

# Splitting and Cone Avoidance in the D.C.E. Degrees

S. Barry Cooper\* and Angsheng Li†

## Abstract

It is shown that the Cooper splitting theorem for the  $n$ -c.e. degrees is not compatible with cone avoidance: For any  $n > 1$ , there exist  $n$ -c.e. degree  $\mathbf{a}$ , c.e. degree  $\mathbf{b}$  such that  $\mathbf{0} < \mathbf{b} < \mathbf{a}$  and such that for any  $n$ -c.e. degrees  $\mathbf{x}, \mathbf{y}$ , if  $\mathbf{x} \vee \mathbf{y} = \mathbf{a}$ , then either  $\mathbf{b} \leq \mathbf{x}$  or  $\mathbf{b} \leq \mathbf{y}$ . This provides a new type of elementary difference between the classes of c.e. and d.c.e. degrees, implementable at lower levels of the high/low hierarchy.

## 1 Introduction

The  $n$ -c.e. Turing degrees (the degree theoretic counterpart of the finite levels of the Ershov hierarchy of sets) provide a natural generalisation of the class of computably enumerable degrees. Recent years have seen much research centred on the comparisons between the structures of the c.e. and the  $n$ -c.e. degrees for  $n > 1$ , and various similarities and differences have been found.

An important first result was provided by Arslanov [1985]: Every nonzero c.e. degree joins to  $\mathbf{0}'$  via an incomplete 2-c.e. degree. From this it easily followed, via Lachlan's observation that every nonzero  $n$ -c.e. degree bounds a nonzero c.e. degree, that every nonzero  $n$ -c.e. degree joins to  $\mathbf{0}'$  via some incomplete 2-c.e. degree. Since (Cooper [1974] and Yates) there exists a nonzero c.e. degree which is not cuppable to  $\mathbf{0}'$  in the c.e. degrees, one immediately gets an elementary difference between the structures of the c.e. and the  $n$ -c.e. degrees for every  $n > 1$ . Other such differences were provided by the Downey [1989] diamond theorem, which contrasts with the Lachlan [1966] nondiamond theorem for the c.e. degrees (for further extensions see Cooper and Li [ta1]), and by the Cooper, Harrington, Lachlan, Lempp and Soare [1991] nondensity theorem for the d.c.e. degrees. However, these differences have not pointed the way to distinctions (which Downey [1989] conjectured did

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not exist) between higher levels of the n.c.e. hierarchy, nor such distinctions within important subclasses, such as at lower levels of the high/low hierarchy. For instance, even within the low degrees one encounters strong noncupping (see Cooper [1989] or Slaman and Steel [1989]): There exist low c.e. degrees  $\mathbf{a} > \mathbf{b} > \mathbf{0}$  such that  $\mathbf{b}$  is not cuppable to  $\mathbf{a}$ , even within the  $\Delta_2^0$  degrees.

As for similarities, Cooper [1992] showed that the Sacks [1963] splitting theorem for the c.e. degrees can be generalised to  $n$ -c.e. degrees: For any  $n$ -c.e. degree  $\mathbf{a}$ , if  $\mathbf{a} \neq \mathbf{0}$ , then there are  $n$ -c.e. degrees  $\mathbf{b}, \mathbf{c}$  such that  $\mathbf{b} < \mathbf{a}, \mathbf{c} < \mathbf{a}$  and  $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$ . Given that at the  $\text{low}_2$  level, splitting for the  $n$ -c.e. degrees can even be relativised (see Cooper [1991]), there would appear to be little help to be got here in extending the known elementary differences.

However, as noted in Cooper and Li [ta3], Cooper's proof of his splitting theorem is non-uniform, the c.e. case (covered by Sacks' proof) and the properly 2-c.e. case being quite different. This non-uniformity can be shown (Cooper and Li [ta2]) to be unavoidable. We show below how the non-uniformity here actually leads to a nonsplitting theorem:

**THEOREM** Given  $n > 1$ , there exists an  $n$ -c.e. degree  $\mathbf{a}$ , and a c.e. degree  $\mathbf{b}$ , with the following properties:

- (1)  $\mathbf{0} < \mathbf{b} < \mathbf{a}$ ,
- (2) for any  $n$ -c.e. degrees  $\mathbf{x}, \mathbf{y}$ , if  $\mathbf{x} \vee \mathbf{y} = \mathbf{a}$ , then either  $\mathbf{b} \leq \mathbf{x}$  or  $\mathbf{b} \leq \mathbf{y}$ .

It is of course very unexpected to find such a basic construction to be incompatible with cone avoidance, and one suspects there may be a number of important consequences arising from further analysis.<sup>1</sup> We would only observe here that the proof for the above theorem is a  $\emptyset''$ -priority argument, in that there is no injury along the true path, while the decision regarding the satisfaction of an  $\mathcal{R}$ -requirement is a  $\emptyset'''$ -proposition. Hence, it is straightforward to ensure that  $\mathbf{a}$  in the above theorem is  $\text{low}_3$ , so providing an elementary difference between the  $\text{low}_3$  c.e. and the  $\text{low}_3$  d.c.e. degrees.

## 2 Requirements and Strategies

We construct a 2-c.e. set  $A$  and a c.e. set  $B$  to satisfy the following requirements:

$$\begin{aligned} \mathcal{S}_e &: B \neq \xi_e, \\ \mathcal{R}_e &: [X_e^0 = \Phi_e(A) \ \& \ X_e^1 = \Psi_e(A) \ \& \ A = \Theta_e(X_e^0, X_e^1)] \Rightarrow [B \leq_T X_e^0 \ \text{or} \ B \leq_T X_e^1] \end{aligned}$$

where  $e \in \omega$ ,  $\{(\xi_e, X_e^0, \Phi_e, X_e^1, \Psi_e, \Theta_e) \mid e \in \omega\}$  is a standard listing of all 6-tuples  $(\xi, X^0, \Phi, X^1, \Psi, \Theta)$ , with  $\xi$  a p.c. function,  $X^0, X^1$  2-c.e. sets, and  $\Phi, \Psi$  and  $\Theta$  Turing functionals.

