# DISCONTINUOUS PHENOMENA AND TURING DEFINABILITY[1]

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ABSTRACT. We discuss the relationship between discontinuity and definability in the Turing degrees, with particular reference to degree invariant solutions to Post's Problem. Proofs of new results concerning definability in lower cones are outlined.

## 1. Introduction

The work of Gödel [1931], [1934], Turing [1936], Church [1936] and others in the 1930's showed that in mathematics many basic decision problems can be reduced to those for particular noncomputable, computably enumerable (c.e.) sets. However (see for example Myhill [1955]) particular examples tend to lie in $\mathbf{0}'$ (the Turing degree of first-order Peano arithmetic and of the halting problem for Turing machines). So it was natural to ask whether there are other c.e. degrees $> \mathbf{0}$ (the degree of the computable sets).

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1.1 Post's Problem [1944]. Does there exist a property $P$ of (indices) of c.e. sets such that

$$e \in P \Rightarrow \emptyset <_T W_e <_T \emptyset'?$$

(See Soare [1987] or Odifreddi [1989] for basic terminology and notation.)

An affirmative answer to this question was independently provided by Friedberg [1957] and Mučnik [1956]. But the property $P$ was not of the form originally envisaged by Post, $P(e)$ saying "$e$ is the index corresponding to a particular finite-injury priority construction". So the question arises as to how *natural* a property we can expect $P$ to be. That is, given that the Friedberg-Mučnik theorem shows the existence of intermediate c.e. degrees, does there exist a description of such a class of degrees which invests them with more than a pathological significance?

One may distinguish three current criteria for describing $P$ to be 'natural':

**(a)** $P$ might be of the form "$e$ is the index of a particular naturally occuring c.e. set". That is, $P$ might show some naturally occuring class of sets in mathematics to have members of intermediate c.e. degree.

There are some striking results in this direction. Feferman [1957] showed that every c.e. degree contains a formal system (which, by Shoenfield [1958], can be chosen to be essentially undecidable), and even (Hanf [1965]) contains a *finitely* axiomatisable first-order theory. Every consistently axiomatisable theory has a model of degree $< \mathbf{0}'$ (Shoenfield [1960]). Matijasevič [1970] et al, in solving Hilbert's tenth problem, showed that every c.e. set is diophantine.

**(b)** The description $P$ may in itself may be particularly simple or natural in form (that is, a $P$ in the spirit of Post's original question).

The first such example is due to Marchenkov [1976], followed by other increasingly convincing solutions to Post's problem in its original form. Most conclusive is the Harrington-Soare [1991] description of such a $P$ in the theory of the lattice of c.e. sets.

Finally, extending criterion (b) to include descriptions of $P$ framed within the language of computability theory itself:

**(c)** $P$ may be recursion theoretically *robust.*

There are two closely related open questions with claims to provide such a property $P$.

1.2 QUESTION (SACKS [1967]). Is there a degree invariant solution to Post's problem? That is, is there an index $e$ such that for any $A, B \subseteq \omega$ we have

$$A <_T W_e^A <_T A'$$

and

$$A \equiv_T B \Rightarrow W_e^A \equiv_T W_e^B?$$

1.3 QUESTION (SLAMAN, see Sacks [1985]). Is there an intermediate c.e. $\mathbf{a}$ which is *definable* (in the language of the theory of $\boldsymbol{\mathcal{C}}$)?

A positive solution to Sacks' longstanding question would be provided by any *relativisable* construction of a definable intermediate c.e. $\mathbf{a}$. Of course, there are definitions of $\mathbf{a}$ outside the theory of $\boldsymbol{\mathcal{C}}$ which would suffice for 1.2.

Sacks' question is not just of interest to computability theorists, but is closely connected to two very general conjectures in set theory (see Slaman [ta]).

Let $\mathcal{F}$ be the class of all degree invariant functions from $2^\omega$ to $2^\omega$, with the ordering $\leq_M$ given by:

$$f \leq_M g \Leftrightarrow (\exists X)(\forall Y \geq_T X)[f(Y) \leq_T g(Y)].$$

(That is, $f \leq_M g$ if $f \leq_T g$ a.e.)

1.4 MARTIN'S CONJECTURE. Assume AD. Then

(a) If $f \not\geq_M id$ then $f$ is constant a.e.
(b) On those elements of $\mathcal{F}$ which are $\geq_M id$, $\leq_M$ is a quasi-wellordering with successor equal to the Turing jump.

We say that a Borel equivalence relation $U$ on Polish spaces, with countable equivalence classes, is *universal* if and only if for every other such equivalence relation $E$ there is a Borel function $f$ such that for all $x$ and $y$

$$x \sim_E y \Leftrightarrow f(x) \sim_U f(y).$$

1.5 KECHRIS'S CONJECTURE. $\equiv_T$ is universal.

Slaman and Steel [1988] have shown that arithmetic equivalence is universal. Roughly speaking, Martin's conjecture says that (assuming AD) the only nontrivial, definable, degree invariant operators are the known jump operators and their iterates. It implies the failure Kechris's conjecture and (in particular) a negative answer to Sack's question.

We note that for a given index $e$, any counterexample $A, B$ to 1.2 may have to be quite complicated.

1.6 THEOREM (SLAMAN AND STEEL, unpublished). *There is an index $e$ such that for any $A, B \in \Delta_2^0$ we have*

$$A <_T W_e^A <_T A'$$

*and*

$$A \equiv_T B \Rightarrow W_e^A \equiv_T W_e^B$$

Also, Lachlan [1975a] showed that there is no index $e$ as in question 1.2 for which the Turing reductions between $W_e^A$ and $W_e^B$ can be obtained *uniformly* from those for the reductions between $A$ and $B$.

