, enumerate the least element which is not in C_α into C_α, and
    - Let $\alpha\hat{}\langle -1\rangle$ be eligible to act next.

This completes the description of the construction.

8 The Verification

In this section, we verify that the construction satisfies all of the requirements. We first list some basic facts concerning the construction.

8.1 Proposition (Basic Facts Proposition). Given a stage s:

(i) If a hard link (τ, α) is created during stage s, then there is no element which is enumerated into A during stage s, and there is at most one $\mathcal{L}$-strategy α such that hard links of the form (τ, α) are created during stage s.

(ii) It is impossible to create both a hard link and a soft link during stage s, and there is at most one soft link (τ, α) for a pair τ, α which is created during stage s.

(iii) For an $\mathcal{L}$-strategy α, if both α and $\delta_j\hat{}\langle(\widehat{c}, e_j)\rangle$ are visited at stage s, then a soft link (τ_j, α) is travelled at stage s.

Proof. (i). If an element, a say, is enumerated into A at stage s, then at the last substage of stage s, one of the following cases occurs:

Case 1a. An implementation of step 4 of program τ for some $\mathcal{R}$-strategy τ.

Case 1b. An implementation of step 6 of cycle $\vec{c}$ of α for some $\vec{c}$ and some $\mathcal{L}$-strategy α.

Case 1c. An implementation of case 8a of cycle $\vec{c}$ of α for some $\vec{c}$ and some $\mathcal{L}$-strategy α.

Case 1d. An implementation of step 9a of cycle $\vec{c}$ of α for some $\vec{c}$ and some $\mathcal{L}$-strategy α.

Case 1e. An implementation of step 9b of cycle $\vec{c}$ of α for some $\vec{c}$ and some $\mathcal{L}$-strategy α.

However, if a hard link (τ, α) for some τ, α is created during stage s, then at the last substage of stage s, step 7 of cycle $\vec{c}$ of α for some $\vec{c}$ and some $\mathcal{L}$-strategy α occurs. Therefore it is impossible that a hard link is created during stage s, and at the same stage s, an element is enumerated into A.

And by the construction, if a hard link (τ, α) is created at stage s, then all the hard links created during stage s are of the form (τ_j, α) for the same α and for some j. (i) follows.

(ii). If a soft link (τ, α) is created during stage s, then at the last substage of stage s, one of the following cases occurs:

Case 2a. An implementation of case 8a of cycle $\vec{c}$ of α for some $\vec{c}$ and some $\mathcal{L}$-strategy α.

Case 2b. An implementation of step 9b of cycle $\vec{c}$ of α for some $\vec{c}$ and some $\mathcal{L}$-strategy α.

So step 7 of cycle $\vec{c}$ of α for any $\vec{c}$ and any $\mathcal{L}$-strategy α does not occur at the last substage of stage s.

And by the construction, in either case 2a or case 2b, there is at most one soft link which is created during stage s. (ii) follows.

(iii). By the assumption that α is visited at stage s, we implement program α. We then observe that $\delta_j\hat{\ }\langle(\widehat{c}, e_j)\rangle$ is visited at stage s if and only if step 2 of program α occurs. (iii) follows. $\square$

It is convenient for the verification to introduce the following:

8.2 Definition. (i) Given a strategy ξ, we say that a 1-node β is *valid at* ξ, if and only if:

(a) $\beta\hat{\ }\langle(c, e)\rangle \subseteq \xi$ for some e, and

(b) For the e in (a), there is no γ such that $\beta\hat{\ }\langle(c, e)\rangle \subseteq \gamma\hat{\ }\langle(\widehat{c}, e')\rangle \subseteq \xi$ for any $e' \leq e$.

(ii) Assume given an $\mathcal{L}$-strategy α. Assume the notation in definition 5.6. Let $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$ be the current cycle of α, let $x(\vec{c}, \alpha) = x$ be the current witness of cycle $\vec{c}$ of α, and let $[t, s]$ be a state.

We say that δ_j is *well defined* at $[t, s]$, if:

(a) β_j is 1-active at α,

(b) $\Psi_\alpha^A(x)[t, s] \downarrow$ via an α-believable computation,

(c) For any $i > j$, if β_i is 0-active at α, then $\psi_\alpha(x)[t, s] < \gamma_{\tau_i}(k_i^*)$, and

(d) For any $i > j$, if β_i is 1-active at α, then $\widehat{\gamma}_{\beta_i}(k_i^*) > \phi_{\tau_i}(d(\tau_i, k_i^*, \vec{c}))$.

We now prove some basic properties of the construction.

8.3 Proposition. (Basic Properties Proposition) Given a state $[t, s]$:

(i) If an $\mathcal{R}$-strategy τ is visited at substage t of stage s, then for any 1-node β, if β is valid at τ, then β is well defined at state $[t, s]$.

(ii) If an $\mathcal{L}$-strategy α is visited at substage t of stage s, and there is no hard link and no soft link (τ, α) which is travelled at substage t of stage s, then for any 1-node β, if β is valid at α, then β is well defined at state $[t, s]$.

(iii) If a soft link (τ_j, α) is travelled at substage t of stage s, then for any β, if β is valid at $\delta_j\hat{\ }\langle(\widehat{c}, e_j)\rangle$, then β is well defined at state $[t, s]$.

(iv) If a hard link (τ_j, α) is created at substage t of stage s, then for any β, if β is valid at α, then β is well defined at both $[t, s]$ and at the end of stage s.

(v) If a hard link (τ_j, α) is travelled successfully at substage t of stage s, then for any β, if β is valid at α, then β is well defined at $[t, s]$.

(vi) If a hard link (τ_j, α) is travelled unsuccessfully at substage t of stage s, then for any β, if β is valid at β_j, then β is well defined at $[t, s]$.

Proof. Suppose to the contrary that s is the least stage at which one of the clauses (i)–(vi) fails to hold at a substage of that stage. Let t be the least substage of stage s at which one of (i)–(vi) fails to hold. There are 6 corresponding cases to consider:

Case 1. (i) fails to hold at state $[t, s]$.

First we prove that at substage $t - 1$ of stage s, there is no $\mathcal{L}$-strategy α such that step 1 of cycle $\vec{c}$ of α for some cycle $\vec{c}$ of α occurs, unless step 12 of program α occurs at substage $t - 1$ of stage s.

Suppose to the contrary that step 1 of program α occurs but step 12 of program α does not occur at substage $t - 1$ of stage s. By the construction at substage $t - 1$ of stage s, either $t - 1$ is the last substage of stage s, or an $\mathcal{L}$-strategy is visited via step 9d at substage t of stage s. In either case, τ is not visited at substage t of stage s.

Hence there are just three subcases:

Subcase (1,a). There is a cycle $\vec{c}$ and an $\mathcal{L}$-strategy α such that one of the following steps of cycle $\vec{c}$ of α occurs at substage $t - 1$ of stage s.

(A). Step 2.

(B). Step 3.

(C). Step 9d.

(D). Step 12.

By the construction, in any of these cases, there is no element which is enumerated into A during substage $t - 1$ of stage s, and τ^- is visited at substage $t - 1$ of stage s, where τ^- is the longest strategy $\subset \tau$. We now consider the following subsubcases (for $\alpha = \tau^-$):

Subsubcase (1,a,A). (A) occurs.

In this subsubcase, $\tau = \alpha\hat{}\langle(i_n, k_n^*, e_n)\rangle\hat{}\cdots\hat{}\langle(i_j, k_j^*, e_j)\rangle\hat{}\langle(\widehat{c}, e_j)\rangle$ for some j, and a soft link (τ_j, α) is travelled at substage $t-1$ of stage s.

By definition 8.2, if β is valid at τ, then β is valid at τ_j, by (iii), at substage $t-2$ of stage s. By the construction, there is no element which is enumerated into A during substages $t-2$ or $t-1$ of stage s, so that β is well defined at $[t, s]$. So (i) holds in subsubcase (1,a,A).