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<sup>1</sup>For instance, Arslanov, Kalimullin and Lempp have independently claimed that, using more complicated methods than those in this paper, there are 2-c.e. degree  $\mathbf{a}$  and c.e. degree  $\mathbf{b}$  such that  $\mathbf{0} < \mathbf{b} < \mathbf{a}$  and such that every 2-c.e. degree  $\mathbf{x}$  below  $\mathbf{a}$  is comparable with  $\mathbf{b}$ .

**An  $\mathcal{S}$ -Module:** An  $\mathcal{S}$ -module will be a usual Friedberg-Muchnik procedure to satisfy an  $\mathcal{S}$ -requirement,  $B \neq \xi$ , say (we drop the index):

1. Appoint a witness,  $b$  say, which is fresh.
2. Wait for a stage at which  $\xi(b) = 0 = B(b)$ . Then:
  - enumerate  $b$  into  $B$  and stop.

$\mathcal{S}$  will be satisfied by one of the two cases below:

**Case 1.**  $\xi(b) \neq 0 = B(b)$ .

**Case 2.**  $\xi(b) = 0 \neq 1 = B(b)$ .

**An  $\mathcal{R}$ -Module.** Given an  $\mathcal{R}$ -requirement,  $\mathcal{R}$  say, we define the length of agreement functions  $l(X^0, \Phi(A))$ ,  $l(X^1, \Psi(A))$ ,  $l(A, \Theta(X^0, X^1))$  as usual. And we define  $L = \max\{x \mid (\forall y < x)[\Theta(X^0, X^1; y) \downarrow = A(y) \ \& \ l(X^0, \Phi(A)) > \theta(y) \ \& \ l(X^1, \Psi(A)) > \theta(y)]\}$ . We say that  $s$  is  $\mathcal{R}$ -expansionary, if  $L[s] > L[v]$  for all  $v < s$ . Clearly if there are only finitely many  $\mathcal{R}$ -expansionary stages, then  $\mathcal{R}$  is satisfied.

Suppose that there are infinitely many  $\mathcal{R}$ -expansionary stages. Then we will build Turing functionals  $\Gamma^0$  and  $\Gamma^1$  such that one of the following holds:

- (i)  $\Gamma^0(X^0)$  is total and  $\Gamma^0(X^0) =^* B$ ,
- (ii)  $\Gamma^1(X^1)$  is total and  $\Gamma^1(X^1) =^* B$ .

The definition of  $\Gamma^i$  will be distributed to nodes of a *priority tree*,  $T$  say.

**Satisfying**  $(\mathcal{R}_0, \mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \dots)$

An  $\mathcal{S}$ -strategy assumes that  $X_0^0 = \Phi_0(A)$ ,  $X_0^1 = \Psi_0(A)$  and  $A = \Theta_0(X_0^0, X_0^1)$ . So we will build Turing functionals  $\Gamma_0^0$  and  $\Gamma_0^1$ .

**An  $\mathcal{S}_0$ -Strategy.** This will satisfy  $B \neq \xi_0$ , and will proceed as follows:

1. Define a *tester*,  $a$  say, which is fresh.
2. (Testing Outcomes) Wait for a stage,  $v$  say, at which
  - (2a)  $l(A, \Theta_0(X_0^0, X_0^1)) > a$ ,
  - (2b)  $l(X_0^0, \Phi_0(A)) > \theta_0(a)$ , and
  - (2c)  $l(X_0^1, \Psi_0(A)) > \theta_0(a)$ . Then:
    - enumerate  $a$  into  $A$ ,
    - define a *relative restraint*  $r(a)$  to be the current stage, which means that if  $x \leq r(a)$ , then  $x$  can be enumerated into or extracted from  $A$  only if  $x = a$ .
3. (Defining *e-State*) Wait for a stage,  $s$  say, which is greater than the  $v$  at which conditions (2a)–(2c) above reappear. There are now three cases:
 

**Case 3a.** (Global Win) There is an  $x \leq \theta_0(a)[v]$  which has been extracted from  $X^i$ , some  $i \leq 1$ , since stage  $v$ . Then:
 
  - extract  $a$  from  $A$ ,

– define an *absolute restraint*  $r = v$ , intending that no  $x \leq r$  can be enumerated into  $A$  or extracted from  $A$ .

[We have now created a permanent inequality  $\Phi_0(A; x) \downarrow = 1 \neq 0 = X_0^0(x)$  or  $\Psi_0(A; x) \downarrow = 1 \neq 0 = X_0^1(x)$  for some  $x \leq \theta_0(a)[v]$ .]

**Case 3b.** (0-State) There is an  $x \leq \theta_0(a)[v]$  such that  $x \in X^0 - X_v^0$ . Then:

- Define  $e(0, a) = 0$ ,
- Define a *witness*,  $b$  say, as fresh,
- We say that  $a$  is the *agitator* of  $b$ , denoted by  $a(b) = a$ , and define  $e(0, b) = 0$ ,
- Define a relative restraint  $r(a) = s$ , the current stage,
- For every  $x \leq b$ , if  $\Gamma^0(X^0; x)$  is undefined, define  $\Gamma^0(X^0; x) \downarrow = B(x)$  with  $\gamma^0(x) = b$ .

**Case 3c.** (1-State) Otherwise. Then:

- Define  $e(0, a) = 1$ ,
- Define a *witness*,  $b$  say, as fresh,
- We say that  $a$  is the *agitator* of  $b$ , written by  $a(b) = a$ , and define  $e(0, b) = 1$ ,
- Define a relative restraint  $r(a) = s$ , the current stage,
- For every  $x \leq b$ , if  $\Gamma^1(X^1; x)$  is undefined, define  $\Gamma^1(X^1; x) \downarrow = B(x)$  with  $\gamma^1(x) = b$ .

4. (Diagonalising Witness) Wait for a stage  $u > s$  at which  $\xi_0(b) \downarrow = 0 = B(b)$ . Then:

- enumerate  $b$  into  $B$ ,
- extract  $a$  from  $A$ .