Any discussion of definability of computably enumerable Turing degrees is incomplete without some mention of the notion of *bi-interpretability*. The Bi-Interpretability Conjecture for $\boldsymbol{\mathcal{C}}$ (due to Harrington and Slaman) posits the existence of a kind of pseudo-isomorphism between $\omega$ (the standard model of first-order arithmetic) and $\boldsymbol{\mathcal{C}}$, sufficient to invest $\boldsymbol{\mathcal{C}}$ with some familiar properties of $\omega$ (such as rigidity, definability of arithmetically describable sets – in particular, of all individual members of the domain). Roughly speaking, a bi-interpretation between $\omega$ and $\boldsymbol{\mathcal{C}}$ consists of:

(1) A coding of $\omega$ into $\boldsymbol{\mathcal{C}}$ (involving specifying a collection of c.e. degrees, and relations on those degrees which represent addition and multiplication),
(2) A mapping of each $x \in \boldsymbol{\mathcal{C}}$ to an index $e$ for a representative $W_e$ of $\mathbf{x}$, and
(3) A uniform definition in $\boldsymbol{\mathcal{C}}$ of the relationship between any $\mathbf{x}$ and the code of the index $e$ for the representative $W_e$ of $\mathbf{x}$.

The best result in this direction so far is:

1.7 Theorem (Harrington and Slaman). *$Th(\boldsymbol{\mathcal{C}}) \equiv_{1-1}$ the first-order theory of $\omega$ (and so $\in \mathbf{0}^{(\omega)}$).*

Unfortunately, a proof of bi-interpretability would not necessarily solve question 1.3 so as to give a degree invariant solution to Post's problem. The definition of any intermediate c.e. degree $\mathbf{x}$ would just describe (degree theoretically) the index $e$ for $W_e \in \mathbf{x}$, and requiring this to give us a relativisable definition of an intermediate c.e. degree effectively returns us to the original question.

## 2. Continuity in $\boldsymbol{\mathcal{C}}$

A first step in the search for a *natural* definition of a singleton in $\boldsymbol{\mathcal{C}}$ must be to consider questions of the form:

- Is there a definable $P \subset \boldsymbol{\mathcal{C}} - \{\mathbf{0}, \mathbf{0}'\}$ with a *least* or *greatest* member? Or even:
- Is there such a $P$ with *minimal* or *maximal* elements?

In doing this one must take account of known *continuity* properties of $\boldsymbol{\mathcal{C}}$.

Let $F(x, y)$ be a formula in the language $\mathcal{L}$ for $\langle \boldsymbol{\mathcal{C}}, \leq, \cup, \cap, \mathbf{0}, \mathbf{0}' \rangle$. The strongest such continuity notion (see Harrington and Soare [ta]) is:

2.1 Definition. If $\mathbf{a}, \mathbf{b} \in \boldsymbol{\mathcal{C}}$, we say $F(x, y)$ *holds in a neighbourhood of* $\mathbf{a}, \mathbf{b}$ if there exist c.e. degrees $\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1$ with $\mathbf{a} \in (\mathbf{a}_0, \mathbf{a}_1)$ and $\mathbf{b} \in (\mathbf{b}_0, \mathbf{b}_1)$ such that $F(\mathbf{x}, \mathbf{y})$ holds for all c.e. $\mathbf{x} \in (\mathbf{a}_0, \mathbf{a}_1)$ and $\mathbf{y} \in (\mathbf{b}_0, \mathbf{b}_1)$.

Then:

2.2 Theorem (Harrington-Soare). *Let $F(x, y)$ be any open formula of $\mathcal{L}$ with two variables, such that $F(\mathbf{a}, \mathbf{b})$ holds for c.e. degrees $\mathbf{a} \neq \mathbf{b}$. Then $F(x, y)$ holds in a neighbourhood of $\mathbf{a}, \mathbf{b}$.*

Proof. This reduces to considering four basic cases:

**1)** If $F(\mathbf{x}, \mathbf{y})$ says that $\mathbf{0} < \mathbf{x} < \mathbf{y} < \mathbf{0}'$, then the required intervals $(\mathbf{a}_0, \mathbf{a}_1)$, $(\mathbf{b}_0, \mathbf{b}_1)$ are easily got via the Sacks [1964] Density Theorem.

**2)** $F(\mathbf{x}, \mathbf{y})$ says that $\mathbf{x}, \mathbf{y}$ form a minimal pair (with $\mathbf{x} \cup \mathbf{y} < \mathbf{0}'$ by the Lachlan [1966] Nondiamond Theorem). Then the result follows using the Harrington-Soare [ta] 'continuity of capping' theorem to get $\mathbf{a}_1, \mathbf{b}_1$ and Sacks Density to get $\mathbf{a}_0, \mathbf{b}_0$.

**3)** $F(\mathbf{x}, \mathbf{y})$ says that $\mathbf{x}, \mathbf{y}$ form a splitting of $\mathbf{0}'$. This is the dual of case 2, here using the 'continuity of cupping' result of Ambos-Spies, Lachlan and Soare [ta].