Subsubcase (1,a,B). (B) occurs.

In this subsubcase, $\tau = \alpha\hat{}\langle 2\rangle$, and by definition 8.2, given any β, β is valid at τ if and only if β is valid at α.

As in the argument given in the first paragraph of Case 1, we do not travel any hard link (τ_j, α) at substage $t-1$ of stage s (remembering that subsubcase (1,a,A) does not hold). And by the construction, there is no element which is enumerated into A during substage $t-1$ of stage s. So (i) at $[t, s]$ follows from (ii) at $[t-1, s]$.

Subsubcase (1,a,C). (C) occurs.

By the construction, in this subsubcase there is no soft link which is travelled at substage $t-1$ of stage s. Therefore there are only two possibilities.

The first of these is that $\tau^- = \alpha_j$, and a hard link (τ_j, α) is travelled unsuccessfully at substage $t-2$ of stage s for some α_j such that $\tau_j \subset \alpha_j \subset \beta_j \subset \alpha$, and that $\tau <_{\mathrm{L}} \beta_j$. Now if $\beta \subset \tau$ and β is valid at τ, then β is valid at β_j. By the choice of $[t, s]$, and by (vi) at $[t-2, s]$, and by the construction, there is no element which is enumerated into A during substages $t-2, t-1$ of stage s. So β is well defined at $[t, s]$. If $\beta = \tau$, then by the construction α_j receives τ'-destruction at substage $t-1$ of stage s for some τ'. Then by definition 6.7 and by the construction at substage $t-2$ of stage s, β is well defined at $[t, s]$.

The second possibility is that we travelled no hard link during stage $t-2$ of stage s. In this case, for any $\beta \subset \tau$, if β is valid at τ, then by (ii) at $[t-1, s]$ and by the construction at substage $t-1$ of stage s, β is well defined at $[t, s]$. For $\beta = \tau$, by the construction at substage $t-1$ of stage s, and by definition 6.7, β is again well defined at $[t, s]$.

It follows that (i) holds in subsubcase (1,a,C).

Subsubcase (1,a,D). (D) occurs.

If there is no hard link travelled at substage $t-1$ of stage s, then (i) at $[t, s]$ follows from (ii) at $[t-1, s]$.

Otherwise, for $\tau^- = \alpha$, there is a j such that a hard link (τ_j, α) is travelled successfully at substage $t-1$ of stage s. So (i) at $[t, s]$ follows from (v) at $[t-1, s]$.

It now follows that (i) holds in subcase (1,a).

Subcase (1,b). $\tau^- = \alpha$ is an $\mathcal{R}$-strategy and either step 1 or step 6 of program α occurs at substage $t-1$ of stage s.

By the construction, there is no element which is enumerated into A during substage $t-1$ of stage s. So (i) at $[t,s]$ follows from (i) at $[t-1,s]$.

Subcase (1,c). Otherwise.

In this subcase, $\tau = \lambda\hat{}\langle 0\rangle$, where λ is the root node. Therefore there is no 1-node which is valid at τ, so (i) holds at $[t,s]$.

Therefore, in any subcase, (i) holds at $[t,s]$, and Case 1 does not occur.

Case 2. (ii) fails to hold at $[t,s]$.

Subcase (2,a). $\alpha^- = \tau$ is an $\mathcal{R}$-strategy and τ is visited at substage $t-1$ of stage s.

By the construction, there is no element which is enumerated into A during substage $t-1$ of stage s. By definition 8.2, β is valid at τ if and only if β is valid at α. Therefore (ii) follows from (i) at $[t-1,s]$.

Subcase (2,b). Otherwise. Then there is an $\mathcal{L}$-strategy α' such that α' is visited at substage $t-1$ of stage s. Examining program α' in the construction, we see that one of the following events occurs at substage $t-1$ of stage s:

(A). Both step 12 and case 1a of step 1 of program α'.

(B). Case 1b of step 1 of program α' occurs, and $\alpha = \alpha_j$ is defined.

(C). Step 2 of program α'.

(D). Step 3 of program α'.

(E). step 12 of program α'.

(F). step 9d of program α'.

We consider these 6 subsubcases in turn.

Subsubcase (2,b,A). (A) occurs.

Now $\alpha = \alpha'\hat{}\langle -1\rangle$. By the construction, there is no element which is enumerated into A during substage $t-1$ of stage s. And by definition 8.2, β is valid at α if and only if β is valid at α'. So the required instance of (ii) follows from (ii) at $[t-1,s]$.

Subsubcase (2,b,B). (B) occurs.

Suppose that a hard link (τ_j, α') is travelled unsuccessfully at substage $t-1$ of stage s, and $\alpha = \alpha_j$. By the construction, there is no element which is enumerated into A during substage $t-1$ of stage s, so that (ii) at $[t,s]$ follows from (i) at $[t-1,s]$.

Subsubcase (2,b,C). (C) occurs.

Suppose that a soft link (τ_j, α') is travelled at substage $t-1$ of stage s. Then $\alpha = \delta_j\hat{}\langle(\widehat{c}, e_j)\rangle$ for some δ_j and some e_j. For any β, if β is valid at α, then α is valid at τ_j. By the construction, there is no element which is enumerated into A during substages $t-2$, $t-1$ of stage s. So (ii) at $[t,s]$ follows from (i) at $[t-2,s]$.

Subsubcase (2,b,D). (D) occurs.

Now in this case we have $\alpha = \alpha'\hat{}\langle 2\rangle$. By definition 8.2, β is valid at α if and only if β is valid at α'. By the construction, there is no element which is enumerated into A during substage $t-1$ of stage s. So (ii) at $[t,s]$ follows from (ii) at $[t-1,s]$.

Subsubcase (2,b,E). (E) occurs.

The proof for this case is the same as that for subsubcase (2,b,A).

Subsubcase (2,b,F). (F) occurs.

Let $\alpha = \alpha'\hat{}\langle(i_n', k_n', e_n')\rangle\hat{}\cdots\hat{}\langle(i_j', k_j', e_j')\rangle\hat{}\langle(c, e_j')\rangle$ for some j. If $\beta \subseteq \alpha'$ and β is valid at α', then by (ii) at $[t-1,s]$ and by the construction at substage $t-1$ of stage s, β is well defined at $[t,s]$. If $\beta \subseteq \alpha$ and $\beta \not\subseteq \alpha'$, and β is valid at α, then by definition 6.7 and by the construction at substage $t-1$ of stage s, β is well defined at $[t,s]$. So (ii) holds at $[t,s]$.

It follows that in any case, (ii) holds at $[t,s]$, and Case 2 does not occur.

Case 3. (iii) fails to hold at $[t,s]$.

This cannot hold, by (i) at $[t-1,s]$ and by the construction at substage $t-1$ of stage s.

Case 4. (iv) fails to hold at $[t,s]$.

Let α' be the strategy which is visited at substage $t-1$ of stage s. Then there are two subcases:

Subcase 4a. $\alpha' = \tau$ is an $\mathcal{R}$-strategy.

Then either (A) or (B) below applies:

(A). Step 1 of program τ occurs and $\alpha = \tau\hat{}\langle 1\rangle$.

(B). Step 6 of program τ occurs and $\alpha = \tau\hat{}\langle 0\rangle$.

Subsubcase (4,a,A). (A) occurs.

By definition 8.2, β is valid at α if and only if β is valid at τ, and by the construction, there is no element which is enumerated into A at substage $t-1$ of stage s. So (iv) at $[t,s]$ follows from (i) at $[t-1,s]$.

Subsubcase (4,a,B). (B) occurs.

This is covered by the same argument as that for subsubcase (4,a,A).

Hence (iv) at $[t,s]$ holds in subcase 4a.

Subcase 4b. α' is an $\mathcal{L}$-strategy.

