Genericity and enumeration reducibility

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Real World Computing

- Turing - importance of computer interaction
- New computational paradigms (Grid etc)
- Experimental/ analog computing
- Modelling non-linear natural phenomena
- Key role of imitation in AI

An Improbable Story . . .

- You are having trouble with a math problem, and go to your dept. for help
- You knock on Prof. A's door
- Unfortunately Prof. A has taken early retirement - he will never again be at work
- You wait forever . . .
- Your math problem remains unsolved . . .

The real world?

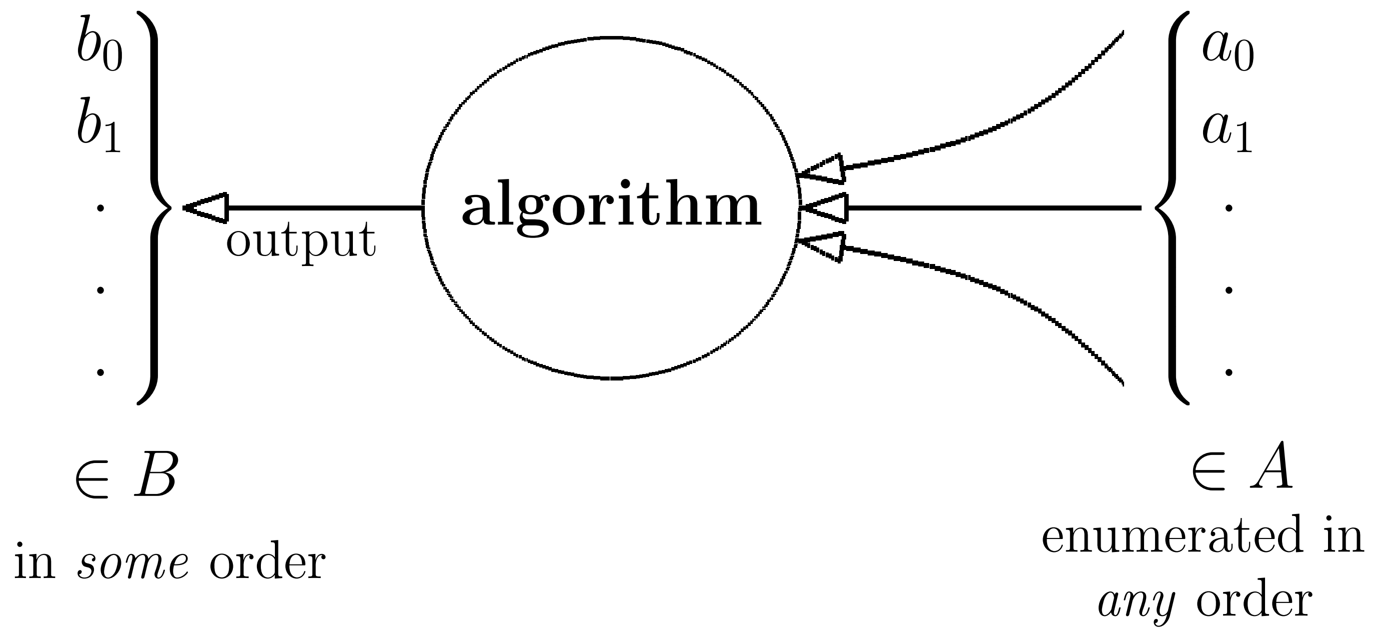
Back in the real world

- You knock on Prof. A's door
- Getting no answer, you try Prof. B (at lunch)
- You then move on to Prof. C, who is in her office . . .
- Always helpful and knowledgeable, she gives you the information you need, and you complete your project . . .

Get two different models of relative computability :

- ★ Compute using an oracle - i.e., call deterministically on values of a total function
- ★ Compute using emergent (or enumerated) information - i.e., call non-deterministically on values of a partial function

Enumeration reducibility



- $n \in \Psi_i^A \iff_{\text{defn}} (\exists \text{ a finite } D \subseteq A)[\langle n, D\rangle \in \Psi]$.