**4)** $F(\mathbf{x}, \mathbf{y})$ says $\mathbf{x} \mid \mathbf{y}$ with $\mathbf{x} \cup \mathbf{y} \neq \mathbf{0}'$ and $\mathbf{x} \cap \mathbf{y} \neq \mathbf{0}$. In this case, one gets $\mathbf{a}_0, \mathbf{b}_0$ by applying the Robinson [1971] Low Splitting Theorem to $\mathbf{a}$ and $\mathbf{b}$ over some low $\mathbf{d} > \mathbf{0}$, incorporating avoidance of the appropriate lower cone below $\mathbf{a}$ or $\mathbf{b}$. And then get $\mathbf{a}_1, \mathbf{b}_1$ by the Robinson [1971] extension of Sacks Density applied to intervals $(\mathbf{a}, \mathbf{a} \cup \mathbf{b})$ and $(\mathbf{b}, \mathbf{a} \cup \mathbf{b})$, avoiding the respective upper cone above $\mathbf{b}_0$ or $\mathbf{a}_0$. $\square$

Athough this general continuity result cannot be extended to open formulas with more than two variables, or to formulas with quantifiers, further examples of continuity include (1) (Seetapun [ta]) the continuity of Lachlan [1979] nonbounding, and (2) (arising from recent work of Cooper, Lachlan, Seetapun, Slaman and Yi) the continuity of Harrington nonsplitting base.

Weaker notions of continuity (framed in terms of limits) provide further limitations on such definable properties $P$. One of the longest-standing continuity problems, that concerning the existence of *major subdegrees* (see Harrington and Soare [ta]) for intermediate c.e. degrees, asks about lower continuity of $\varphi(a,x)$, where $a$ is a parameter and

$$\varphi(\mathbf{a},\mathbf{x}) \Leftrightarrow \forall \mathbf{y}(\mathbf{a} \cup \mathbf{y} = \mathbf{0}' \Leftrightarrow \mathbf{x} \cup \mathbf{y} = \mathbf{0}').$$

Definability relative to other structures looks more promising in that there are known nontrivial discontinuity properties. For instance density fails in both the d.r.e. degrees $\mathbf{D}_2$ (Cooper-Harrington-Lachlan-Lempp-Soare [1991]) and in the enumeration degrees (see Cooper [1990] or Calhoun-Slaman [ta]). See Arslanov [ta] for results relating to definability of c.e. degrees in $\mathbf{D}_2$.

## 3. Discontinuity and definability

Initial segment results (see Lerman [1983]) provided the technical background to early definability results such as those (over $\boldsymbol{\mathcal{D}}'$) of Jockusch and Simpson [1976], and:

3.1 THEOREM (SIMPSON [1977]). *The degree of Th($\boldsymbol{\mathcal{D}}$) (= the first-order theory of $\boldsymbol{\mathcal{D}}$) = the degree of Th($\mathcal{N}$) (the theory of second-order arithmetic).*

These have largely been superseded by the technically simpler ad hoc coding techniques of Nerode and Shore [1980], [1980a], and more recently of Slaman and Woodin [1986] (see also Slaman [1991]), giving rise to an impressive body of definability results for $\boldsymbol{\mathcal{D}}$ and other degree structures. Examples in addition to Theorem 1.7 above are given by:

3.2 THEOREM.

(1) (SLAMAN-WOODIN [ta]) $\mathbf{0}''$ *is definable in* $\boldsymbol{\mathcal{D}}$.

(2) (SHORE [1988]) *High$_n$, Low$_n$ is definable in* $\boldsymbol{\mathcal{D}}(\leq \mathbf{0}')$ *for each* $n \geq 3$.

(3) (SLAMAN, private communication) *High$_n$, Low$_n$ are definable in* $\boldsymbol{\mathcal{C}}$ *for* $n \geq 2, 3$ *(respectively).*

Improvements on these results seem to require discontinuity derived from the sort of local degree theory involving higher priority, in particular Lachlan [1975] and Harrington nonsplitting and $\emptyset'''$-priority. For example, splitting and relative nonsplitting techniques in $\boldsymbol{\mathcal{D}}(\leq \mathbf{0}')$ and the pseudo-jump machinery of Jockusch and Shore [1984] yield:

3.3 Theorem (Cooper [ta1], [1994]). *There is a set $P$ of degrees definable in $\boldsymbol{\mathcal{D}}$ which has greatest element $\mathbf{0}'$. Moreover, we can obtain $P = \boldsymbol{\mathcal{C}}$.*

Slaman and Woodin [1986] coding had previously led to the definability of $\boldsymbol{\mathcal{C}}$ in $\boldsymbol{\mathcal{D}}(\leq \mathbf{0}')$ from a finite number of parameters. The discontinuity here arises in showing the definability of uniformly low sets $R, S$ of c.e. degrees which (by Welch [1981]) pairwise generate $\boldsymbol{\mathcal{C}}$. $R$, say, is defined by constructing $\mathbf{a}, \mathbf{b} < \mathbf{0}'$ such that

$$\mathbf{x} \in R \Leftrightarrow \ \mathbf{x} \text{ is a minimal solution of } \mathbf{x} \neq (\mathbf{x} \cup \mathbf{a}) \cap (\mathbf{x} \cup \mathbf{b}).$$

The above results seem close to the limits of what is possible with currently known techniques. Definability is dependent upon structural discontinuity, concerning which there are a number of intractible questions, which may turn out to be crucial.

## 4. Discontinuity in cones

Despite Theorem 2.2 above, the following discontinuity result is obtainable via an ingenious application of distributivity (Lachlan [1972]) of $\boldsymbol{\mathcal{D}}_{wtt}$ (the upper semi-lattice of c.e. wtt-degrees).

4.1 Theorem (Stob [1983]). *There is a lower cone $\boldsymbol{\mathcal{C}}(\leq \mathbf{a})$ in which continuity of cupping fails.*

Proof. Construct a minimal pair $\mathbf{a}_0, \mathbf{a}_1$ of c.e. degrees whose join is *contiguous* (i.e. consists of just one wtt-degree).