In this subcase, one of (A)–(F) below applies:

(A). Case 1a of step 1 of program α' occurs, $\alpha = \alpha'\hat{}\langle -1\rangle$, and step 12 of program α' occurs at substage $t-1$ of stage s.

(B). Case 1b of step 1 of program α' occurs, and α is the α_j defined in case 1b of step 1 of program α'.

(C). A soft link (τ_j, α') is travelled at substage $t-2$ of stage s, and $\alpha = \alpha'\hat{}\langle a\rangle\hat{}\langle(\widehat{c}, e_j^{'})\rangle$ for some a and some $e_j^{'}$.

(D). Step 3 of program α' occurs and $\alpha = \alpha'\hat{}\langle 2\rangle$.

(E). Step 9d of program α' occurs, and $\alpha = \alpha'\hat{}\langle a\rangle\hat{}\langle(c, e_j^{'})\rangle$ for some a and some $e_j^{'}$.

(F). Step 12 of program α' occurs and $\alpha = \alpha'\hat{}\langle -1\rangle$.

Subsubcase (4,b,A). (A) occurs.

By the construction, there is no element which is enumerated into A during substage $t-1$ of stage s. By definition 8.2, β is valid at α if and only if β is valid at α'. Therefore (iv) at $[t, s]$ follows from (iv) at $[t-1, s]$.

Subsubcase (4,b,B). (B) occurs.

By the construction, there is no element which is enumerated into A during substage $t-1$ of stage s. So (iv) at $[t, s]$ follows from (vi) at $[t-1, s]$.

Subsubcase (4,b,C). (C) occurs.

In this case (iv) at $[t, s]$ follows from (i) at $[t-2, s]$ and from the construction at substages $t-2, t-1$ of stage s.

Subsubcase (4,b,D). (D) occurs.

This is the same as subsubcase (4,b,A).

Subsubcase (4,b,E). (E) occurs.

Now (iv) at $[t, s]$ follows from (iv) at $[t-1, s]$, and from the construction at substage $t-1$ of stage s, and from definition 6.7.

Subsubcase (4,b,F). (F) occurs.

The proof for this subsubcase is the same as that for subsubcase (4,b,A).

Therefore in any case, (iv) holds at $[t, s]$, and Case 4 does not occur.

Case 5. (v) fails to hold at $[t, s]$.

Suppose that a hard link (τ_j, α) is travelled successfully at substage t of stage s. Let v_0 be the stage at which the current hard link (τ_j, α) was created.

By the construction, for every $i \leq j$, if β_i is 1-active at α, then a hard link (τ_i, α) was created at stage v_0. Suppose that $(\tau_1, \alpha), \cdots, (\tau_j, \alpha), \cdots$ are all hard links created at stage v_0.

For every i, if $1 \leq i \leq j$, suppose that the hard link $(\tau_i, \alpha)[v_0]$ was travelled at substage t_i of stage v_i. Then $v_j = s$, $t_j = t$. By the construction, all these hard links were travelled successfully, since otherwise the hard link (τ_j, α) would already have been cancelled.

We now note that by (iv) at stage v_0, for any β, if β is valid at α, then β is well defined during and at the end of stage v_0.

We need the following:

8.4 Lemma. (i) There is no $\mathcal{R}$-strategy τ with $\tau \subset \tau\hat{}\langle 0\rangle \subseteq \tau_1$, and $\tau\hat{}\langle 0\rangle$ active at α_1, but not active at α.

(ii) There is no 1-node $\beta \subseteq \tau_1$ which is active at α_1, but not active at α.

(iii) For any $\mathcal{R}$-strategy τ, if $\tau \subset \tau\hat{}\langle 0\rangle \subseteq \tau_1$ and $\tau\hat{}\langle 0\rangle$ is active at α, then $\gamma_\tau(k_\tau)$ is undefined during and at the end of stage v_0, where k_τ is the threshold of cycle $\vec{c}$ of α for Γ_τ.

Proof. For (i): Suppose to the contrary that (i) fails to hold. Then there is an $\mathcal{L}$-strategy α' such that:

(a) $\beta_1 \subseteq \alpha' \subset \alpha'\hat{}\langle a\rangle\hat{}\langle (b, e)\rangle \subseteq \alpha$ for some a and some $b \in \{c, \widehat{c}\}$, and for $e = e(\tau)$, and

(b) α' is destroying the p.c. functional Γ_τ.

By proposition 5.7, $e < e(\tau_1)$. Therefore β_1 is no longer 1-active at α, and we never create a hard link (τ_1, α) during the course of the construction, contradicting the choice of s.

So (i) follows.

And (ii) follows by the proof of (i).

For (iii): Suppose to the contrary that there is an $\mathcal{R}$-strategy τ such that $\tau \subset \tau\hat{}\langle 0\rangle \subseteq \tau_1$, $\tau\hat{}\langle 0\rangle$ is active at α and $\gamma_\tau(k_\tau)$ is defined during stage v_0, where k_τ is the threshold of cycle $\vec{c}$ of α for Γ_τ.

By the choice of v_0, α was visited at stage v_0, and hard links (τ_1, α), (τ_2, α), $\cdots, (\tau_j, \alpha), \cdots$ were created at stage v_0. That is to say, step 7 of program α occurs at stage v_0.

By the well-ordering in step 6, we have that $\gamma_\tau(k_\tau)\downarrow < \gamma^*_{\beta_1}(k_1^*)$, where k_1^* is the threshold of cycle $\vec{c}$ of α for the p.c. functional $\widehat{\Gamma}_{\beta_1}$. Because a hard link (τ_1, α) is created at stage v_0, we have that $\gamma^*_{\beta_1}(k_1^*) \leq \psi_\alpha(x)[v_0]$, where x is the witness of cycle $\vec{c}$ of α. By definition 6.8, cycle $\vec{c}$ of α does not require a switch of strategy at stage v_0, contradicting the choice of v_0.

Lemma 8.4 follows. □.

We now verify that at $[t_1, v_1]$, we have:

8.5 Lemma. (i) For any β, if $\tau_1 \subseteq \beta \subseteq \alpha$, and β is valid at α, then for $\alpha_0 = \beta^-$, we have:

- β is well defined during and at the end of substage t_1 of stage v_1,
- If x_0 is the witness for the current cycle of α_0, then

$$A_{v_0} \upharpoonright (\psi_{\alpha_0}(x_0)[v_0] + 1) = A_{t_1,v_1} \upharpoonright (\psi_{\alpha_0}(x_0)[v_0] + 1).$$

(ii) For any $\beta \subseteq \tau_1$, if β is valid at α, then β is well defined at $[t_1, v_1]$.

(iii) $\widehat{\gamma}_{\beta_1}(k_1^*)$ is set to be undefined at stage v_1.

(iv) If $1 < j$, then there is no element which is enumerated into A at a substage $t > t_1$ of stage v_1.

Proof. For (i): By proposition 8.1 (i), there is no element which is enumerated into A during stage v_0. By (iv) at the end of stage v_0, by the hard link $(\tau_1, \alpha)[v_0]$ and by lemma 8.4, (i) holds immediately before the hard link $(\tau_1, \alpha)[v_0]$ is travelled at stage v_1. And by the construction at substage t_1 of stage v_1, there is no element which is enumerated into A during substage t_1 of stage v_1, so that (i) follows.

For (ii): This follows from (i) at substage $t_1 - 1$ of stage v_1 and from the construction at substage t_1 of stage v_1.

For (iii): By the construction at substage t_1 of stage v_1, $\widehat{\gamma}_{\beta_1}(k_1^*)$ is set to be undefined at substage t_1 of stage v_1. (iii) immediately follows.