The link with NT-reducibility

If g is total, we have $f \leq_{NT} g \iff f \leq_T g$.

THEOREM
Let f, g be partial functions. Then

$$f \leq_{NT} g \iff \mathrm{Graph}(f) \leq_e \mathrm{Graph}(g).$$

THEOREM ***(Selman's Theorem, 1971)***

For any $A, B \subseteq \mathbb{N}$

$$A \leq_e B \iff \forall X\,[\,B \text{ c.e. in } X \Rightarrow A \text{ c.e. in } X\,].$$

PROOF (sketch) The left-to-right implication I *will* leave to you.

Conversely, assume that $A \not\leq_e B$. We will construct a $C = \cup_{s \geq 0} C_s$ such that B is c.e. in C but A is not c.e. in C.

We satisfy "B c.e. in C" by imposing an overall requirement

$$\exists \langle x, y \rangle \in C \iff x \in B \tag{0.1}$$

for each $x \geq 0$. Call a finite $D \supseteq C_s$ *admissible* if it satisfies Equation (0.1) with D in place of C, but with the right-to-left half of Equation (0.1) restricted to $x \leq s$ (so that the admissible D's can be enumerated from an enumeration of B and a *finite* amount of information about $\overline{B}$).

We satisfy $A \neq W_s^C$ (at stage $s+1$) by looking for some admissible $D \supseteq C_s$ with $x \in W_s^D - A$. If D exists, choose $C_{s+1} = D$ giving $A \neq W_s^C$. Otherwise, either $x \in A - W_s^D$ for some x, all admissible D (so $A \neq W_s^C$ again), or

$$\forall x\,(x \in A \iff \exists \text{ an admissible } D \text{ such that } x \in W_s^D),$$

giving $A \leq_e B$, a contradiction. ∎

Define $A \equiv_e B \iff_{\text{defn}} A \leq_e B \,\&\, B \leq_e A$.

The **enumeration degree** — *or* **e-degree**, *written* $deg_e(A)$ — *of* A *is*

$$\deg_e(A) =_{\text{defn}} \{X \mid X \equiv_e A\}.$$

We define $deg_e(A) \leq deg_e(B) \iff_{\text{defn}} A \leq_e B$.

We write $\boldsymbol{\mathcal{D}}_e =_{\text{defn}}$ *the set of all e-degrees with the ordering* $\leq$.

The **partial degree** *of a partial function* f *is*

$$\begin{aligned} \deg(f) &=_{\text{defn}} \{g \mid \text{Graph}(f) \equiv_e \text{Graph}(g)\} \\ &= \{g \mid f \equiv_{NT} g\} \end{aligned}$$

We write $\boldsymbol{\mathcal{P}}$ = *the set of all partial degrees, with ordering* $\leq$ *defined by* $deg(f) \leq deg(g) \iff_{\text{defn}} \text{Graph}(f) \leq_e \text{Graph}(g) \iff f \leq_{NT} g$.

We say that an e-degree $\mathbf{a}_e$ *is* **total** *if there is a total function* f *with* $\text{Graph}(f) \in \mathbf{a}_e$.

