SPLITTING AND JUMP INVERSION IN THE TURING DEGREES

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Abstract. It is shown that for any computably enumerable degree $\boldsymbol{a} \neq \mathbf{0}$, any degree $\boldsymbol{c} \neq \mathbf{0}$, and any Turing degree $\boldsymbol{s}$, if $\boldsymbol{s} \geq \mathbf{0}'$, and c.e. in $\boldsymbol{a}$, then there exists a c.e. degree x with the following properties,

(1) $\boldsymbol{x} < \boldsymbol{a}$, $\boldsymbol{c} \not\leq \boldsymbol{x}$,

(2) $\boldsymbol{a}$ is splittable over $\boldsymbol{x}$, and

(3) $\boldsymbol{x}' = \boldsymbol{s}$.

This implies that the Sacks' splitting theorem and the Sacks' jump theorem can be uniformly combined. A corollary is that there is no atomic jump class consisting entirely of Harrington non-splitting bases.

1. Introduction

Basic to the theory of the computably enumerable (c.e.) Turing degrees, with wide-ranging applications, are three theorems from the early 1960s with proofs using the newly discovered infinite injury priority method: The Sacks [1963a] Splitting Theorem, the Sacks [1964] Density Theorem, and the Jump Theorem (Sacks [1963b]).

Key questions for the characterising of the theory of the computably enumerable degrees concern the compatibility of these fundamental constructions. Robinson [1971b] found a number of partial results dealing with density and jump inversion, and left four open questions in the form of conjectures. Lachlan [1975] developed more advanced techniques to show the incompatibility, in general, of density and splitting. This paper shows that splitting and jump inversion *can* be combined, and in so doing shows that there is no atomic jump class consisting entirely of Harrington non-splitting bases. (We say that a c.e. degree $\boldsymbol{a}$ is a *Harrington non-splitting base* if $\mathbf{0}'$ cannot be split over $\boldsymbol{a}$, and that the set of all c.e. degrees jumping to a given $\boldsymbol{u} \geq \mathbf{0}'$ forms an *atomic jump class.*) This leaves open the question of whether there exists an atomic jump class consisting of Lachlan non-splitting bases. Of course, we already know from the Robinson [1971a] Splitting Theorem that the low c.e. degrees cannot form such an atomic jump class.

The difficulty in splitting computably enumerable Turing degrees is due to a very strong non-extension feature (see for instance, Cooper and Li [2002]). Intuitively speaking, the splitting construction is incompatible with infinite injury arguments. The Sacks Jump theorem is nevertheless an infinite injury proposition. A deeper understanding of the theory of the Turing degrees calls for further examination of some subtle relationships between the basic notions and the fundamental results of the theory, such as those of Sacks. In the present paper, we investigate the relationship between the Sacks Jump theorem and the splitting theorem. In particular, we show that:

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THEOREM. Let $\boldsymbol{a} \neq \mathbf{0}$ be a c.e. degree, $\boldsymbol{c} \neq \mathbf{0}$ and $\boldsymbol{s}$ be Turing degrees. If $\boldsymbol{s} \geq \mathbf{0}'$ and c.e. in $\boldsymbol{a}$, then there exists a c.e. degree $\boldsymbol{x}$ with the following properties,

(1) $\boldsymbol{c} \not\leq \boldsymbol{x} < \boldsymbol{a}$,
(2) $\boldsymbol{x}' = \boldsymbol{s}$, and
(3) $\boldsymbol{a}$ is splittable above $\boldsymbol{x}$.

COROLLARY. Let $\boldsymbol{s} \geq \mathbf{0}'$ and c.e. in $\mathbf{0}'$. Then there exists a c.e. degree $\boldsymbol{a}$ with the following properties,

(1) $\boldsymbol{a} < \mathbf{0}'$,
(2) $\boldsymbol{a}' = \boldsymbol{s}$, and
(3) $\mathbf{0}'$ is splittable above $\boldsymbol{a}$.