For (iv): By the construction, if $1 < j$, then substage t_1 is the last substage of stage v_1, and (iv) follows. □

Suppose we have inductively that for $i \leq j$:

(a) For any β, if $\tau_i \subseteq \beta \subseteq \alpha$, and β is valid at α, then for $\alpha_0 = \beta^-$, we have that:

- β is well defined during and at the end of substage t_i of stage v_i, and
- If x_0 is the witness for the current cycle of α_0, then

$$A_{v_0} \upharpoonright (\psi_{\alpha_0}(x_0)[v_0] + 1) = A_{t_i,v_i} \upharpoonright (\psi_{\alpha_0}(x_0)[v_0] + 1).$$

(b) For any $\beta \subseteq \tau_i$, if β is valid at α, then β is well defined during and at the end of substage t_i of stage v_i.

(c) $\widehat{\gamma}_{\beta_i}(k_i^*)$ is undefined at the end of substage t_i of stage v_i.

(d) If $i < j$, then substage t_i is the last substage of stage v_i.

With these conditions for i, the same argument as that in lemma 8.4 and lemma 8.5 establishes properties (a)–(d) for $i + 1$.

Now if $i = j$, then (v) at $[t, s]$ is established. So (v) holds at $[t, s]$, and Case 5 does not occur.

Case 6. (vi) fails to hold at $[t, s]$.

By the same aggument as that for case 5, we have that (a)–(d) above hold at $[t_{j-1}, v_{j-1}]$. By the proof of lemma 8.4 and lemma 8.5, (vi) holds immediately before the hard link (τ_j, α) is travelled unsuccessfully at substage

t of stage s. So by the construction at substage $t-1$ of stage s, (vi) holds at $[t, s]$.

Therefore, case 6 does not occur.

It follows that in each of the cases considered we get a contradiction.

Proposition 8.3 follows. □

We note that sometimes it is not necessary to distinguish between hard and soft links, in which case we simply refer to each as *links*.

We need the following:

8.6 Definition. (i) Let δ_s be the last node to have been visited at stage s.

(ii) Define the *true path* TP *of the construction* to be

$$TP = \liminf_s \delta_s.$$

8.7 Proposition. (True Path TP Proposition) For every $\alpha \in TP$, there is a possible outcome a of α such that:

(i) $\alpha\hat{\ }\langle a\rangle \in TP$.

(ii) $\alpha\hat{\ }\langle a\rangle \in TP$ is visited infinitely often.

(iii) $\alpha\hat{\ }\langle a\rangle \in TP$ is initialised only finitely many times.

(iv) If either $\alpha\hat{\ }\langle 1\rangle \in TP$ or $\alpha\hat{\ }\langle 2\rangle \in TP$, then α acts only finitely many times.

(v) If an $\mathcal{R}$-strategy α such that $\alpha\hat{\ }\langle 0\rangle \in TP$, then:

(a) Step 6 of program α occurs infinitely many times, and

(b) Any link (α, β) which is created will be either cancelled or travelled.

(vi) If α is an $\mathcal{L}$-strategy and $\alpha\hat{\ }\langle -1\rangle \in TP$, then:

(a) Step 12 of program α occurs infinitely often, and

(b) $C_\alpha = \omega$.

(vii) If α is an $\mathcal{L}$-strategy and $\gamma_j = \delta_j\hat{\ }\langle (a, e_j)\rangle \in TP$ for some $a \in \{c, \widehat{c}\}$, then:

(a) There are only finitely many stages at which step 7 of program α occurs, and

(b) $U(\gamma_j)[s]$ will be unbounded over the course of the construction.

Proof. By induction. For the root node $\lambda \in TP$, λ will never be initialised, and λ is visited at every stage $s > 0$. We consider two cases:

Case 1. There are only finitely many λ-expansionary stages.

By the assumption of this case, step 6 of program λ occurs only finitely many times. By definition 6.6 and definition 6.8 (vi), there are only finitely many stages at which either a hard or soft link (λ, α), some α with $\lambda\hat{}\langle 0\rangle \subseteq \alpha$, is created during the course of the construction.

Let s_0 be the least stage after which there is no λ-expansionary stage, and let s_1 be least $> s_0$ after which there is no link (λ, α) which is either created or travelled. By the choice of s_1, $\lambda\hat{}\langle 1\rangle$ will never be initialised after stage s_1, $\lambda\hat{}\langle 1\rangle$ will be visited at every stage $> s_1$, and λ will never act after stage s_1. So each of (i), (ii), (iii) and (iv) holds for λ.

Case 2. There are infinitely many λ-expansionary stages.

By the construction, $\lambda\hat{}\langle 0\rangle$ will never be initialised during the course of the construction. By definition 6.8 (vi), and by the construction, step 6 of program λ occurs infinitely many times, so that every hard or soft link (λ, α) which is created will be either cancelled or travelled. So (i), (ii), (iii) and (v) again hold for λ.

Assume inductively that the proposition holds for all $\alpha' \subset \alpha$, and that $\alpha \in TP$. We have:

1. α is visited infinitely often.

2. There is a stage s_0 with the following properties:

   (a) No $\beta \subseteq \alpha$ is initialised after stage s_0,

   (b) For all δ, if either $\delta\hat{}\langle 2\rangle \subseteq \alpha$ or $\alpha\hat{}\langle 1\rangle \subseteq \alpha$, then δ does not act after stage s_0,

   (c) For any $\mathcal{L}$-strategy $\alpha' \subset \alpha$, if $\alpha' \subset \delta\hat{}\langle (a, e)\rangle \subseteq \alpha$ for some $a \in \{c, \widehat{c}\}$, and some e, then step 7 of program α' will never occur after stage s_0.

3. If $\beta\hat{}\langle -1\rangle \subseteq \alpha$, then $C_\beta = \omega$.

4. If $\tau\hat{}\langle 0\rangle \subseteq \alpha$, then:

   (a) Step 6 of program τ occurs infinitely many times,

   (b) Every hard or soft link (τ, α) which is created will be either cancelled or travelled.

5. $U(\alpha)[s]$ will be unbounded during the course of the construction.

We now have two cases:

Case 1. $\alpha = \tau$ is an $\mathcal{R}$-strategy.

By the choice of s_0, and by the assumption of this case, the boundary $b(\tau)$ of τ has a finite limit, i.e. $\lim_s b(\tau)[s] = b(\tau)$ exists $< \omega$. Let s_1 be the stage at which $b(\tau)$ is defined. We consider two subcases.

Subcase 1a. There are only finitely many τ-expansionary stages.

Let s_2 be the least stage $> s_1$ after which there is no τ-expansionary stage. By the assumption of this subcase, step 6 of program τ occurs only finitely many times, and by definitions 6.6 and 6.8 and by the construction, there are only finitely many stages at which either a hard or a soft link (τ', α), some τ', α with $\tau' \subset \tau \subset \tau\hat{}\langle 0\rangle \subseteq \alpha$, is created. So we can choose a (least) stage $s_3 > s_2$ after which we neither create nor travel any link (τ', α) with $\tau' \subset \tau \subset \tau\hat{}\langle 0\rangle \subseteq \alpha$. By the choice of s_3, $\tau\hat{}\langle 1\rangle$ will never be initialised after stage s_3, $\tau\hat{}\langle 1\rangle$ is visited at every stage $s > s_3$ at which τ is visited, and τ will not act at any stage greater than s_3. (i), (ii), (iii) and (iv) immediately follow, and the proposition holds in this subcase.

Subcase 1b. Otherwise.

By the construction, it suffices to prove that step 6 of program τ occurs infinitely often. Suppose to the contrary that step 6 of program τ occurs only finitely many times.

Let s_2 be minimal $> s_1$ after which step 6 of program τ never occurs. By the construction and by definitions 6.6 and 6.8, there are only finitely many stages at which either a hard link or a soft link (τ', α), for some τ', α with $\tau' \subset \tau \subset \tau\hat{}\langle 0\rangle \subseteq \alpha$, is created. So let s_3 be minimal $> s_2$ after which no link (τ', α) with $\tau' \subset \tau \subset \tau\hat{}\langle 0\rangle \subseteq \alpha$ is either created or travelled.

By the choice of s_3, for any $s > s_3$, if s is τ-expansionary, then one of the steps 4, 5 or 7 of program τ occurs at stage s.