To see that splitting of $\mathbf{a}_0 \cup \mathbf{a}_1$ is discontinuous at $\mathbf{a}_0, \mathbf{a}_1$, let $A_0, A_1 \in \mathbf{a}_0, \mathbf{a}_1$ respectively, and say $B \in \mathbf{b}$ with $\mathbf{b} \cup \mathbf{a}_1 = \mathbf{a}_0 \cup \mathbf{a}_1$. Then $A_0 \leq_{wtt} A_0 \oplus A_1 \leq_{wtt} B \oplus A_1$ (since $\mathbf{b} \cup \mathbf{a}_1$ is contiguous). Then by distributivity we have $A_0 \equiv_{wtt} B_0 \oplus A_{10}$, say, with $B_0 \leq_{wtt} B$, $A_{10} \leq_{wtt} A_1$. But $A_{10} \leq_{wtt} A_0, A_1 \Rightarrow A_{10}$ computable. So $A_0 \equiv_{wtt} B_0 \leq_{wtt} B$, giving $\mathbf{a}_0 \leq \mathbf{b}$. $\square$

Although contiguous degrees are necessarily $\text{low}_2$, it can be shown that degrees $\mathbf{a}$ with discontinuous splittings are widely distributed in $\boldsymbol{\mathcal{C}}$ (see Cooper and Yi [ta]). For example, there exist such degrees $\mathbf{a}$ which are high.

For upper cones of $\mathcal{C}$ there are some remarkable recent results.

We say that $\mathbf{a}$ *branches uniquely in* $\mathcal{C}$ if and only if there exist c.e. $\mathbf{b}_0, \mathbf{b}_1 > \mathbf{a}$ for which

(i) $\mathbf{b}_0 \mid \mathbf{b}_1$,
(ii) $\mathbf{b}_0 \cap \mathbf{b}_1 = \mathbf{a}$, and
(iii) $\mathbf{c}_0 \cap \mathbf{c}_1 = \mathbf{a} \Rightarrow (\exists j \leq 1)[\mathbf{c}_j \leq \mathbf{b}_0 \,\&\, \mathbf{c}_{1-j} \leq \mathbf{b}_1]$.

A particular case of a more general result is:

4.2 THEOREM (AMBOS-SPIES, SHORE [ta]. *There is a c.e.* $\mathbf{a}$ *which branches uniquely in* $\mathcal{C}$.

PROOF. A combination of Lachlan [1966] branching and Fejer [1983] nonbranching. □

It follows from the proof (Shore, private communication) that the constructed branchings $\mathbf{b}_0, \mathbf{b}_1$ of $\mathbf{a}$ are already *individually* definable in $\mathcal{C}(\geq \mathbf{a})$.

One can also define a notion of *unique splitting* in intervals of $\mathcal{C}$. The following combines Sacks [1963] splitting with Harrington nonsplitting to extend a recent result of Leonhardi [1994].

4.3 THEOREM (COLES [ta]). *There is a c.e.* $\mathbf{a}$ *above which* $\mathbf{0}'$ *splits uniquely in* $\mathcal{C}$.

## 5. Definability in lower cones of $\mathcal{C}$

The following gives a special significance to definability in lower cones of $\mathcal{C}$.

5.1 THEOREM (AMBOS-SPIES, private communication). *Every lower cone* $\mathcal{C}(\leq \mathbf{a})$ *(*$\mathbf{a} \neq \mathbf{0}$*) is an automorphism base for* $\mathcal{C}$.

PROOF. Let $\mathbf{a} > \mathbf{0}$, and let $f$ be a nontrivial automorphism of $\mathcal{C}$ (with $\mathbf{a} = f(\mathbf{a})$, otherwise there is nothing to prove). Find a $\mathbf{c} \in \mathbf{PS}$ such that $\mathbf{c} \mid \mathbf{a}, f(\mathbf{c})$ and $\mathbf{c}$ has the '*downward separation property*' (that is, for all distinct $\mathbf{d}_0, \mathbf{d}_1 \mid \mathbf{c}$, $\mathcal{C}(< \mathbf{d}_0, \mathbf{c}) \neq \mathcal{C}(< \mathbf{d}_0, \mathbf{d}_1)$). Then $\mathcal{C}(< \mathbf{a}, \mathbf{c}) \neq \mathcal{C}(< \mathbf{a}, f(\mathbf{c})) = \mathcal{C}(< f(\mathbf{a}), f(\mathbf{c}))$. Since $f$ is an automorphism there is some $\mathbf{d} \in \mathcal{C}(< \mathbf{a}, \mathbf{c})$ with $\mathbf{d} \neq f(\mathbf{d}) \in \mathcal{C}(< f(\mathbf{a}), f(\mathbf{c}))$. □

5.2 COROLLARY. *If there is a definable* $\mathbf{a} > \mathbf{0}$ *for which* $\boldsymbol{\mathcal{C}}(\leq \mathbf{a})$ *is rigid, then* $\boldsymbol{\mathcal{D}}$ *is rigid.*

PROOF. By Slaman and Woodin [ta], if $\boldsymbol{\mathcal{C}}$ is rigid so is $\boldsymbol{\mathcal{D}}$. Similarly, the definability and rigidity of $\boldsymbol{\mathcal{C}}(\leq \mathbf{a})$ gives the rigidity of $\boldsymbol{\mathcal{C}}$, and hence of $\boldsymbol{\mathcal{D}}$. $\square$

A first step, both towards finding a rigid $\boldsymbol{\mathcal{C}}(\leq \mathbf{a})$ *and* a definable $\mathbf{b} \in \boldsymbol{\mathcal{C}}$, is to find a definable $\mathbf{b}$ in some lower cone $\boldsymbol{\mathcal{C}}(\leq \mathbf{a})$.