The theorem simultaneously combines the Sacks splitting theorem and the Sacks jump theorem. The proof of the theorem is an infinite injury tree argument, in which there are some interesting new ideas — for instance, the use of the decreasing sequence of permitting markers of a γ-marker for the Turing functional Γ we build; and the feature involving finite injury from nodes to the right of the true path, due to the impossibility of initialization of γ-markers.

The theorem entails refined analysis of the structure of the c.e. Turing degrees, while its proof offers extended computability-theoretic techniques of construction.

The terminology and the notations are standard and generally can be found in Cooper [2004] or Soare [1987] (pending the forthcoming revised edition).

2. The Requirements and Strategies

The Requirements. Let A be a c.e. set of degree $\boldsymbol{a}$, C be a set of degree $\boldsymbol{c}$, and $S = W^A$ for some c.e. set W. We construct c.e. sets X, X_0, X_1, and B, and Turing functionals Γ, and Ω to satisfy the following conditions and requirements,

(1) For all e, y, $\Gamma(X; e, y) \downarrow$
(2) For all e, $\lim_y \Gamma(X; e, y) \downarrow$
$\mathcal{T}$: $X \leq A$, $X_0 \cap X_1 = \emptyset$, and $X_0 \cup X_1 = A$
$\mathcal{N}_{2e}$: $A \neq \Phi_e(X_0, X)$
$\mathcal{N}_{2e+1}$: $A \neq \Phi_e(X_1, X)$
$\mathcal{Q}_e$: $C \neq \Phi_e(X)$
$\mathcal{R}_e$: $W^A(e) = \lim_s \Gamma(X; e, y)$
$\mathcal{S}_e$: $X^{'}(e) = \Omega(B, W^A; e)$

where $e \in \omega$, $X^{'} = \{e \mid \Phi_e(X; e) \downarrow\}$, $\{\Phi_e \mid e \in \omega\}$ is an effective enumeration of all Turing functionals Φ.

It is clear that meeting the requirements is sufficient to prove the theorem.

The $\mathcal{T}$-Strategy. To satisfy $X \leq_{\mathrm{T}} A$, we build a Turing reduction from X to A using standard permitting. We ensure that at any stage $s+1$, an element x can be enumerated into X only if there is a natural number $a \leq x$ such that $a \in A_{s+1} \setminus A_s$. From this, we can argue that X is computable in A.

To satisfy $A \leq_{\mathrm{T}} X_0 \oplus X_1$, we split A into X_0 and X_1 such that

(i) If $x \in A$, then $X_0(x) \neq X_1(x)$,
(ii) If $x \notin A$, then $X_0(x) = X_1(x) = 0$.

An $\mathcal{N}$-Strategy. Suppose that we want to satisfy the following $\mathcal{N}$-requirement:

$$\mathcal{N}: \quad A \neq \Phi(X_0, X).$$

We will build a computable partial function f such that if $A = \Phi(X_0, X)$, then f is total, and $f = A$. f will be built as follows.

1. Let x be the least y such that $f(y)\uparrow$.
2. If $\Phi(X_0, X; x) \downarrow= A(x)$, then define $f(x) = A(x)$, and define the X_0-, and X-restraints by $r^{X_0} = r^X = \phi(x)$, respectively.

The X_0-, and X-restraints will ensure that for any x, if $f(x)$ is defined, then $f(x) = \Phi(X_0, X; x)$, so that if $A(x) = \Phi(X_0, X; x)$, then $f(x) = A(x)$. Therefore if f is built infinitely often, then f is total, and $f = A$.

A $\mathcal{Q}$-Strategy. Given a $\mathcal{Q}$-requirement $\mathcal{Q}$: $C \neq \Phi(X)$, say, we build a computable partial function g, such that if $C = \Phi(X)$, then g is total, and $g = \Phi(X)$. g will be built as follows.