Since Γ_τ is finite, step 4 of program τ occurs only finitely many times. Let s_4 be minimal $> s_3$ after which step 4 of program τ never occurs.

Since Γ_τ is finite, let y be the least x for which $\Gamma_\tau^{X_\tau, A}(x)$ remains eventually undefined, and let m be the number of times that $\Gamma_\tau^{X_\tau, A}(y)$ has been defined since stage s_1. Then $l = y + m + b(\tau)$ is a fixed number.

By the inductive hypothesis, we can choose s_5 to be the least stage $> s_4$ for which, for any β and any s, if $\beta\hat{}\langle -1\rangle \subseteq \tau$ and $s > s_5$, then $l \in C_\beta[s]$.

By the assumption of this subcase, let s_6 be the least τ-expansionary stage which is greater than s_5. By the choice of s_5, and by the construction, step 6 of program τ occurs at stage s_6, contradicting the choice of s_2.

Therefore, in this subcase, step 6 of program τ occurs infinitely often, and so every link (τ, α) which is created will either be cancelled or travelled. Again, by the construction, (i), (ii), (iii) and (v) hold.

Case 2. α is an $\mathcal{L}$-strategy.

We first prove:

8.8 Lemma. If step 12 of program α occurs only finitely many times, then $\alpha\hat{\ }\langle -1\rangle$ is to the left of the true path TP.

Proof. By the assumption of the lemma, C_α is a finite set.

By definitions 6.6 and 6.8 and by the construction, there are only finitely many stages at which either a hard or a soft link (τ, β) for some τ, β, with $\tau \subset \alpha \subset \alpha\hat{\ }\langle -1\rangle \subseteq \beta$, is created. Let s_1 be minimal $> s_0$ after which there is no link (τ, β) with $\tau \subset \alpha \subset \alpha\hat{\ }\langle -1\rangle \subseteq \beta$ which is either created or travelled.

By the assumption of the lemma, let s_2 be the least stage $> s_1$ after which step 12 of program α never occurs. By the choice of s_2, there is no node $\beta \supseteq \alpha\hat{\ }\langle -1\rangle$ which is visited at a stage $s > s_2$. Therefore $\alpha\hat{\ }\langle -1\rangle <_{\mathrm{L}} TP$, and the lemma follows. □

8.9 Lemma. If $\alpha\hat{\ }\langle -1\rangle \in TP$, then (i), (ii), (iii) and (vi) hold.

Proof. We divide the proof into two cases.

Case 1. $n(\alpha) = 0$.

By the construction, for any s, if α is visited at stage s, then either step 3 or step 12 of program α occurs. If there are only finitely many α-expansionary stages, then the following argument indicates that $\alpha\hat{\ }\langle -1\rangle$ is to the left of the true path TP. By the construction and by definitions 6.6 and 6.8, there are only finitely many stages at which either a hard link or a soft link (τ, β), for some $\tau \subset \alpha\hat{\ }\langle -1\rangle \subseteq \beta$, is created during the course of the construction. Let $s_1 > s_0$ be minimal after which no link (τ, β), with $\tau \subset \alpha\hat{\ }\langle -1\rangle \subseteq \beta$, is either created or travelled. By the assumption of this subcase, let $s_2 > s_1$ be minimal after which there is no α-expansionary stage. By the choice of s_2, $\alpha\hat{\ }\langle 2\rangle$ will not be initialised, and will not act after stage s_2, and will be visited at every stage $s > s_2$ at which α is visited.

Therefore, there are infinitely many α-expansionary stages, and so by the construction, step 12 of program α occurs infinitely often. By the choice of s_0, $\alpha\hat{\ }\langle -1\rangle$ will not be initialised after stage s_0, so (i), (ii) and (iii) are immediate from the construction. We look at (vi).

By the definition of C_α, we only need to prove that C_α is a Δ^0_2 set. Suppose to the contrary that x is the least y such that y is enumerated into and extracted from C_α infinitely often. By the construction, if x is enumerated into C_α at a stage $s > s_0$, then the computation of $\Psi^A_\alpha(x)[s] \downarrow$ will not be injured by strategies $\xi \not\supseteq \alpha\hat{\ }\langle -1\rangle$. Since $n(\alpha) = 0$, if a strategy $\xi \supseteq \alpha\hat{\ }\langle -1\rangle$ enumerates an element $z \leq \psi_\alpha(x)[s]$ into A at a stage $v > s$, then z is the use function of some p.c. functional built by a strategy $\xi \supseteq \alpha\hat{\ }\langle -1\rangle$, say. By the slowing down of the definition of p.c. functionals, $z = \gamma^{(m)}(y)$ less than

$\psi_\alpha(x)$ only if $x > m + y + b$, where m is the number of times $\Gamma(y)$ has been defined, and b is the boundary of the strategy ξ. By the definition of the boundaries, for a fixed x, there are only finitely many triples (m, y, b) such that $x > m+y+b$ and such that $\gamma^{(m)}(y)$ is defined by a strategy below $\alpha\hat{\ }\langle -1\rangle$. Therefore, let $s_2 > s_1$ be minimal such that for any (m, y, b), if $x > m+y+b$ and $\gamma^{(m)}(y)$ is defined by a strategy below $\alpha\hat{\ }\langle -1\rangle$, then $\gamma^{(m)}(y)$ will never be enumerated into A after stage s_2. By the assumption, let $s_3 > s_2$ be minimal such that x is enumerated into C_α at stage s_3. Then by the choice of s_3, and by the definition of C_α, for any $s > s_3$ we have $x \in C_\alpha[s]$, contradicting the choice of x. (vi) follows. Hence the lemma holds in this case.

Case 2. $n(\alpha) \neq 0$.

(i) is obvious. (ii), (iii) follow from lemma 8.8 and from the construction. So it suffices to prove (vi). (vi)(a) follows from lemma 8.8, and we need only prove (vi)(b).

Suppose to the contrary that there is an x such that $x \notin C_\alpha$. Let x_0 be the least such x. By (vi)(a), and by the definition of C_α, x_0 is enumerated into C_α infinitely often.

By the same argument as above, and by the slowing down action of the $\mathcal{R}$-strategies and step 9d of program α for $\mathcal{L}$-strategy α, $\Psi^A_\alpha(x_0)$ is injured by use functions of p.c. functionals built by strategies below $\alpha\hat{\ }\langle -1\rangle$ only finitely many times. So let $s_1 > s_0$ be minimal after which $\Psi^A_\alpha(x_0)$ will never be injured by any use function of a p.c. functional built by a strategy below $\alpha\hat{\ }\langle -1\rangle$.

By the definition of C_α, if x is enumerated in C_α by some cycle $\vec{c}$ of α at a stage $s > s_1$, and cycle $\vec{c}$ is reset at a stage $v > s$, then x is extracted from C_α immediately and automatically.

By the way in which cycles $\vec{c}$ of α are opened, we can take $s_2 > s_1$ be the least such stage at which a permanent cycle $\vec{c} = (k^*_1, k^*_2, \cdots, k^*_n)$ of α is initiated. This means that for any $k < k^*_j$, and any $s > s_2$, $\gamma^*_{\beta_j}(k)$ will not be enumerated into A at stage s.

Clearly the witness of cycle $\vec{c}$ of α is x_0. Let $s_3 > s_2$ be the stage at which cycle $\vec{c}$ of α enumerates x_0 into C_α. By the construction, the computation of $\Psi^A_\alpha(x_0)[s_3]\downarrow= y$ for some y is α-believable, and for any j, $\gamma^*_{\beta_j}(k^*_j) > \psi_\alpha(x_0)[s_3]$. By the construction at stage s_3, any ξ to the right of $\alpha\hat{\ }\langle -1\rangle$ is initialised. Therefore by the choice of s_1, s_2, s_3, $\psi^A_\alpha(x_0)[s_3]\downarrow$ is preserved for ever and for any $s > s_3$, $x_0 \in C_\alpha[s]$, contradicting the choice of x_0. (vi) follows.