We write **TOT** = *the set of total e-degrees.*

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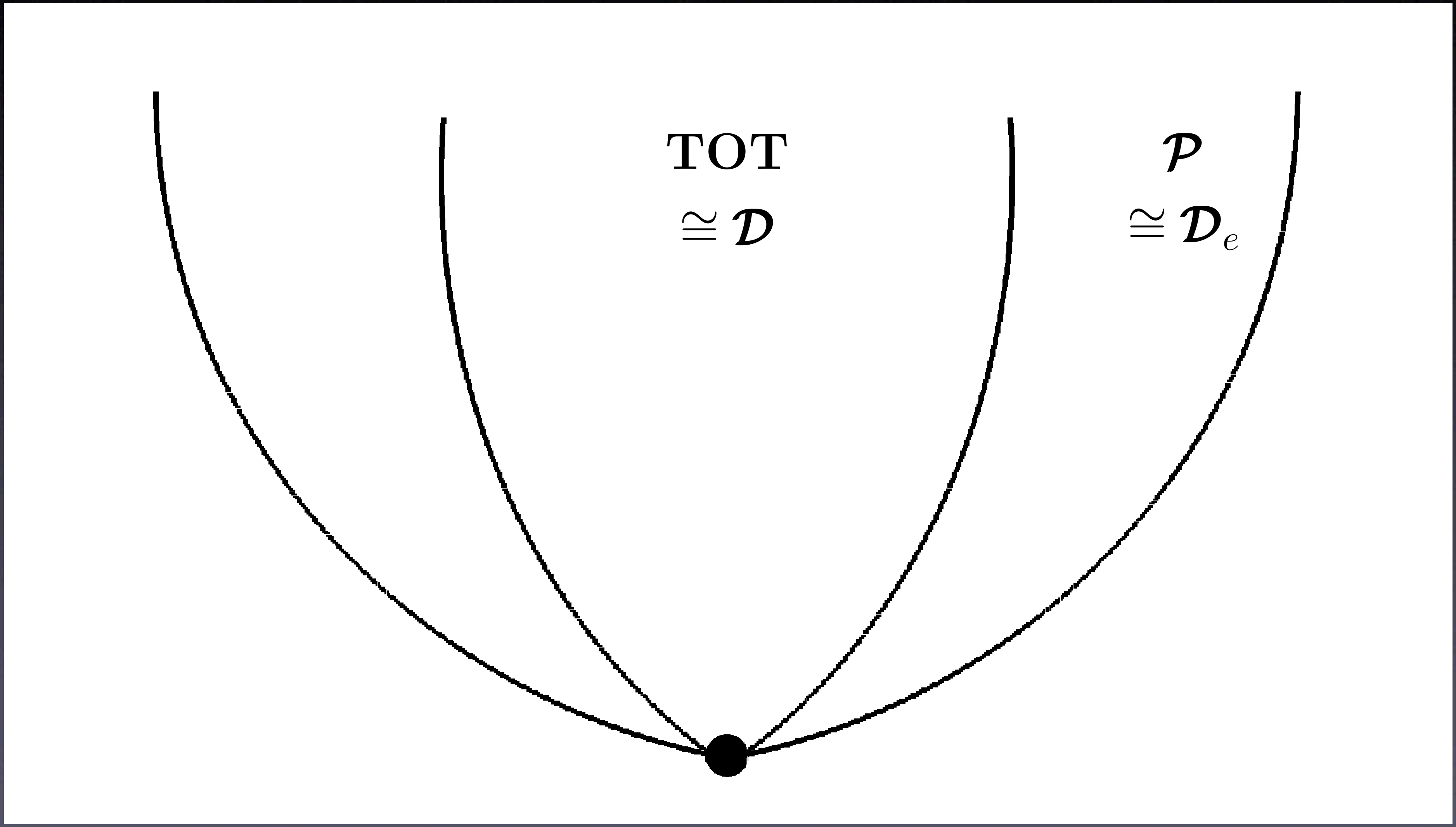
The Natural Embedding



Extending the Jump Operator

Let $K_A = \{x \mid x \in \Psi_x^A\}$.

Then the **e-jump** *of a set* A *is* $J_e^A =_{\text{defn}} A \oplus \overline{K}_A$. *And the* **jump** *of an e-degree* $\mathbf{a} = \deg_e(A)$ *is defined to be* $\mathbf{a}' = \deg_e(A \oplus \overline{K}_A) = \deg_e(J_e^A)$.

We iterate the jump in the usual way to obtain the n^{th} *jump* $\mathbf{a}^{(n)}$ *of* $\mathbf{a}$.

PROPOSITION 0.3.19
The e-jump agrees with the natural embedding of the Turing jump — that is, for each $A \subseteq \mathbb{N}$ *we have* $\iota(\deg(A')) = \deg_e(J_e(\chi_A))$.