5.3 DEFINITION. $\mathbf{b}$ is a *nonsplitting base* in $\boldsymbol{\mathcal{C}}(\leq \mathbf{a})$ if and only if $\mathbf{b} < \mathbf{a}$ and for all $\mathbf{c} \cup \mathbf{d} = \mathbf{a}$ *either* $\mathbf{c} \cup \mathbf{b} = \mathbf{a}$ *or* $\mathbf{d} \cup \mathbf{b} = \mathbf{a}$.

5.4 THEOREM. *There is a lower cone* $\boldsymbol{\mathcal{C}}(\leq \mathbf{c})$ *containing a least nonsplitting base* $\mathbf{a}$ *(so* $\mathbf{a}$ *is definable in* $\boldsymbol{\mathcal{C}}(\leq \mathbf{c})$ *).*

(We note that $\mathbf{c} \neq \mathbf{0}'$ by an unpublished result of Lachlan, Seetapun and Soare.)

PROOF. We give a module for a typical batch of requirements. Let $(W, U_0, U_1)$ and $(\Theta, \Phi, \Xi, \widehat{\Theta}_0, \widehat{\Theta}_1, \Psi)$ be typical entries in respective standard lists of all triples of c.e. sets and 6-tuples of p.c. functionals. We construct c.e. sets $A$, $C$, $B_{0,W,\Theta}$, $B_{1,W,\Theta}$ and p.c. functionals $\Delta$, $\Omega_{W,\Theta}$, $\Upsilon_{0,W,\Theta}$, $\Upsilon_{1,W,\Theta}$, $\Gamma_{0,\Phi,W,\Theta}$, $\Gamma_{1,\Phi,W,\Theta}$, $\Lambda_{U,\Xi,\widehat{\Theta}}$, $\widehat{\Lambda}_{U,\Xi,\widehat{\Theta}}$ satisfying an overall requirement $A = \Delta^C$ and all requirements of the form:

$$\mathcal{S}_{W,\Theta}: \quad W = \Theta^C \Rightarrow C = \Omega_{W,\Theta}^{B_{0,W,\Theta}, B_{1,W,\Theta}},\ B_{i,W,\Theta} = \Upsilon_{i,W,\Theta}^C\, texteach\, i \leq 1,$$

(A requirement saying: If $\mathbf{w} \leq \mathbf{c}$ prepare a splitting $\mathbf{c} = \mathbf{b}_0 \cup \mathbf{b}_1$ )

$$\mathcal{P}_{i,\Phi,W,\Theta}: \quad C = \Phi^{W, B_{i,W,\Theta}} \Rightarrow A = \Gamma_{i,\Phi,W,\Theta}^W,\ (i \leq 1),$$

(If $\mathbf{a} \not\leq \mathbf{w}$ then $\mathbf{w} \cup \mathbf{b}_0$, $\mathbf{w} \cup \mathbf{b}_1$ is a *nontrivial* splitting of $\mathbf{c}$)

$$\mathcal{Q}_{U,\Xi,\widehat{\Theta}}: \quad \left.\begin{array}{l} C = \Xi^{U_0,U_1} \\ \& \, U_0 = \widehat{\Theta}_0^C \, \& \, U_1 = \widehat{\Theta}_1^C \end{array}\right\} \Rightarrow C = \Lambda_{U,\Xi,\widehat{\Theta}}^{U_0,A} \vee C = \widehat{\Lambda}_{U,\Xi,\widehat{\Theta}}^{U_1,A},$$

(Allow no nontrivial splittings of $\mathbf{c}$ above $\mathbf{a}$)

$$\mathcal{R}_\Psi: \quad C \neq \Psi^A.$$

**The atomic strategies:**

**(a) For $\mathcal{S}_{W,\Theta}$:** We arrange all $\Theta$-*expansionary* stages (that is, those stages at which the observed length of agreement $\ell(W,\Theta^C)$ of $W$ and $\Theta^C$ attains a new maximum) to be $\Omega_{W,\Theta}$-expansionary stages. Following a $C$-change (at $z$ say) we *rectify* $\Omega_{W,\Theta}$ (that is, redefine $\Omega^{B_{0,W,\Theta},B_{1,W,\Theta}}_{W,\Theta}(z)=C(z)$) using a $B_{i,W,\Theta}\upharpoonright\omega_{W,\Theta}(z)$-change, some $i=0$ or 1 (permitted via $\Upsilon_{i,W,\Theta}$).

**(b) For $\mathcal{P}_{i,\Phi,W,\Theta}$:** All $\Phi$-expansionary stages are arranged to be $\Gamma_{i,\Phi,W,\Theta}$-expansionary stages. Associate with each candidate $w$ for membership of $A$ an *agitator* $a_w$, and keep $\Gamma_{i,\Phi,W,\Theta}$ *honest* as far as possible by defining $\gamma(w)>\varphi(a_w)$. We respond to $w\searrow A$ by prompting $a_w\searrow C$ and imposing a $B_{i,W,\Theta}$-restraint (maintained until the next $\Phi$-expansionary stage), using the consequent $W\upharpoonright\varphi(a_w)$-change to rectify $\Gamma_{i,\Phi,W,\Theta}$.