Lemma 8.9 follows. □

8.10 Lemma. If there are only finitely many α-expansionary stages, then:

(a) $\alpha\hat{\ }\langle 2\rangle \in TP$,

(b) $\alpha\hat{\ }\langle 2\rangle \in TP$ is visited infinitely many times,

(c) $\alpha\hat{}\langle 2\rangle \in TP$ is initialised only finitely many times, and

(d) α acts only finitely often.

Proof. By the assumption of the lemma, step 12 of program α occurs only finitely many times. Let s_1 be the least stage $> s_0$ for which there is no α-expansionary stage after stage s_1, and for which there is no node below $\alpha\hat{}\langle -1\rangle$ which is visited after stage s_1.

This means there is a cycle $\vec{c}$ of α which is permanent. Let $s_2 > s_1$ be minimal such that:

(1) A cycle $\vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$ of α is defined,

(2) There is no $k < k_j^*$, any j, for which $\gamma_{\beta_j}^*(k)$ is enumerated after stage s_2.

By definitions 6.6 and 6.8, and by the construction, there are only finitely many stages at which either a hard link or a soft link (τ, β), for some τ, β with $\tau \subset \alpha \subset \beta <_{\mathrm{L}} \alpha\hat{}\langle 2\rangle$, is created. Let $s_3 > s_2$ be the least such stage after which no link (τ, β), with $\tau \subset \alpha \subset \beta <_{\mathrm{L}} \alpha\hat{}\langle 2\rangle$, is either created or travelled.

By the choice of s_3, $\alpha\hat{}\langle 2\rangle$ will never be initialised after stage s_3, and for any $s > s_3$, if α is visited at stage s, then $\alpha\hat{}\langle 2\rangle$ is visited at stage s, and α will never act after stage s_3. So (a)–(d) hold, and the lemma follows. □

8.11 Lemma. If there are infinitely many α-expansionary stages, but $\alpha\hat{}\langle -1\rangle \notin TP$, then proposition 8.3 holds.

Proof. First we note that by the construction, $n(\alpha) \neq 0$. This is because, if $n(\alpha) = 0$, then $\alpha\hat{}\langle -1\rangle$ is visited at every α-expansionary stage s, so that $\alpha\hat{}\langle -1\rangle \in TP$.

Since $\alpha\hat{}\langle -1\rangle \notin TP$, step 12 of program α acts only finitely often, and so there is a fixed cycle $\vec{c}$ of α which is permanent and which acts infinitely often.

Let s_1 be minimal such that none of the steps 11, 12 of program α occurs after stage s_1.

We first prove:

8.12 Sublemma. There are only finitely many stages at which step 7 of program α occurs.

Proof. Suppose to the contrary that there are infinitely many stages at which step 7 of program α occurs.

Let s_2 be a stage $> s_1$ at which step 7 of program α occurs. Suppose that (τ_1, α), (τ_2, α), $\cdots$, (τ_m, α) are all hard links which are created at stage s_2. Let $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_m$. By the inductive hypothesis, and by the choice of s_1, each hard link (τ_j, α) will be travelled, and there is a j such that the hard link (τ_j, α) will be travelled unsuccessfully.

Suppose that j_0 is the least j such that there are infinitely many stages at which a hard link (τ_j, α) is travelled unsuccessfully.

By the choice of j_0, we can choose s_3 to be the least stage $> s_1$ after which, for all $j < j_0$, any hard link (τ_j, α) will be travelled successfully.

We now examine β_{j_0}. Let α_{j_0} be the longest $\mathcal{L}$-strategy $\subset \beta_{j_0}$.

Suppose that $\beta_1^0 \subset \beta_2^0 \subset \cdots \subset \beta_l^0$ lists every β active at α_{j_0}. Let $\vec{c}_{j_0} = (k_1^0, k_2^0, \cdots, k_l^0)$ be the current cycle of α_{j_0}. Let τ_j^0 be the longest $\mathcal{R}_{\Phi,X,Y}$-strategy $\subset \beta_j^0$ for $e = E(\beta_j^0)$, $j = 1, 2, \cdots, l$, let i_j^0 be the i such that β_j^0 is i-active at α_{j_0}, and let $e_j^0 = E(\beta_j^0)$. Set $\delta_j^0 = \alpha_{j_0}\hat{\ }\langle(i_l^0, k_l^0, e_l^0)\rangle\hat{\ }\cdots\hat{\ }\langle(i_j^0, k_j^0, e_j^0)\rangle$. Clearly $\beta_{\jmath_0} = \delta_k^0$ for some k.

We consider the nodes $\beta_1^0, \beta_2^0, \cdots, \beta_l^0$.

Let x_0 be the witness for cycle $\vec{c}_{j_0}$ of α_{j_0}. By the assumption, let v_{j_0} be a stage at which a hard link (τ_{j_0}, α) which was created at a stage $> s_3$ is travelled unsuccessfully. Let v_0 be the stage at which the current hard link (τ_{j_0}, α) was created, and let v_j be the stage at which we travel the hard link $(\tau_j, \alpha)[v_0]$ for $j \in \{1, 2, \cdots, j_0\}$.

By proposition 8.3 (iv), during and at the end of stage v_0 we have that:

(a) Cycle $\vec{c}_{j_0}$ of α_{j_0} is preservable,

(b) For any $j > k$, if β_j^0 is 0-active at α_{j_0}, then cycle $\vec{c}_{j_0}$ of α_{j_0} is δ_j^0-clear, and

(c) For any $j > k$, if β_j^0 is 1-active at α_{j_0}, then cycle $\vec{c}_{j_0}$ of α_{j_0} has been τ_j^0-honestified.

We inductively verify that (a)–(c) above hold with the same $p(\vec{c}_{j_0}, \alpha_{j_0})[v_0]$ during stages v_j for all $j \in \{0, 1, \cdots, j_0\}$.

For v_1: We consider the β_j^0 with $\beta_j^0 \subset \tau_1$. If β_j^0 is 0-active at α_{j_0}, but not 0-active at α, then let $e = E(\beta_j^0)$, so there is a node γ such that $\alpha_{j_0} \subseteq \gamma\hat{\ }\langle(a, e)\rangle \subseteq \alpha$ for some $a \in \{c, \widehat{c}\}$ and some e. By proposition 5.7, $e < E(\beta_1)$. Therefore β_1 is no longer active at α. If β_j^0 is 1-active at α_{j_0} but not 1-active at α, then the same argument leads to a contradiction.

By the choice of v_1, we only need to consider the nodes $\beta_j^0 \subset \tau_1$ which are 0-active at both α_{j_0} and α, and by the choice of v_0, for any $\beta_j^0 \subset \tau_1$, if β_j^0 is 0-active at α, then $\gamma_{\tau_j^0}(k^*)$ has been set to be undefined before the hard links are created at stage v_0, where k^* is the threshold of cycle $\vec{c}$ of α for the p.c. functional $\Gamma_{\tau_j^0}$. By the choice of s_1, there is no $k < k^*$ such that $\gamma_{\tau_j^0}(k)$ will be enumerated into A after stage s_1.

Therefore at stage v_1, we have that (a), (b) and (c) hold, and that by the construction, $\gamma^*_{\beta_1}(k_1^*)$ is set to be undefined.

Suppose inductively that for $j < j_0$, at stage v_j, we have that (a), (b) and (c) hold and that $\gamma^*_{\beta_j}(k_j^*)$ is undefined. Then by the same argument as above we have that at stage v_{j+1}:

- (a), (b) and (c) hold,
- If $j+1 < j_0$, then $\gamma^*_{\beta_{j+1}}(k^*_{j+1})$ is set to be undefined,
- If $j+1 = j_0$, then $\gamma_{\tau_{j_0}}(k^0_k)$ is set to be undefined.