5. (Rectifying  $\Gamma^i$ ) Wait for a stage  $t > u$  at which conditions (2a)–(2c) reappear. Then:

- Let  $i = e(0, b)$ ,
- for every  $x \leq b$ , if  $\Gamma^i(X^i; x)$  is undefined, then define  $\Gamma^i(X^i; x) \downarrow = B(x)$  with  $\gamma^i(x) = b$ , and
- define an absolute restraint  $r$  to be the current stage.

**The Possible Outcomes.** If there are infinitely many  $\mathcal{R}_0$ -expansionary stages, then the  $\mathcal{S}_0$ -strategy will eventually get a witness,  $b$  say. Then  $\mathcal{S}_0$  will be satisfied by either  $\xi_0(b) \neq 0 = B(b)$  or  $\xi_0(b) \downarrow = 0 \neq 1 = B(b)$ .

**An  $\mathcal{S}_1$ -Strategy.** An  $\mathcal{S}_1$ -strategy is similar to the  $\mathcal{S}_0$ -strategy. The only difference is in step 3. If  $\mathcal{S}_1$  can get a 0-state witness, but  $\mathcal{S}_0$  can only get a 1-state witness, then the  $\mathcal{S}_1$ -strategy will allow the  $\mathcal{S}_0$ -strategy to choose a 0-state witness. That is to say, whenever a strategy gets a 0-state agitator,  $a$  say, we will find the highest  $\mathcal{S}_e$ -strategy which has no 0-state witness, and define a fresh number  $b$  to be a 0-state

witness of  $\mathcal{S}_e$  with agitator  $a$ . If we get a 1-state agitator  $a$ , we define a 1-state witness for the highest  $\mathcal{S}_e$ -strategy which has no witness available.

During the course of the construction, once an  $\mathcal{S}_e$ -strategy is initialised, any witness associated with it is cancelled. And if a witness  $b$  is cancelled, then the agitator  $a$  of  $b$  is cancelled simultaneously, so that both  $A(a)$  for  $a = a(b)$ , and  $B(b)$  will never change. This ensures that the cancellation of witnesses will never make  $\Gamma^i(X_0^0)$  and  $B$  unequal at any argument. The definition of witnesses ensures that if a witness,  $b$  say, is enumerated into  $B$ , and the corresponding agitator  $a(b)$  is extracted from  $A$ , then for  $i = e(0, b)$ , for every  $x \geq b$ , the extraction of  $a(b)$  from  $A$  will create a permanent  $X_0^i$ -permission for  $x$  via  $\Gamma^i$ . This means that the diagonalisation of an  $\mathcal{S}$ -strategy will never make  $\Gamma^i(X_0^i)$  and  $B$  unequal. The definition of witnesses and the rectification of  $\mathcal{S}$ -strategies ensure that if  $\Gamma^0$  is built infinitely often by strategies on the true path  $TP$ , then  $\Gamma^0(X_0^0)$  is total. And otherwise,  $\Gamma^1(X_0^1)$  is total.

And by the definition and cancellation of witnesses, for every  $i \leq 1$ , if  $\Gamma^i(X_0^i)$  is total, then  $\Gamma^i(X_0^i) =^* B$ .

We say that the state 0 is higher than the state 1. Intuitively speaking, every  $\mathcal{S}$ -strategy will try to build a maximal state witness. Clearly, every  $\mathcal{S}$ -requirement is satisfied by an  $\mathcal{S}$ -strategy, and  $\mathcal{R}_0$  will be satisfied via one of the following cases:

**Case 1.** There are infinitely many 0-state witnesses which are built during the construction. In this case, we have that  $\Gamma^0(X_0^0)$  is total, so that  $\Gamma^0(X_0^0) =^* B$ .

**Case 2.** Otherwise. Then  $\Gamma^1(X_0^1)$  is total, giving  $\Gamma^1(X_0^1) =^* B$ .

In either case,  $\mathcal{R}_0$  is satisfied.

**Satisfying**  $(\mathcal{R}_0, \mathcal{R}_1, \mathcal{S}_1, \mathcal{S}_2, \dots)$ . In this case, an  $\mathcal{S}$ -strategy assumes that Turing functionals  $\Gamma_j^0$  and  $\Gamma_j^1$  are built by an  $\mathcal{R}_j$ -strategy for each  $j \in \{0, 1\}$ . So an  $\mathcal{S}$ -strategy will define  $e$ -states  $e(0, b)$  and  $e(1, b)$  for its witness  $b$ . Let  $\sigma(b) = e(0, b)e(1, b)$ , and say that  $\sigma$  is the  $e$ -state of  $b$ . We can define a witness  $b$ , only if we have tested the  $e$ -state of an agitator  $a$ . As before, whenever we enumerate a witness  $b$  into  $B$ , we extract the corresponding agitator  $a(b)$  from  $A$ , which must create a permanent  $X_j^i$ -change to rectify  $\Gamma_j^i$ . We now look at the details of an  $\mathcal{S}$ -strategy:

**An  $\mathcal{S}_1$ -Strategy.** This will proceed as follows:

1. (Defining a Tester) Define a *tester*,  $a$  say, as fresh.
2. (Testing Outcomes) Wait for a stage  $v$  say, at which
  - (2a)  $l(A, \Theta_j(X_j^0, X_j^1)) > a$ ,  $j = 0, 1$ ,
  - (2b)  $l(X_j^0, \Phi_j(A)) > \theta_j(a)$ ,  $j = 0, 1$ ,
  - (2c)  $l(X_j^1, \Psi_j(A)) > \theta_j(a)$ ,  $j = 0, 1$ . Then:
    - enumerate  $a$  into  $A$ ,
    - define a relative restraint  $r(a) = v$ , the current stage.
3. (Defining  $e$ -States) Wait for the least stage  $s_0 > v$  at which  $L(0) > a$ . Then there are three cases:

**Case 3a.** There is an  $x \leq \theta_0(a)[v]$  which has been extracted from  $X_0^i$  since stage  $v$ , for some  $i \leq 1$ . Then:

- extract  $a$  from  $A$ ,
- define an absolute restraint  $r = v$ .

**Case 3b.** There is an  $x \leq \theta_0(a)[v]$  which has been enumerated into  $X_0^0$  since stage  $v$ . Then:

- define  $e(0, a) = 0$ ,
- define a relative restraint as  $r(a) = s_0$ .