**(c) For $\mathcal{Q}_{U,\Xi,\widehat{\Theta}}$:** Application of the Lachlan [1975] nonsplitting strategy. That is, we use $A$-changes (below $\lambda(x)$ say) and $C$-restraints with the equations $U_i=\widehat{\Theta}^C_i$ to ensure to ensure that $\Lambda_{U,\Xi,\widehat{\Theta}}$ (and $\widehat{\Lambda}_{U,\Xi,\widehat{\Theta}}$) is honest at $x$. We rely on $x\searrow C$ producing a $U_0\upharpoonright\xi(x)$- or $U_1\upharpoonright\xi(x)$-change. We use a $U_0$-change to rectify $\Lambda_{U,\Xi,\widehat{\Theta}}$. We respond to a $U_1$-change with an $A\upharpoonright\lambda(x)$-change to rectify $\Lambda_{U,\Xi,\widehat{\Theta}}$, while implementing the auxiliary $\widehat{\Lambda}_{U,\Xi,\widehat{\Theta}}$-strategy.

As usual, the $\widehat{\Lambda}_{U,\Xi,\widehat{\Theta}}$-strategy will only in fact arise in response to unsuccessful attempts to satisfy a particular lower priority $\mathcal{R}_\Psi$-requirement, in which case the $\widehat{\Lambda}_{U,\Xi,\widehat{\Theta}}$-strategy is passed down to $\mathcal{R}_\Psi$-requirements of even lower priority in synchronisation with the $\Lambda_{U,\Xi,\widehat{\Theta}}$-activity in relation to the instigating $\mathcal{R}_\Psi$-requirement, and acts similarly to the $\Lambda_{U,\Xi,\widehat{\Theta}}$-strategy except in so far as it is assured (by astute choice of $\widehat{\Lambda}_{U,\Xi,\widehat{\Theta}}$-uses) of $\widehat{\Lambda}_{U,\Xi,\widehat{\Theta}}$-rectifying $U_1$-changes. Harrington honestification (to retain the proper relationship between $\Xi$- and $\Lambda_{U,\Xi,\widehat{\Theta}}$-, $\widehat{\Lambda}_{U,\Xi,\widehat{\Theta}}$-uses) is avoided through controls on $U_0,U_1$ via the equations $U_0=\widehat{\Theta}^C_0$, $U_1=\widehat{\Theta}^C_1$.

**(d) For $\mathcal{R}_\Psi$:** A standard Friedberg-Mučnik strategy suffices.

There is a main conflict between these strategies which we address in the module below. In order to satisfy $\mathcal{R}_\Psi$, we may need to enumerate some $x$ into $C$, while imposing an $A \upharpoonright \psi(x)$-restraint. *But* this may:

(i) lead to an $A \upharpoonright \lambda(x)$-change on behalf of $\mathcal{Q}_{U,\Xi,\widehat{\Theta}}$ as a result of an instance of $\Lambda_{U,\Xi,\widehat{\Theta}}$-strategy failure (so injuring the $A$-restraint for $\mathcal{R}_\Psi$), and
(ii) force a $B_{i,W,\Theta}$-change for $\mathcal{S}_{W,\Theta}$ which damages honesty in $\mathcal{P}_{i,\Phi,W,\Theta}$. *Then*
(iii) the $A$-change from (i) cannot be recorded by $\Gamma_{i,\Phi,W,\Theta}$ due to loss of honesty in (ii).

**Basic module for $\mathcal{R}_\Psi$ below the $\Lambda_{U,\Xi,\widehat{\Theta}}$-strategy for $\mathcal{Q}_{U,\Xi,\widehat{\Theta}}$ operating below $\mathcal{S}_{W,\Theta}$ and $\mathcal{P}_{i,\Phi,W,\Theta}$**

Assume infinitely many $\Theta$-, $\Phi$- and $\Xi$-*expansionary stages*, organised in such a way that all $\Xi$-expansionary stages are $\Theta$- and $\Phi$-expansionary stages, and assume the module to only act at $\Xi$-expansionary stages.

**(1)** *Choose* a $\Gamma_{i,\Phi,W,\Theta}$*-rectification level* $w$ for $\mathcal{R}_\Psi$.

**(2)** *Choose* a $\Phi$*-agitator* $a_w > w$, and *define* $\delta(w) > a_w$.

**(3)** *Wait* for $\varphi(a_w)\downarrow$. Then *restrain* $B_{i,W,\Theta} \upharpoonright \varphi(a_w)$ and $C \upharpoonright \vartheta(\varphi(a_w))$ (both at $\mathcal{R}_\Psi$), and *define* $A = \Gamma^W_{i,\Phi,W,\Theta}$ up to $a_w$ with $\gamma(w) > \varphi(a_w)$.

**(4)** *Choose* a $\Lambda_{U,\Xi,\widehat{\Theta}}$*-threshold* $t_w > \varphi(a_w)$ with $\Lambda^{U_0,A}_{U,\Xi,\widehat{\Theta}}(t_w)\uparrow$ and *wait* for $\Xi^{U_0,U_1}(t_w)\downarrow$.

**(5)** *Define* $\Lambda^{U_0,A}_{U,\Xi,\widehat{\Theta}}(t_w) = C(t_w)$ with $\xi(t_w) < \lambda(t_w)$ and $\Delta^C(\lambda(t_w))\uparrow$. *Restrain* $C \upharpoonright \widehat{\vartheta}_i(\xi(t_w))$, each $i \le 1$, at $\mathcal{R}_\Psi$.

**(6)** *Choose* a potential witness $x > t_w$ (so $\omega(x) > \varphi(a_w)$) for $\mathcal{R}_\Psi$.

**(7)** *Wait* for $\Psi^A(x)\downarrow$.

**(8)** *Define* $C(x) \neq \Psi^A(x)$ and *restrain* $A \restriction \psi(x)$. Then *ask:* Is $\lambda(t_w) > \psi(x)$?

**(a)** If *yes* - *Change* $B_{1-i} \restriction \omega(x)$ and *proceed* to (9)(b). *Outcome:* $\mathcal{R}_\Psi$ satisfied.