By the construction at stage v_{j_0}, we allow some node β with $\alpha_{j_0} \subset \beta$ and $\beta <_{\mathrm{L}} \beta_{j_0}$ to act, *unless* one of the following cases applies:

(A) The definition of $\widehat{\Gamma}_\beta$ is delayed in step 9e of program α_{j_0},

(B) A hard link (τ, α_{j_0}) is created at stage v_{j_0}, i.e. step 7 of program α_{j_0} occurs, or

(C) Step 10 of program α_{j_0} occurs.

By the choice of s_0, and by proposition 8.3, (B) does not occur, and neither does (C). By the proof of subcase 1b for the $\mathcal{R}$-strategies and by the assumption that there are infinitely many α_{j_0}-expansionary stages, (A) occurs only finitely many times. Therefore if v_{j_0} is sufficiently large, then neither (A) nor (B) nor (C) occurs, so that there is a β such that $\alpha_{j_0} \subset \beta$ and such that $\beta <_{\mathrm{L}} \beta_{j_0}$ acts at stage v_{j_0}, contradicting the choice of s_0.

So Sublemma 8.12 holds. □

By sublemma 8.12, let s_2 be minimal $> s_1$ after which step 7 of program α will never occur.

To complete the proof of lemma 8.11, we now need the following:

8.13 Sublemma. Suppose that there is an i such that cycle $\vec{c}$ of α receives τ_j-attention infinitely many times. Then proposition 8.3 holds.

Proof. Let j be the least i such that cycle $\vec{c}$ of α receives τ_i-attention infinitely often. We look at two cases.

Case 1. Cycle $\vec{c}$ of α receives τ_j-destruction infinitely often. We divide this into subcases.

Subcase 1a. Step 9d occurs infinitely often.

By the construction, $\delta_j\hat{\ }\langle(c, e_j)\rangle$ is visited infinitely often, and by the construction, it is initialised only finitely many times. (i), (ii) and (iii) follow.

By the construction, $\widehat{\Gamma}_{\delta_j}$ is built infinitely often, so that if $\widehat{\Gamma}^{Y_{\tau_j}, A}_{\delta_j}$ is total, then $K = \widehat{\Gamma}^{Y_{\tau_j}, A}_{\delta_j}$. By steps 9a and 9b of the construction, $\gamma_{\tau_j}(k^*_j)$ will be unbounded over the course of the construction. So (vii) holds in this subcase.

Subcase 1b. Step 9b occurs infinitely often.

By the construction, in this subcase we have $\delta_j\hat{\ }\langle(\widehat{c}, e_j)\rangle$ visited infinitely often, and hence initialised only finitely many times, and so $\gamma_{\tau_j}(k^*_j)$ will be unbounded.

Subcase 1c. Otherwise.

By the assumption of this subcase, the l defined in step 9c is a fixed number. By the inductive hypothesis, there is a stage $v_0 > s_0$ say, after which l is in C_β for every β with $\beta\hat{\ }\langle -1\rangle \subseteq \alpha$. Let $v_1 > v_0$ be the least such stage after which none of steps 9a – 9d occurs. By the assumption of this subcase, we can take $s > v_1$ to be a stage at which step 9 of cycle $\vec{c}$ of α occurs. Then by the choice of s, step 9d occurs, contradicting the choice of v_1. Again, this subcase does not occur.

Case 2. Cycle $\vec{c}$ of α receives τ_j-honestification infinitely often.

In this case, by the construction, it is easy to see that (i), (ii) (iii) and (vii) hold, and Sublemma 8.13 follows. □

8.14 Sublemma. There is a j such that $\vec{c}$ of α receives τ_j-attention infinitely often.

Proof. Suppose to the contrary that cycle $\vec{c}$ of α receives τ_j-attention only finitely many times during the course of the construction, for each j.

Let v_1 be a stage after which none of steps 1, 2, 4, 5, 7, 8, 9, 11 or 12 occurs. Since $\alpha\hat{\ }\langle -1\rangle$ is to the left of the true path TP, for any j, cycle k_j^* of α for $\Gamma_{\beta_j}^*$ is awakened only finitely many times, and any strategy ξ with $\alpha\hat{\ }\langle 2\rangle \leq \xi$ will have initial value greater than k_j^* for every j. Therefore, step 6 of cycle $\vec{c}$ of α occurs only finitely many times. So, for any $s > v_1$, if α is visited at stage s, then either step 3 or step 10 occurs. This contradicts the assumption that there are infinitely many α-expansionary stages.

The sublemma follows. □

The Lemma 8.11 also follows. □

And this completes the inductive proof of proposition 8.7, as required. □

Next we observe a basic fact:

8.15 Proposition. For any p.c. functional Γ^*, say, and any given k, there are only finitely many strategies which can define k as their threshold for Γ^* during the course of the construction.

Proof. By the definition of the thresholds, an $\mathcal{L}$-strategy with initial value $> k$ will never define k as its threshold. And by the definition of the initial values of the $\mathcal{L}$-strategies, there are only finitely many strategies with initial values less than k during the course of the construction.

The proposition follows. □

We now prove a proposition concerning the computation of the true path TP.

8.16 Proposition. (Computation of the True Path Proposition) The true path TP of the construction is a total function from ω into the priority tree T, and $TP \leq_{\mathrm{T}} \emptyset''$.

Proof. We describe an algorithm for computing TP.

1. Define $f(0) = \lambda$.

   Suppose that $f(n) \downarrow = \alpha$.

2. If $\alpha = \tau$ is an $\mathcal{R}$-strategy, then:

   Case 2a. If there are infinitely many τ-expansionary stages, define $f(n+1) = \tau\hat{\ }\langle 0 \rangle$.

   Case 2b. Otherwise. Then define $f(n+1) = \tau\hat{\ }\langle 1 \rangle$.

3. If α is an $\mathcal{L}$-strategy, then:

   Case 3a. If there are infinitely many stages at which C_α is built, define $f(n+1) = \alpha\hat{\ }\langle -1 \rangle$.

   Case 3b. If there are only finitely many α-expansionary stages, then define $f(n+1) = \alpha\hat{\ }\langle 2 \rangle$.

   Case 3c. Otherwise. Then let $\lim_s \vec{c}(\alpha)[s] = \vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$. Let δ_j be the $<$-least δ_i such that δ_i is visited infinitely many times. If $\widehat{\Gamma}_{\delta_j}$ is built infinitely often, then define $f(n+1) = \delta_j\hat{\ }\langle (c, e_j) \rangle$, and otherwise define $f(n+1) = \delta_j\hat{\ }\langle (\widehat{c}, e_j) \rangle$.

By the definition of f, $f \leq_{\mathrm{T}} \emptyset''$. By the proof of proposition 8.7, it is easy to see that $f = TP$, giving $TP \leq_{\mathrm{T}} \emptyset''$.

Proposition 8.16 follows. □

8.17 Proposition. (Outcomes Along TP Proposition) Given a strategy $\alpha \in TP$:

(i) If α is an $\mathcal{R}$-strategy and $\alpha\hat{\ }\langle 0 \rangle \in TP$, then:

(a) Γ_α is built infinitely many times,

(b) If $\Gamma_\alpha(X_\alpha, A)$ is total, then $K = \Gamma_\alpha(X_\alpha, A)$.

(ii) If α is an $\mathcal{R}$-strategy and $\alpha\hat{\ }\langle 1 \rangle \in TP$, then $l(D, \Phi_\alpha(X_\alpha, Y_\alpha))[s]$ will be bounded during the course of the construction.

(iii) If α is an $\mathcal{L}$-strategy and $\alpha\hat{\ }\langle -1 \rangle \in TP$, then:

(a) $C_\alpha = \omega$,

(b) Ψ_α^A is total.

(iv) If α is an $\mathcal{L}$-strategy and $\alpha\hat{\ }\langle 2 \rangle \in TP$, then Ψ_α^A is partial.