**Case 3c.** Otherwise. Then:

- define  $e(0, a) = 1$ ,
- define a relative restraint by  $r(a) = s_0$ .

4. Let  $s_1$  be the least stage greater than  $s_0$  at which  $L(1) > a$ . Then there are three cases:

**Case 4a.** There is an  $x \leq \theta_1(a)[v]$  which has been extracted from  $X_1^i$  since stage  $v$ , for some  $i \leq 1$ . Then:

- extract  $a$  from  $A$ ,
- define an absolute restraint by  $r = v$ .

**Case 4b.** There is an  $x \leq \theta_1(a)[v]$  which is enumerated into  $X_1^0$  since stage  $v$ . Then:

- define  $e(1, a) = 0$ ,
- define a relative restraint by  $r(a) = s_1$ .

**Case 4c.** Otherwise. Then:

- define  $e(1, a) = 1$ ,
- define a relative restraint  $r(a) = s_1$ .

5. (Defining Witnesses) Then:

- Let  $\sigma = e(0, a)e(1, a)$ ,
- Define a witness  $b$  as fresh,
- Define the  $e$ -state of  $b$  to be  $\sigma$ , and the agitator of  $b$  to be  $a(b) = a$ ,
- If  $e(j, a) = i$ ,  $x \leq b$  and  $\Gamma_j^i(x) \uparrow$ , then define  $\Gamma_j^i(X_j^i; x) = B(x)$  with  $\gamma_j^i(x) = b$  for each  $j \in \{0, 1\}$ .

6. (Diagonalisation) Wait for a stage  $s > s_1$  at which  $\xi_1(b) \downarrow = 0 = B(b)$ . Then:

- enumerate  $b$  into  $B$ ,
- extract  $a = a(b)$  from  $A$ .

7. (Rectifying  $\Gamma_0$ ) Wait for the least stage  $v_0 > s$  at which  $L(0) > a$ . Then for  $i = e(0, b)$ , for each  $x \leq b$ , if  $\Gamma_0^i(X_0^i; x) \uparrow$ , then define it to be  $B(x)$  with use function  $\gamma_0^i(x) = b$ .

8. (Rectifying  $\Gamma_1^i$ ) Wait for the least stage  $v_1 > s$  at which  $L(1) > a$ . Then for  $i = e(1, b)$  and for  $x \leq b$ , if we have  $\Gamma_1^i(X_1^i; x) \uparrow$ , then define it to be  $B(x)$  with use function value  $= b$ .

For  $\sigma, \tau \in 2^{<\omega}$ , we define  $\sigma < \tau$  according to the lexicographical ordering, in which case, we say that  $\sigma$  is greater than  $\tau$ . As before, an  $\mathcal{S}$ -strategy will demand a witness with maximal  $e$ -state.

Suppose that  $\beta_1 \subset \beta_2 \subset \dots \subset \beta_m$  are  $\mathcal{S}$ -strategies, that  $b_1, b_2, \dots, b_m$  are the current witnesses of  $\beta_1, \beta_2, \dots, \beta_m$  respectively, and that  $\sigma_1, \sigma_2, \dots, \sigma_m$  are the  $e$ -states of  $b_1, b_2, \dots, b_m$  respectively. Then we will have that:

$$\sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_m.$$

A general  $\mathcal{S}$ -strategy  $\beta$  is the same as an  $\mathcal{S}_1$ -strategy, the only difference being that if  $\beta$  gets a witness  $b$  say, with  $e$ -state  $\sigma$  and there is an  $\mathcal{S}$ -strategy  $\beta' \subset \beta$  such that the  $e$ -states of witnesses of  $\beta'$  are all less than  $\sigma$ , then  $\beta$  lets the shortest such  $\beta'$  get a new witness with  $e$ -state  $\sigma$ . This occurs only finitely often, so any such  $\beta$  on the true path  $TP$  will eventually get a permanent witness.

Clearly every  $\mathcal{S}$ -requirement will eventually be satisfied.

We say that a witness  $b$  is *permanent*, if there is a stage  $s_0$  say, at which  $b$  is defined to be a witness of an  $\mathcal{S}$ -strategy  $\beta$  such that:

- (a)  $\beta$  is on the true path  $TP$ , and
- (b)  $b$  is the greatest witness of  $\beta$  which is defined during the construction.

By the strategies, once a witness  $b$  is defined, its  $e$ -state is defined and kept forever. The point is that the satisfaction of  $\mathcal{R}_j$  will be determined by the  $e$ -states of permanent witnesses. First,  $\mathcal{R}_0$  will be satisfied by one of the following cases:

**Case 1.** There are infinitely many permanent witnesses  $b$  with  $e(0, b) = 0$ . In this case, we have that  $\Gamma_0^0$  is built infinitely often, that  $\Gamma_0^0(X_0^0)$  is total and  $\Gamma_0^0(X_0^0) = B$ .

**Case 2.** Otherwise. Then we ensure that  $\Gamma_0^1(X_0^1)$  is total and that  $\Gamma_0^1(X_0^1) =^* B$ .

Suppose  $i_0$  to be the least  $i \in \{0, 1\}$  for which there are infinitely many permanent witnesses  $b$  with  $e(0, b) = i_0$ . Then  $\mathcal{R}_1$  is satisfied via one of the following:

**Case 1.** There are infinitely many permanent witnesses  $b$  with  $e(0, b) = i_0$  and  $e(1, b) = 0$ . In this case, we have that  $\Gamma_1^0(X_1^0)$  is total, and  $\Gamma_1^0(X_1^0) =^* B$ .

**Case 2.** Otherwise. Then we have that  $\Gamma_1^1(X_1^1)$  is total and  $\Gamma_1^1(X_1^1) =^* B$ .

In each case, every  $\mathcal{R}_j$ , with  $j \leq 1$ , is satisfied.

### 3 The Priority Tree and the Construction

In this section, we define the tree of strategies and describe a construction.