**(b)** *No* - *Enumerate* $t_w$ into $C$ and *change* $B_{1-i} \restriction \omega(t_w)$. *Proceed* to (9).

**(9)** *Ask:* Has $U_0 \restriction \xi(t_w)$ changed?

**(a)** Yes - Use the $U_0$-change to *rectify* $\Lambda_{U,\Xi,\widehat{\Theta}}$. $\mathcal{R}_\Psi$ again satisfied.

**(b)** *No* - *Enumerate* $\lambda(t_w)$ into $A$ and $a_w$ into $C$. *Proceed* to (10), while *implementing* the auxiliary $\widehat{\Lambda}_{U,\Xi,\widehat{\Theta}}$-strategy.

**(10)** Use the $W \restriction \varphi(t_w)$-change to *rectify* $\Gamma_{i,\Phi,W,\Theta}$ and then *return* to (1).

For further discussion of interactions along and off the true path see Cooper [ta2]

## 6. Definability in $\boldsymbol{\mathcal{D}}(\leq \mathbf{a}\,\text{c.e.})$

Richer structure, often accessible via simpler techniques, is a feature of c.e. topped lower cones of $\Delta^0_2$ degrees. There is an expectation that definability in such lower cones may be correspondingly more approachable, and that there may be an eventual dividend in the form of definability of c.e. singletons in $\boldsymbol{\mathcal{D}}(\leq \mathbf{0}')$.

Bounding 1-generic degrees with non-zero c.e. $\mathbf{a}$'s (see Odifreddi [ta], Jockusch [1980]) gives the rich basic structure for $\boldsymbol{\mathcal{D}}(\leq \mathbf{a}\,\text{c.e.})$ (e.g., embeddings, minimal pairs, non-c.e. elements, not a lattice, etc) originally discovered by Sacks [1963] and Yates [1970]. The development of full-approximation tree constructions (Yates [1970] and Cooper [1973]) resulted in very strong initial segment embedding results for $\boldsymbol{\mathcal{D}}(\leq \mathbf{a}\,\text{c.e.})$. Building on the Yates [1970] embedding of a minimal degree in any non-trivial c.e. topped lower cone, the most comprehensive such result is:

6.1 THEOREM (LERMAN [1983]). *Every* $\emptyset'$*-presentable upper semi-lattice (with least element) is embeddable as an initial segment of* $\boldsymbol{\mathcal{D}}(\leq \mathbf{a}\,c.e.)$ $(\mathbf{a} \neq \mathbf{0})$*.*

(If $\mathbf{a} \leq \mathbf{0}'$ then $\boldsymbol{\mathcal{D}}(\leq \mathbf{a})$ is $\mathbf{a}^{(3)} \leq \mathbf{0}^{(4)}$ presentable, so there is still considerable scope for improvement here. One should note however that any *lattice* embeddable as an initial segment of $\boldsymbol{\mathcal{D}}(\leq \mathbf{a})$ must be $\Sigma^0_3$-presentable.)

Shore [1981] converts the structure provided by theorem 6.1 into definability.

6.2 THEOREM (SHORE [1981]). *The theory of* $\boldsymbol{\mathcal{D}}(\leq \mathbf{a}\,c.e.)$ $(\mathbf{a} \neq \mathbf{0}) \equiv_1$ *the theory of first-order Peano arithmetic (so has degree* $\mathbf{0}^{(\omega)}$*). (But if* $\boldsymbol{\mathcal{D}}(\leq \mathbf{a}) \equiv \boldsymbol{\mathcal{D}}(\leq \mathbf{0}')$ *then* $\mathbf{a}^{(3)} = \mathbf{0}^{(4)}$*.)*

The most promising source of structure in the search for natural definitions of c.e. singletons in some $\boldsymbol{\mathcal{D}}(\mathbf{0}, \mathbf{a}\,\text{c.e.})$ is:

6.3 CONJECTURE (EPSTEIN [1979]). Every c.e. $\mathbf{b} \in \boldsymbol{\mathcal{D}}(\mathbf{0}, \mathbf{a}\,\text{c.e.})$ has a minimal complement $\mathbf{m} < \mathbf{a}$.

Epstein [1981] confirmed the conjecture for $\mathbf{a}$ high c.e., while Seetapun and Slaman [ta] have constructed a minimal complement for *any* $\mathbf{b} \in \boldsymbol{\mathcal{D}}(\mathbf{0}, \mathbf{0}')$ (so extending the Posner [1981] Complementation Theorem). But for the general case, Cooper and Epstein [1987] pointed to a more interesting situation:

- If $\mathbf{b} < \mathbf{a}$ is low, then one can find a minimal $\mathbf{m} < \mathbf{a}$ with $\mathbf{m} \mid \mathbf{b}$. But there exists a $\boldsymbol{\mathcal{D}}(\leq \mathbf{a}\,\text{c.e.})$ in which complementation fails (where by Cooper [1986], $\mathbf{a}$ can be chosen to be high).

  In fact:
- (Slaman, private communication) There exists a $\boldsymbol{\mathcal{D}}(\leq \mathbf{a}\,\text{c.e.})$ in which *no* $\mathbf{b} \in \boldsymbol{\mathcal{D}}(\mathbf{0}, \mathbf{a})$ is complemented.
- (Cooper [1989],Slaman and Steel [1988]) There exists a $\boldsymbol{\mathcal{D}}(\leq \mathbf{a}\,\text{c.e.})$ in which cupping of some c.e. $\mathbf{b}$ fails. (For an interesting alternative proof of this see Downey [1987].) And:
- (Cooper [1986]) There exists a $\boldsymbol{\mathcal{D}}(\leq \mathbf{a}\,\text{c.e.})$ in which capping of some $\mathbf{b}$ fails.