(v) If α is an $\mathcal{L}$-strategy and $\delta_j\hat{\ }\langle(c, e_j)\rangle \in TP$ for some j, then for $\vec{c}(\alpha) = \vec{c} = (k_1^*, k_2^*, \cdots, k_n^*)$, and for $x = x(\vec{c}, \alpha)$, we have that:

(a) $\widehat{\Gamma}_{\delta_j}$ is built infinitely often,

(b) If $\widehat{\Gamma}_{\delta_j}(Y_{\tau_j}, A)$ is total, then $\widehat{\Gamma}_{\delta_j}(Y_{\tau_j}, A) = K$,

(c) $\gamma^*_{\beta_j}(k_j^*)[s]$ will be unbounded during the course of the construction,

(d) There are infinitely many stages at which $\gamma^*_{\beta_j}(k_j^*) \downarrow \leq \psi_\alpha(x)$ holds.

(vi) If α is an $\mathcal{L}$-strategy and $\delta_j\hat{\ }\langle(\widehat{c}, e_j)\rangle \in TP$ for some j, then:

(a) $\gamma^*_{\beta_j}(k_j^*)[s]$ will be unbounded during the course of the construction,

(b) There are infinitely many stages at which $\gamma^*_{\beta_j}(k_j^*) \downarrow \leq \psi_\alpha(x)$ holds,

(c) There is a fixed d such that there are infinitely many stages at which $\gamma^*_{\beta_j}(k_j^*) \downarrow \leq \phi_{\tau_j}(d)$ holds.

Proof. (i) follows from proposition 8.7 (v) and from the construction.

For (ii): By the proof of proposition 8.7 for α, we have that either there is a fixed d such that $l(D, \Phi_\alpha(X_\alpha, Y_\alpha)) \not> d$ at almost every stage at which α is visited, or there is a fixed stage s_0 such that for any $s > s_0$, if α is visited at stage s, then $l(D, \Phi_\alpha(X_\alpha, Y_\alpha)) \not> l[s_0]$. In either case, $l(D, \Phi_\alpha(X_\alpha, Y_\alpha))$ will be bounded over the course of the construction, and (ii) follows.

(iii) follows from proposition 8.7 (vi) and from the definition of C_α.

(iv) follows from proposition 8.7 (iv) for all $\mathcal{L}$-strategies $\subset \alpha$.

For (v): By the proof of proposition 8.7 (vii), there are infinitely many stages at which both α and $\delta_j\hat{\ }\langle(c, e_j)\rangle$ are visited. So (v) follows.

Property (vi) is immediate from proposition 8.7(vii) and from the construction.

The proposition follows. □

8.18 Proposition. $A'' \leq_{\mathrm{T}} \emptyset''$.

Proof. By proposition 8.17, for every $\mathcal{L}$-strategy $\alpha \in TP$, Ψ^A_α is total if and only if $\alpha\hat{\ }\langle -1\rangle \in TP$. By the definition of the priority tree T, for every Ψ, there is a unique $\mathcal{L}_\Psi$-strategy $\alpha \in TP$ which can be computed effectively. By proposition 8.16, $TP \leq_{\mathrm{T}} \emptyset''$. Therefore $\mathrm{Tot}^A \leq<_{\mathrm{T}} \emptyset''$, giving $A'' \leq_{\mathrm{T}} \emptyset''$.

The proposition follows. □

8.19 Proposition. ($\mathcal{R}$-Satisfaction Proposition) For every Φ, X, Y, $\mathcal{R}_{\Phi,X,Y}$ is satisfied.

Proof. Given e, let α be the longest $\mathcal{R}_e$ $(= \mathcal{R}_{\Phi,X,Y})$ -strategy $\in TP$. By proposition 8.17 (ii), if $\alpha\hat{\ }\langle 1\rangle \in TP$, then $D \neq \Phi^{X,Y}$, giving $\mathcal{R}_{\Phi,X,Y}$ trivially satisfied. Otherwise, there are three cases:

Case 1. There is a node γ such that $\alpha\hat{}\langle 0\rangle \subseteq \gamma\hat{}\langle(\widehat{c}, e)\rangle \in TP$.

In which case, by proposition 8.17 (vi), $\Phi^{X,Y}$ is partial, and $\mathcal{R}_{\Phi,X,Y}$ is satisfied.

Case 2. Case 1 does not hold, and there is an $\mathcal{L}$-strategy β say, such that $\beta = \langle\gamma\rangle\hat{}\langle(c, e)\rangle$ for some γ.

Therefore, there is a node $\delta \supset \alpha$ such that $\widehat{\Gamma}^{Y,A}_{\delta}$ is built infinitely often. By proposition 8.17, if $\widehat{\Gamma}^{Y,A}_{\delta}$ is total, then $K = \widehat{\Gamma}^{Y,A}_{\delta}$, giving $\mathcal{R}_{\Phi,X,Y}$ satisfied. So we only need to prove the totality of $\widehat{\Gamma}^{Y,A}_{\delta}$. Suppose to the contrary that k is the least x such that $\widehat{\Gamma}^{Y,A}_{\delta}(x)$ is (re)defined infinitely often. By proposition 8.15 and by the construction, there is a fixed strategy γ say which enumerates $\widehat{\gamma}^{Y,A}_{\delta}(k)$ into A infinitely many times. By the choice of α, the only possibility is that there is a node ξ say such that $\beta \subset \xi\hat{}\langle(\widehat{c}, e)\rangle \in TP$, contradicting the assumption of this case. Therefore $\widehat{\Gamma}^{Y,A}_{\delta}$ is total.

Case 3. Otherwise.

By a similar argument to that of case 2, we have that $\Gamma^{X,A}_{\alpha}$ is total. By proposition 8.17 (i), $\Gamma^{X,A}_{\alpha} = K$. So again $\mathcal{R}_{\Phi,X,Y}$ is satisfied.

The proposition follows. □

This completes the proof of the main theorem. □

References

1. M.M. Arslanov, S.B. Cooper and A. Li [2000], There is no low maximal d.c.e. degree, Math. Log. Quart. 46 (2000)3,409–416.

2. S.B. Cooper and A. Li [ta], Splitting and Nonsplitting, I: A Low$_3$ Harrington Nonsplitting Base, to appear.

3. S.B. Cooper, A. Li and X. Yi [ta], On the distribution of Lachlan nonsplitting bases, Archive for Mathematical Logic, to appear.

4. R.G. Friedberg [1957], Two recursively enumerable sets of incomparable degrees of unsolvability, Proc. Natl. Acad. Sci. USA **43**, 236–238.

5. L. Harrington [1980], 'Understanding Lachlan's monster paper' (handwritten notes), Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720.

6. A.H. Lachlan [1975], A recursively enumerable degree which will not split over all lesser ones, Ann. Math. Logic **9**, 307–365.

7. A.A. Muchnik [1956], On the unsolvability of the problem of reducibility in the theory of algorithms, Dokl. Akad. Nauk SSSR, NS. *108*, 194–197 (Russian).

8. P. Odifreddi [1989], *Classical Recursion Theory*, North-Holland, Amsterdam, New York, Oxford.

9. E.L. Post [1944], Recursively enumerable sets of positive integers and their decision problems, Bull. Amer. Math. Soc. **50**, 284–316.

10. R.W. Robinson [1971], Interpolation and embedding in the recursively enumerable degrees, Ann. of Math. (2) **93**, 285–314.

11. G.E. Sacks [1963], On the degrees less than $\mathbf{0}'$, Ann. of Math. (2) **77**, 211–231.

12. G.E. Sacks [1964], The recursively enumerable degrees are dense, Ann. of Math. (2) **80**, 300–312.

13. R.A. Shore and T.A. Slaman [1990], Working below low_2 recursively enumerable degrees, Archive for Math. Logic **29**, 201–211.

14. R.I. Soare [1987], *Recursively Enumerable Sets and Degrees*, Springer-Verlag, Berlin, Heidelberg, London, New York.

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