**Definition 3.1.** (i) Define the *priority ranking of requirements* by:

$$\mathcal{R}_0, \mathcal{S}_0, \mathcal{R}_1, \mathcal{S}_1, \dots$$

(ii) The *possible outcomes of an  $\mathcal{S}$ -strategy* are given by

$$d <_{\mathbf{L}} w,$$

where  $d$  means that  $\xi(b) \downarrow = 0 \neq 1 = B(b)$ , and  $w$  means that  $\xi(b) \neq 0 = B(b)$  for some witness  $b$ .

(iii) The *possible outcomes of an  $\mathcal{R}$ -strategy* are  $0 <_{\mathbf{L}} 1$  to denote infinite and finite actions respectively.

**Definition 3.2.** (i) Define the root node  $\lambda$  to be an  $\mathcal{R}_0$ -strategy.

(ii) The immediate successors of a node are the possible outcomes of the corresponding strategy.

(iii) A node  $\xi$  works on the highest priority requirement,  $\mathcal{X}$  say, on which there is no  $\xi' \subset \xi$  working.

This defines the priority tree  $T$ .

Next, we introduce some parameters and notations:

**Definition 3.3.** (i) For an  $\mathcal{R}$ -strategy  $\alpha$ :

(a) we will build Turing functionals  $\Gamma_{\alpha}^0$  and  $\Gamma_{\alpha}^1$ ,

(b) we say that  $s$  is  $\alpha$ -*expansionary* if  $L(\alpha) > L(\alpha)[v]$  for all  $v < s$  at which  $\alpha$  was visited and  $L(\alpha) > a(\beta)$  for all  $\beta \supseteq \alpha \hat{\langle} 0 \rangle$  with  $a(\beta) \downarrow$ ,

(c) if  $\alpha$  is initialised, then both  $\Gamma_{\alpha}^0$  and  $\Gamma_{\alpha}^1$  are set to be totally undefined.

(ii) For an  $\mathcal{S}$ -strategy  $\beta$ , let  $\alpha_0 \subset \alpha_1 \subset \dots \subset \alpha_l$  list all  $\mathcal{R}$ -strategies  $\alpha$  with  $\alpha \hat{\langle} 0 \rangle \subseteq \beta$ . Then we will have the following parameters and conventions:

(a)  $a(\beta)$ : the current tester and agitator of  $\beta$ ,

(b)  $b(\beta)$ : the current witness of  $\beta$ ,

(c)  $e(\alpha_j, b(\beta))$ :  $j \leq l$ , the  $\alpha_j$ -*state* of  $b(\beta)$ ,

(d) let  $e(\alpha_j, b(\beta)) = i_j$ , for  $j = 0, 1, \dots, l$ , then we say that  $\sigma(b, j) = i_0 i_1 \dots i_j$  is an  $e$ -state of  $b$  for  $b = b(\beta)$ ,

(e) if  $\beta$  is initialised, then both  $a(\beta)$  and  $b(\beta)$  are cancelled.

We also note that for an  $\mathcal{R}$ -strategy  $\alpha$ , and an  $i \leq 1$ , the Turing functional  $\Gamma_{\alpha}^i$  is built as a c.e. set of axioms  $\langle \sigma, x, y \rangle$ . Then  $\Gamma_{\alpha}^i(X_{\alpha}^i; x) \downarrow = y$  if and only if there is an axiom  $\langle \sigma, x, y \rangle \in \Gamma_{\alpha}^i$  such that  $\sigma \subset X_{\alpha}^i$ .

Finally, we fix some notations which are related to a given  $\mathcal{S}$ -strategy  $\gamma$  and which will be used in the description of the construction and the verification:

(i) Let  $\alpha_1 \subset \alpha_2 \subset \dots \subset \alpha_m$  be all  $\mathcal{R}$ -strategies  $\alpha$  with  $\alpha \hat{\langle} 0 \rangle \subseteq \gamma$ .

(ii) Let  $\beta_1 \subset \beta_2 \subset \dots \subset \beta_{n-1}$  be all  $\mathcal{S}$ -strategies  $\beta$  with  $\beta \subset \gamma$  and let  $\beta_n = \gamma$ .

Now we are ready to describe the construction.

**Definition 3.4.** The construction will proceed by stages as follows:

**Stage  $s = 0$ .** Set  $A = B = \omega$ , initialise all strategies.

**Stage  $s > 0$ .** We say that  $\xi$  is *visited at stage  $s$*  if it is *eligible to act at a substage  $t$  of stage  $s$* . We first allow the root node  $\lambda$  to be eligible to act at substage  $t = 0$  of stage  $s$ .

**Substage  $t$ .** Let  $\xi$  be eligible to act at substage  $t$ . If  $t = s$ , then initialise all nodes  $\gamma$  with  $\xi <_L \gamma$  and go to stage  $s + 1$ . Otherwise, there are two cases:

**Case 1.**  $\xi = \alpha$  is an  $\mathcal{R}$ -strategy. Then run the following:

**Program  $\alpha$**

1. If  $s$  is not  $\alpha$ -expansionary, then let  $\alpha \hat{\langle} 1 \rangle$  be eligible to act next.
2. If there is an  $\mathcal{S}$ -strategy  $\gamma$  with  $d(\gamma) \downarrow = \alpha$ , let  $\alpha \hat{\langle} 1 \rangle$  be eligible to act next.

[The parameter  $d(\gamma)$  denotes that some  $\mathcal{R}$ -strategy is delayed by  $\gamma$ .]

We say that  $\alpha$  *requires  $\Gamma_\alpha$ -rectification at stage  $s$* , if there is an  $\mathcal{S}$ -strategy  $\beta$ , and a stage  $v < s$ , such that:

- (i)  $\alpha \hat{\langle} 0 \rangle \subseteq \beta$ ,
- (ii)  $\beta$  agitated at stage  $v$ , and has never been initialised since stage  $v$ , and
- (iii)  $\alpha$  has not been rectified since stage  $v$ .

In this case, let  $\beta_0$  be the  $\beta$  with  $a(\beta)$  minimal. We say that  $\alpha$  *requires  $\Gamma_\alpha$ -rectification at stage  $s$  via  $\beta_0$* .