The above result of Cooper/ Slaman and Steel is the key to another, technically simpler, lower cone definition of a c.e. singleton.

6.4 THEOREM. *There exist c.e. degrees* $\mathbf{0} < \mathbf{a} < \mathbf{c}$ *such that* $\mathbf{a}$ *is definable as the greatest degree in* $(\mathbf{0}, \mathbf{c})$ *not cuppable to* $\mathbf{c}$ *in* $\boldsymbol{\mathcal{D}}(< \mathbf{c})$.

PROOF. Let $W$ and $(\Theta, \Phi, \Xi, \Psi)$ be typical entries in respective standard lists of all c.e. sets and quadruples of p.c. functionals. We construct c.e. sets $A$, $C$, $B_\Xi$ and p.c. functionals $\Omega$, $\Gamma_{\Theta,\Phi}$, $\Lambda_\Xi$, $\Delta_\Xi$, $\Upsilon_\Xi$ satisfying an overall requirement $A = \Omega^C$ and all requirements of the form:

$$\begin{aligned} \mathcal{P}_{\Theta,\Phi}: &\quad C = \Theta(A, \Phi^C) \Rightarrow A = \Gamma_{\Theta,\Phi}(\Phi^C), \\ \mathcal{Q}_W: &\quad A \neq \overline{W}, \end{aligned}$$

(Two typical requirements asking that $\mathbf{a}$ be noncuppable in $\boldsymbol{\mathcal{D}}(\leq \mathbf{c})$

$$\begin{aligned} \mathcal{S}_\Xi: &\quad C = \Lambda_\Xi(\Xi^C, B_\Xi) \text{ with } B_\Xi = \Delta_\Xi^C, \\ \mathcal{R}_{\Xi,\Psi}: &\quad C \neq \Psi^{B_\Xi} \vee \Xi^C = \Upsilon_\Xi^A \end{aligned}$$

(A pair of requirements asking that $\mathbf{a}$ be the largest such degree.)

**Basic module for $\mathcal{P}_{\Theta,\Phi}$ above $\mathcal{Q}_W$ on the true path**

**(1)** *Choose* an *agitator* $a$ for $\mathcal{P}_{\Theta,\Phi}, \mathcal{Q}_W$.

**(2)** *Wait* for $\ell(C, \Theta(A, \Phi^C)) > a$ and *restrain* $A \restriction \vartheta(a)$ and $C \restriction \varphi(\vartheta(a))$.

**(3)** *Choose* a witness $x$ for $\mathcal{Q}_W$ with $\omega(x) > a$, $x > \vartheta(a)$, and $\Gamma_{\Theta,\Phi}(\Phi^C, x)\uparrow$ – *Define* $\Gamma_{\Theta,\Phi}(\Phi^C, x) = A(x)$ with $\gamma(x) > \vartheta(a)$.

**(4)** *Wait* for $x$ realised (i.e., $x \in W$).

**(5)** *Enumerate* $x$ into $A$ and $a$ into $C$ (so rectifying $\Omega$ at $x$).

**(6)** Use the $\Phi^C \restriction \vartheta(a)$-change to rectify $\Gamma_{\Theta,\Phi}$ at $x$.

**Module for $\mathcal{R}_{\Xi,\Psi}$ with $\mathcal{S}_{\Xi}$**

**(1)** *Choose* a witness $y$ for $\mathcal{R}_{\Xi,\Psi}$ with $\Lambda_{\Xi}(\Xi^C, B_{\Xi})\uparrow$.

**(2)** *Wait* for $\Xi^C \restriction \lambda(y)\downarrow$ and *restrain* it.

**(3)** *Wait* for $y$ realised (i.e., $\ell(C,\Psi^{B_{\Xi}}) > y$).

**(4)** *Change* $C(y)$ and $B_{\Xi}(\lambda(y))$ (where $\delta(\lambda(y)) > y$), defining $\delta(\lambda(y)) > \xi(\lambda(y))$ whenever subsequently $\Xi^C(\lambda(y))\downarrow$. *Restrain* $B_{\Xi} \restriction \psi(y)$ at all values $\neq \lambda(y)$.

**(5)** At each subsequent stage ask: Is $\Lambda_{\Xi}(\Xi^C, B_{\Xi})\downarrow= 0$?

**(a)** *Yes – Define* $B_{\Xi}(\lambda(y)) = 1$ and *initiate* an additional cycle of activity via (1).

**(b)** *No – Define* $B_{\Xi}(\lambda(y)) = 0$ whenever $\Xi^C \restriction \lambda(y)\downarrow$ with $\delta(\lambda(y)) > \xi(\lambda(y))$.

*Infinitary outcome:* We return to (1) infinitely often. In which case we apply the Modulus Lemma to obtain $\Xi^C$ recursive.

**Combining the strategies for the different groups of requirements**

The main problem is that the activity of $\mathcal{Q}_W$ disrupts the application of the Modulus Lemma in the infinitary outcome for $\mathcal{R}_{\Xi,\Psi}$. However, all these $C$-injuries are $A$-related, so we are able to use these $A$-changes for rectification of $\Upsilon_{\Xi}$ with eventual outcome $\Xi^C = \Upsilon_{\Xi}^A$. Again, for a more detailed discussion see Cooper [ta2]. $\square$

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