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Local information content

THEOREM
For each $A \subseteq \mathbb{N}, n \geq 0$ *we have* $\deg_e(A) \leq \mathbf{0}_e^{(n)} \iff A \in \Sigma_{n+1}$.

- The most important case is $n = 1$, telling us that:

$$\mathcal{D}_e(\leq \mathbf{0}'_e) = \text{the set of all } \Sigma_2 \text{ e-degrees}$$

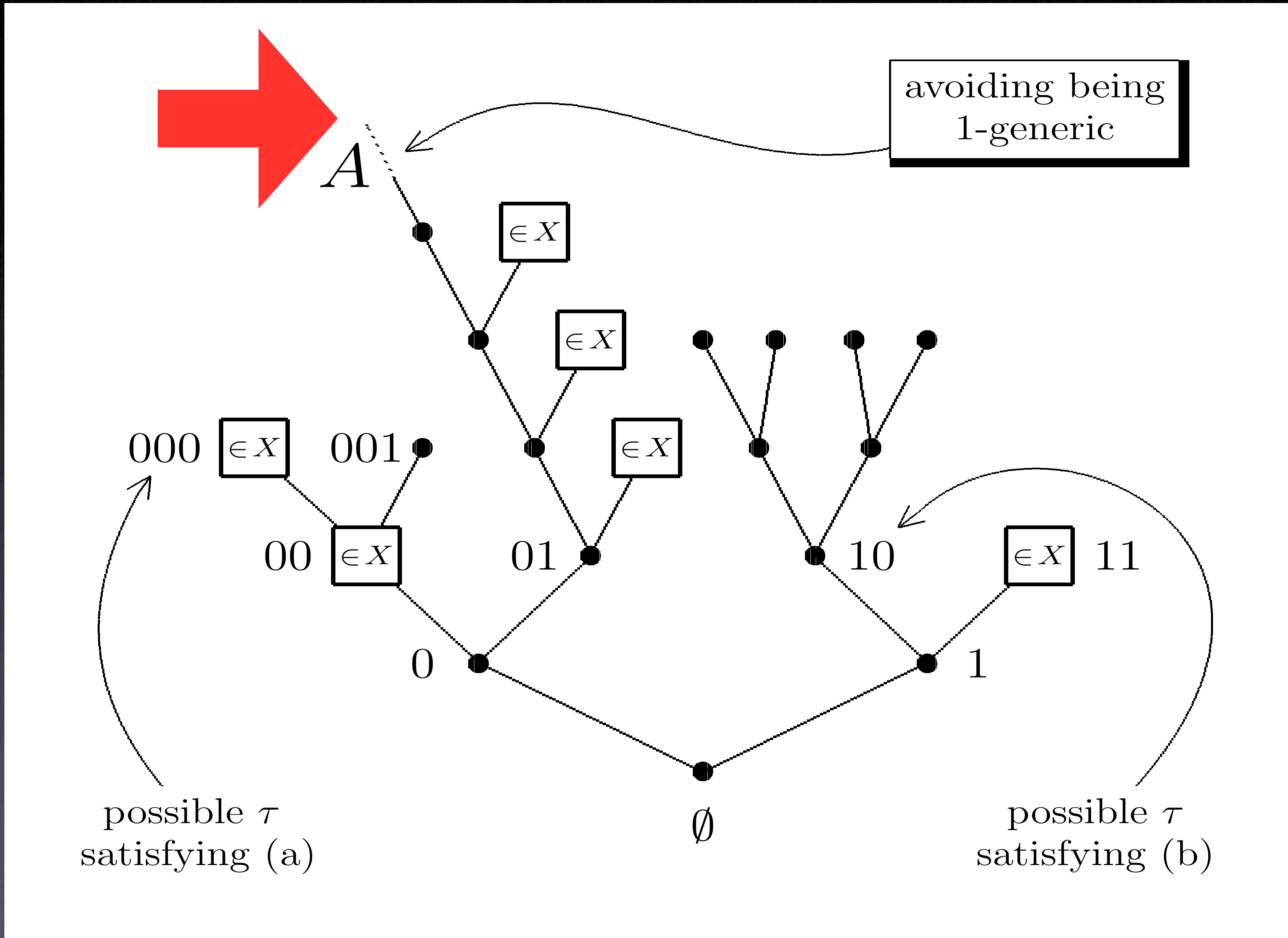
Local forcing

DEFINITION *We say* $A \subseteq \mathbb{N}$ *is* **1-generic** *if for every c.e. set* X *of strings, either*