We say that  $\alpha$  *requires state testing at stage  $s$* , if there is an  $\mathcal{S}$ -strategy  $\beta$ , a stage  $v < s$  such that:

- (a)  $\alpha \hat{\langle} 0 \rangle \subseteq \beta$ ,
- (b)  $\beta$  received outcome testing at stage  $v$ , and has not been initialised since stage  $v$ , and
- (c)  $\alpha$  has not received state testing since stage  $v$ .

In this case, let  $\beta_0$  be the  $\beta$  with  $a(\beta)$  minimal. We say that  $\alpha$  *requires state testing via  $\beta_0$* .

3. If  $\alpha$  requires  $\Gamma_\alpha$ -rectification at stage  $s$  via  $\beta$ , then let  $\alpha$  *receive  $\Gamma_\alpha$ -rectification* as follows:
  - Let  $b = b(\beta)$  and  $i = e(\alpha, b)$ ,
  - for every  $x \leq b$ , if  $\Gamma_\alpha^i(X_\alpha^i; x)$  is undefined, then define  $\Gamma_\alpha^i(X_\alpha^i; x) \downarrow = B(x)$  with  $\gamma_\alpha^i(x) = b$ ,
  - we say that  $\alpha$  *receives  $\Gamma_\alpha$ -rectification at stage  $s$* ,
  - initialise all  $\xi$  with  $\alpha <_L \xi$  and go to stage  $s + 1$ .
4. If  $\alpha$  requires state testing at stage  $s$  via  $\gamma$ , then let  $v$  be the stage at which  $\gamma$  received outcome testing, let  $a = a(\gamma)$ , let  $\alpha = \alpha_j$  for some  $j \in \{1, 2, \dots, m\}$  and let  $\alpha$  *receive state testing* as follows:

**Case 1.** There is an  $x \leq \theta_\alpha(a)[v]$  which has been extracted from  $X_\alpha^i$  since stage  $v$ , for some  $i$ . Then:

- extract  $a$  from  $A$ ,
- define an absolute restraint  $r(\gamma) = s$ , the current stage,

- we say that  $\alpha$  is delayed by  $\gamma$ ,
- define  $d(\gamma) \downarrow = \alpha$ , and go to stage  $s + 1$ .

Suppose  $d(\gamma)$  became defined  $= \alpha$  at stage  $u < s$ . Then  $d(\gamma) = \alpha$  is kept during stage  $s$  if and only if  $r(\gamma)[u]$  has not been injured since stage  $u$  — in which case, we say that  $\alpha$  is delayed by  $\gamma$  at stage  $s$ .

**Case 2.** There is an  $x \leq \theta_\alpha(a)[v]$  enumerated into  $X_\alpha^0$  since stage  $v$ . Then:

- define  $e(\alpha, a) = 0$ ,
- define a relative restraint  $r(a)$  to be the current stage.

**Case 3.** Otherwise. Then:

- define  $e(\alpha, a) = 1$ ,
- define a relative restraint  $r(a)$  to be the current stage.

In case 2 or 3, if  $j = m$ , then we say that  $\gamma$  requires witness at stage  $s$ , and let  $\gamma$  be eligible to act next. If  $j < m$ , we initialise all  $\xi$  with  $\alpha <_L \xi$  and go to stage  $s + 1$ .

5. Otherwise. Then let  $\alpha \hat{\langle} 0 \rangle$  be eligible to act next.

**Case 2.**  $\xi = \gamma$  is an  $\mathcal{S}_e$ -strategy for some  $e$ . Then run the following:

**Program  $\gamma$**

1. If there is a  $b$  such that  $\xi_e(b) \downarrow = 0 \neq B(b)$ , let  $\gamma \hat{\langle} d \rangle$  be eligible to act next.
2. If  $\gamma$  requires witness at stage  $s$ , then:
  - let  $a = a(\gamma)$ ,
  - let  $\sigma(a) = e(\alpha_1, a)e(\alpha_2, a) \cdots e(\alpha_m, a)$ ,
  - let  $j$  be the least  $i \in \{1, 2, \dots, n\}$  such that the  $e$ -state of the current witness of  $\beta_i$  is less than  $\sigma$ , let  $\beta = \beta_j$  and go on to the step 3.
3. Then:
  - define  $b(\beta)$  to be fresh,
  - define  $a$  to be the agitator of  $b(\beta)$ , and let  $a(\beta) = a$ ,
  - define the  $e$ -state of  $a(\beta)$  and  $b(\beta)$  to be the initial segment of  $\sigma$  restricted to the  $\mathcal{R}$ -strategies  $\alpha$  above  $\beta$ ,
  - for every  $\mathcal{R}$ -strategy  $\alpha$  with  $\alpha \hat{\langle} 0 \rangle \subseteq \beta$ , for  $i = e(\alpha, a)$ , and for every  $x \leq b = b(\beta)$ , if  $\Gamma_\alpha^i(X_\alpha^i; x) \uparrow$ , then define it to be  $B(x)$  with use function  $= b$ ,
  - we say that  $\beta$  receives witness at stage  $s$ , initialise any  $\xi$  with  $\xi \not\leq \beta$ , and go to stage  $s + 1$ .
4. (Agitation) If  $b(\gamma) \downarrow = b$ , and  $\xi_e(b) \downarrow = 0 = B(b)$ , then:
  - enumerate  $b$  into  $B$ ,
  - if  $a(\gamma) \downarrow = a$ , then extract  $a$  from  $A$ , and we say that  $\gamma$  agitates at stage  $s$ ,
  - initialise any  $\xi \not\leq \gamma$  and go to stage  $s + 1$ .

5. If  $b(\gamma) \downarrow = b$  and  $\xi_e(b) \neq 0 = B(b)$ , then let  $\gamma \langle w \rangle$  be eligible to act next.
6. (Testing Outcomes) If  $a(\gamma) \downarrow = a$ ,  $a$  has never been enumerated into  $A$ , and for every  $\mathcal{R}$ -strategy  $\alpha$  with  $\alpha \hat{\langle} 0 \rangle \subseteq \gamma$  we currently have  $L(\alpha) > a$ , then:
  - enumerate  $a$  into  $A$ ,
  - we say that  $\gamma$  receives outcome testing at stage  $s$ ,
  - initialise any  $\xi$  with  $\xi \not\leq \gamma$  and go to stage  $s + 1$ .
7. If  $a(\gamma)$  is undefined, then
  - if there is no  $\mathcal{R}$ -strategy  $\alpha$  with  $\alpha \hat{\langle} 0 \rangle \subseteq \gamma$ , then define  $b(\gamma)$  afresh,
  - otherwise, define  $a(\gamma)$  afresh,
  - initialise any  $\xi$  with  $\xi \not\leq \gamma$  and go to stage  $s + 1$ .

This completes the description of the construction.

## 4 The Verification

In this section, we verify that the construction satisfies the requirements.

**Definition 4.1.** (i) Let  $\delta_s$  be the longest node which is visited during stage  $s$ .

(ii) Define the *true path of the construction* by

$$TP = \liminf_s \delta_s.$$

We first need:

**Proposition 4.2.** (Existence of the True Path  $TP$  Proposition) Given a node  $\alpha \in TP$ , there is an  $a$  such that:

- (i)  $\alpha \hat{\langle} a \rangle \in TP$ ,
- (ii)  $\alpha \hat{\langle} a \rangle \in TP$  is initialised only finitely many times,
- (iii)  $\alpha \hat{\langle} a \rangle \in TP$  is visited infinitely many times.

*Proof.* An easy inductive argument on the length of  $\alpha$ .  $\square$

**Proposition 4.3.** (Possible Outcomes Along the True Path  $TP$  Proposition) Given a node  $\alpha \in TP$ , we have that:

- (i) If  $\alpha$  is an  $\mathcal{R}_e$ -strategy for some  $e$ , then:
  - (a) if  $\alpha \hat{\langle} 1 \rangle \in TP$ , then  $X_e^0 \neq \Phi_e(A)$  or  $X_e^1 \neq \Psi_e(A)$  or  $A \neq \Theta_e(X_e^0, X_e^1)$ ,
  - (b) if  $\alpha \hat{\langle} 0 \rangle \in TP$ , then for  $B_\alpha = \{b \in B \mid b \text{ is enumerated by some } \beta \supseteq \alpha \hat{\langle} 0 \rangle\}$ , we have:
    - for each  $i \in \{0, 1\}$ , if  $\Gamma_\alpha^i(X_\alpha^i)$  is total, then  $\Gamma_\alpha^i(X_\alpha^i) =^* B_\alpha$ .
- (ii) If  $\alpha$  is an  $\mathcal{S}_e$ -strategy for some  $e$ , then for  $b = \lim_s b(\alpha)[s]$ , we have:
  - (a) if  $\alpha \hat{\langle} d \rangle \in TP$ , then  $\xi_e(b) \downarrow = 0 \neq 1 = B(b)$ ,
  - (b) if  $\alpha \hat{\langle} w \rangle \in TP$ , then  $\xi_e(b) \neq 0 = B(b)$ .

*Proof.* (i)(a), (ii) are immediate from the construction. We need only prove (i) (b).

By the construction, if  $\alpha \hat{\langle} 0 \rangle \subseteq \beta \in TP$ , then  $\beta$  will provide a chance to define either  $\Gamma_\alpha^0$  or  $\Gamma_\alpha^1$ . Therefore there are infinitely many stages at which either  $\Gamma_\alpha^0$  or  $\Gamma_\alpha^1$  is built.

Suppose that  $\Gamma_\alpha^i(X_\alpha^i)$  is total for some  $i \leq 1$ . We prove that  $\Gamma_\alpha^i(X_\alpha^i) =^* B_\alpha$ . Let  $b = b(\beta)$  for some  $\beta$ . If  $b \notin B_\alpha$ , then whenever we define  $\Gamma_\alpha^i(X_\alpha^i; b)$ , we define it to be 0. Suppose that  $b$  is enumerated into  $B_\alpha$  at a stage  $s_1$ , and that  $b$  is chosen at stage  $s_0 < s_1$ . By the construction,  $r(\beta)[s_0]$  has not been injured at a stage  $s \in [s_0, s_1)$ . Otherwise,  $b(\beta)[s_0]$  will have been cancelled, precluding it being enumerated into  $B$  at stage  $s_1$ . By the construction, the agitator  $a$  of  $b$  is extracted from  $A$  during stage  $s_1$ , and any  $\xi \not\leq \beta$  is initialised at stage  $s_1$ . Let  $s_2$  be the least  $\alpha$ -expansionary stage greater than  $s_1$  at which  $L(\alpha) > a$ . Then there is no element  $\leq s_1$  which is enumerated into or extracted from  $A$  at a stage  $s \in (s_1, s_2]$ , since otherwise  $\alpha$  would have been initialised at a stage  $> s_1$ . By the choice of  $s_0, s_1, s_2$ , there is an  $x \leq b$  which was in  $X_\alpha^i[s_0]$  but which has been extracted from  $X_\alpha^i$  at a stage  $s \in (s_1, s_2]$ . By the construction of  $\Gamma_\alpha^i$ , this ensures that for any  $y$ , if  $\Gamma_\alpha^i(X_\alpha^i; y)$  became defined for the first time at a stage  $\geq s_0$ , then the previous computation  $\Gamma_\alpha^i(y)$  proved to be permanently incorrect at stage  $s_2$ . Therefore, for any such  $y$ , the permanent computation can be defined only after stage  $s_2$ . However, any such  $y$  will have been cancelled at stage  $s_1$ , which means that  $B_\alpha(y)$  never changes at any stage  $> s_1$ , the permanent computation  $\Gamma_\alpha^i(X_\alpha^i; y)$  being one which is defined at a stage  $\geq s_2$ . In particular, this ensures that if  $\Gamma_\alpha^i(X_\alpha^i; b) \downarrow$ , then  $\Gamma_\alpha^i(X_\alpha^i; b) \downarrow = B(b)$ . We now conclude that if  $\Gamma_\alpha^i(X_\alpha^i)$  is total, then  $\Gamma_\alpha^i(X_\alpha^i) =^* B_\alpha$ .