(a) $(\exists \tau \subset A)\,[\,\tau \in X\,]$, *or*

(b) $(\exists \tau \subset A)(\forall \sigma \supseteq \tau)\,[\,\sigma \notin X\,]$.

We say A *— and any* $\tau' \supseteq$ *such a* τ *—* **forces** X. *We write* $A \Vdash X$ *— or* $\tau' \Vdash X$ *— as appropriate.*


avoiding being 1-generic
A
∈X
∈X
000
∈X
001
∈X
00
∈X
01
10
∈X
11
0
1
possible τ satisfying (a)
∅
possible τ satisfying (b)

n-generics are easy to build using an oracle

THEOREM ***(The Existence Theorem for 1-Generic Sets)***
There exists a 1-generic set $A \leq_T \emptyset'$.

PROOF We construct strings $\sigma_0 \subset \sigma_1 \subset \ldots \subset A = \bigcup_{i \geq 0} \sigma_i$ so that for each $i \geq 0$ we have $\sigma_{i+1} \Vdash W_i$.

The idea is that at stage $i+1$ of the construction σ_{i+1} grabs a string in W_i if it possibly can — and if it cannot it blames σ_i for already having satisfied clause (b) of the definition of forcing W_i.

A simple example

- Show that no 1-generic A is computable.

SOLUTION Define $X = \{\sigma \mid \varphi_i$ and σ disagree on some argument $y\}$. Then

$$\sigma \in X \iff (\exists s, y)[\underbrace{y < |\sigma| \,\&\, \varphi_{i,s}(y) \downarrow \neq \sigma(y)}_{\text{computable relation}}],$$

so $X \in \Sigma_1$ and so is c.e.

This means that if A is 1-generic $A \Vdash X$. So by Definition 0.9.1 there is a $\tau \subset A$ such that $\tau \Vdash X$.

Case (a). $\tau \in X$ — then $\tau(y) \neq \varphi_i(y)$ some $y < |\tau|$. So $A \neq \varphi_i$ since $\tau \subset A$.

Case (b). For all $\sigma \supset \tau$ we have $\sigma \notin X$. But then we cannot have $\varphi(\,|\tau|\,)\downarrow$, since otherwise

$$\sigma = \tau^\frown(1 \dot{-} \varphi_i(\,|\tau|\,)) \in X.$$

So φ is not total and so cannot be the characteristic function of A. $\square$

Forcing in the Turing and enumeration degrees

- If a Turing degree $\underline{a}$ contains a 1-generic set we say $\underline{a}$ is 1-generic
- If an enumeration degree $\underline{a}$ contains a 1-generic set we say $\underline{a}$ is set 1-generic

Forcing partial functions

(mainly Case 1971, and Copestake 1988)

A **partial string** σ *is a finite sequence of symbols from* $\mathbb{N} \cup \{\uparrow\}$. *We write* $S^* = (\mathbb{N} \cup \{\uparrow\})^{<\omega}$ *for the set of all partial strings.*

DEFINITION *A partial function* $\psi \in (\mathbb{N} \cup \{\uparrow\})^{\omega}$ *is* **1-generic** *if for every c.e. set* $X \subseteq S^*$, *either*

(a) $(\exists \sigma \subset \psi)\,[\,\sigma \in X\,]$, *or*

(b) $(\exists \sigma \subset \psi)(\forall \tau \in S^*)\,[\,\sigma \subseteq \tau \Rightarrow \tau \notin X\,]$.