The proposition follows.  $\square$

**Proposition 4.4.** ( $\mathcal{R}$ -Satisfaction Proposition). Every  $\mathcal{R}_e$  is satisfied.

*Proof.* We prove it by induction on  $e$ . For  $e = 0$ , if  $\lambda \hat{\langle} 1 \rangle \in TP$ , then by proposition 4.3,  $\mathcal{R}_0$  is satisfied. Suppose that  $\lambda \hat{\langle} 0 \rangle \in TP$ . Let  $B_\lambda$  be the subset of  $B$  enumerated by strategies  $\supseteq \lambda \hat{\langle} 0 \rangle$ , and let  $C_\lambda = B - B_\lambda$ . Then clearly  $C_\lambda$  is computable. So by proposition 4.3 (i), it suffices to show that there is an  $i \leq 1$  such that  $\Gamma_\lambda^i(X_\lambda^i)$  is total. We check the totality of  $\Gamma_\lambda^i(X_\lambda^i)$  by cases:

**Case 1.** There are infinitely many  $\mathcal{S}$ -strategies  $\beta$  such that  $\lambda \hat{\langle} 0 \rangle \subseteq \beta \in TP$ , and  $\beta$  has a permanent witness  $b$  such that  $e(\lambda, b) = 0$ .

Suppose that  $b$  is a permanent witness of some  $\beta \supseteq \lambda \hat{\langle} 0 \rangle$  on the true path  $TP$ . Then by the construction, if  $b \notin B$ , then  $\Gamma_\lambda^0(X_\lambda^0) \uparrow (b+1)$  will be defined eventually and kept forever. Otherwise, after  $b$  is enumerated into  $B$ , there is a  $\lambda$ -expansionary stage at which  $\Gamma_\lambda^0(X_\lambda^0) \uparrow (b+1)$  is rectified and kept forever (otherwise,  $b$  will be cancelled). This means that if  $b = b(\beta)$  is permanent, and  $e(\lambda, b) = 0$ , then  $\Gamma_\lambda^0(X_\lambda^0) \uparrow (b+1)$  converges. Therefore in case 1, we have that  $\Gamma_\lambda^0(X_\lambda^0)$  is total.

**Case 2.** Otherwise. Then clearly there are infinitely many permanent witnesses  $b = b(\beta)$  for  $\beta \supseteq \lambda \hat{\langle} 0 \rangle$  with  $e(\lambda, b) = 1$ .

By the same argument as above, we have that  $\Gamma_\lambda^1(X_\lambda^1)$  is total.

In case 1, we set  $o(\lambda) = 0$ , and  $o(\lambda) = 1$ , otherwise.

Suppose by induction that the proposition holds for all  $e' < e$ , and that  $\alpha$  is the

unique  $\mathcal{R}_e$ -strategy on the true path  $TP$ .

Let  $\alpha_1 \subset \alpha_2 \subset \dots \subset \alpha_n$  list all  $\mathcal{R}$ -strategies  $\alpha'$  with  $\alpha' \hat{\langle} 0 \rangle \subseteq \alpha$ , and let  $\sigma = o(\alpha_1)o(\alpha_2)\dots o(\alpha_n)$ .

By proposition 4.3, if  $\alpha \hat{\langle} 1 \rangle \in TP$ , then  $\mathcal{R}_e$  is satisfied. So suppose that  $\alpha \hat{\langle} 0 \rangle \in TP$ . Similarly to the case  $e = 0$ , we check two cases:

**Case 1.** There are infinitely many  $\beta \supseteq \alpha \hat{\langle} 0 \rangle$  which get permanent witnesses  $b$  with  $e(\alpha_j, b) = o(\alpha_j)$  for every  $j \in \{1, 2, \dots, n\}$ , and such that  $e(\alpha, b) = 0$ .

By the same argument as above, we have that  $\Gamma_\alpha^0(X_\alpha^0)$  is total.

**Case 2.** Otherwise. Then similarly we have that  $\Gamma_\alpha^1(X_\alpha^1)$  is total.

In case 1, we set  $o(\alpha) = 0$ , and 1, otherwise. In each case, by proposition 4.3, we get that  $\mathcal{R}_e$  is satisfied, since  $C_\alpha$  is computable.

The proposition follows.  $\square$

**Proposition 4.5.** ( $\mathcal{S}$ -Satisfaction Proposition) For every  $e$ ,  $\mathcal{S}_e$  is satisfied.

*Proof.* This follows directly from proposition 4.3.  $\square$

This completes the proof of the theorem in the case  $n = 2$ .

For  $n > 2$ , we may restrain  $A$  and agitate  $A(a)$  at most  $(n - 1)$  many times to predict either a permanent  $X^0$ -change or a permanent  $X^1$ -change, if the  $n$ -th  $A(a)$ -change occurs. We define a witness  $b$  for some  $\mathcal{S}$ -strategy,  $\beta$  say, only if we have predicted the possible outcomes of all  $\mathcal{R}$ -strategies of higher priority, so that once the witness  $b$  is enumerated into  $B$ , then the  $n$ -th change of  $A(a)$  will certainly create either a permanent  $X^0$ -change or a permanent  $X^1$ -change, so we can eventually rectify all  $\Gamma$ -functionals at argument  $y \geq b$ . So the proof above can be adapted with some straightforward adjustments to provide a proof of the theorem for all  $n \geq 2$ .

The theorem follows.  $\square$

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Department of Pure Mathematics, School of Mathematics, University of Leeds, Leeds LS2 9JT, UK, Phone: 44-113-2335165, Fax: 44-113-2335145.  
e-mail: s.b.cooper@leeds.ac.uk, <http://www.amsta.leeds.ac.uk/~pmt6sbc>.

Department of Pure Mathematics, School of Mathematics, University of Leeds, Leeds LS2 9JT, UK; Phone: 44-113-2335180, Fax: 44-113-235145.  
e-mail: angsheng@amsta.leeds.ac.uk.

Permanent address: Institute of Software, the Chinese Academy of Sciences, P.O. Box 8718, Beijing, 100080, P.R. CHINA; liang@ox.ios.ac.cn.