A partial degree, or e-degree, **a** *is* **1-generic** *if there is a 1-generic* $\psi \in$ **a**.

Does set 1-generic = generic for e-degrees?

First notice *if the partial function ψ is 1-generic, then it has no p.c. extension.*

- Ask ψ to force each set $X_i \subseteq S^*$ given by

$$\tau \in X \iff (\exists x < |\tau|)\,[\,\varphi_i(x)\downarrow \neq \tau(x)\,].$$

So - no 1-generic function can be p.c.

In fact …

if ψ is n-generic then Graph(ψ) does not contain an infinite Σ_n subset.

there is no $(n+1)$-generic e-degree below $\mathbf{0}_e^{(n)}$.

Also

if ψ is 1-generic then $Dom(\psi) <_e \psi$.

- To show $\psi \neq \Psi_i^{\mathrm{Dom}(\psi)}$ for any i, ask ψ to force each $X_i \subseteq S^*$ given by

$$\tau \in X_i \iff (\exists x < |\tau|, y)\,[\,\tau(x) \neq y \;\&\; \langle x, y\rangle \in \Psi_i^{\mathrm{Dom}(\tau)}\,]$$

if A is a 1-generic set and $\psi <_e A$, then ψ has a p.c. extension and so is not 1-generic.

- Ask A to force the set X_i of binary strings given by

$$\tau \in X_i \iff (\exists x, y, z)\,[\,\langle x, y\rangle, \langle x, z\rangle \in \Psi_i^{\tau^+} \;\&\; y \neq z\,]$$

And

A is an n-generic set $\iff$ $A = \mathrm{Dom}(\psi)$ *for some n-generic function ψ.*

($\Leftarrow$) Assume A does not force some Σ_n set $X \in 2^\omega$, and show ψ does not force $\widehat{X} = \{\tau \in S^* \mid \mathrm{Dom}(\tau) \in X\}$, where $\mathrm{Dom}(\tau) = \chi_{\mathrm{Dom}(\tau)} \restriction |\tau|$.

($\Rightarrow$) Given A n-generic, build ψ as the union of partial strings ψ_i forcing the i^{th} Σ_n set of partial strings, with $A = \mathrm{Dom}(\psi)$.

So . . . set 1-generic e-degrees are never generic!

COROLLARY ***(Copestake,1988)***
Every 1-generic e-degree $\mathbf{a}$ *bounds a set 1-generic* $\mathbf{b} < \mathbf{a}$, *where no* $\mathbf{c} \leq \mathbf{b}$ *is 1-generic.*

Copestake's basics

(1) *There exists an* n*-generic e-degree below* $\mathbf{0}^{(n)}$ *for each* $n \geq 1$.
(2) *There is no* $(n+1)$*-generic e-degree below* $\mathbf{0}^{(n)}$, *any* $n \geq 1$.
(3) *Every* 1*-generic e-degree is quasi-minimal.*
(4) *Every* 2*-generic e-degree bounds a minimal pair of e-degrees.*
(5) *If* $\mathbf{a}$ *is a* 1*-generic e-degree then every r.e. partial ordering can be embedded below* $\mathbf{a}$ *(in the* 1*-generic degrees below* $\mathbf{a}$*).*

And all results hold with 'set 1-generic' in place of '1-generic'

Comparing 1-generic Turing and e-degrees

For the Turing case →

Every 1-generic $\mathbf{a}$ *is generalised low.*
In particular, all the 1-generic degrees $\leq \mathbf{0}'$ *are low.*

But (C, Li, Sorbi and Yang/ Copestake) →

there exists a set 1-generic e-degree which bounds no minimal pair . . . and hence is not low.

Questions

- Characterise the jumps of the (set) 1-generic e-degrees below $\mathbf{0}'$

- Are e-degrees of 1-generic sets (below $\mathbf{0}'$) closed downwards?

- Are the e-degrees of 1-generic sets (below $\mathbf{0}'$) definable?

Thank you!