1. Let x be the least y such that $g(y)$ is undefined.
2. If $\Phi(X; x) \downarrow= C(x)$, then define $g(x) = \Phi(X; x)$, and define the X-restraint by $r^X = \phi(x)$.

Just as for the $\mathcal{N}$-strategy, if g is built infinitely often, then g is total, and $g = C$.

Suppose that $A \not\leq_{\mathrm{T}} \emptyset$, and $C \not\leq \emptyset$. Then there is no $\mathcal{N}$- or $\mathcal{Q}$-strategy acts infinitely often. So we use 1 to denote the unique outcome of an $\mathcal{N}$- or a $\mathcal{Q}$-strategy, which indicates that the strategy acts only finitely often.

An $\mathcal{R}$-Strategy. All $\mathcal{R}_e$-strategies will compete to define $\Gamma(X; e, y)$ for all y such that for each y, $\Gamma(X; e, y)$ is defined eventually and $\lim_s \Gamma(X; e, y) \downarrow= W^A(e)$.

Suppose that α is an $\mathcal{R}_e$-strategy. α will build $\lambda y \Gamma(X; e, y)$ as follows.

1. If $e \in W^A$, then let x be the least y such that $\Gamma(X; e, y)\uparrow$, define $\Gamma(X; e, x) \downarrow= 1$ with $\gamma(e, x)$ fresh, locate $\gamma(e, x)$ at $\alpha\hat{\ }\langle 1\rangle$, and define the *permitting marker* $m(z)$ for $z = \gamma(e, x)$ to be the least u such that $e \in W^{A\upharpoonright(u+1)}$.
2. If $\gamma(e, x) = y$ is located at $\alpha\hat{\ }\langle 1\rangle$, and A changes below the permitting marker $m(y)$ for $y = \gamma(e, x)$, then enumerate $\gamma(e, x)$ into X.
   Let s^- be the greatest stage $v < s$ at which $\alpha\hat{\ }\langle 0\rangle$ was visited.
3. If there is a v such that $s^- < v \leq s$ at which $e \notin W^A$, then let x be the least y such that $\Gamma(X; e, y)$ undefined, define $\Gamma(X; e, x) = 0$ with $\gamma(e, x)$ fresh, and locate it at $\alpha\hat{\ }\langle 0\rangle$.

Before analysing the possible outcomes of the $\mathcal{R}$-strategy, we introduce some rules for W^A. We ensure that the enumeration of W^A will have the following properties,

(i) $e \in W^A$ if and only if there is a $\sigma \subset A$ such that $\langle e, \sigma\rangle \in W$.
(ii) A will be enumerated at odd stages.
(iii) At each odd stage s, there is exactly one element enumerated into A.

By (i) – (iii), we have that $e \notin W^A$ if and only if there are infinitely many s such that $e \notin W^A[s]$.

We define the *possible outcomes* of an $\mathcal{R}_e$-strategy by $0 <_{\mathrm{T}} 1$, where 0 means that $e \notin W^A$, and 1 means that $e \in W^A$.

Suppose that $e \in W^A$, we use $w(e)$ to denote the least u such that $e \in W^{A\upharpoonright(u+1)}$. We call $w(e)$ the *use* of $e \in W^A$.

The Permitting Marker. Let $\gamma(e, y) = x$ be a γ-marker and ξ be a node.

(i) If $\xi = \alpha\hat{\ }\langle 1\rangle$ for some $\mathcal{R}_e$-strategy α for some e, then we define the *permitting marker of* ξ by $p(\xi)$ to be the use $w(e)$.

(ii) If $x = \gamma(e, y)$ is located at ξ, then we define the permitting marker $m(x) = \max\{p(\beta) \mid \beta \leq \xi\}$.

The Enumeration of X. Let $x = \gamma(e, y)[v]$ for some e, y, and v. Then x is enumerated into X at stage $s+1$ if and only if both (i) and (ii) below hold,

(i) $x > R(x) = \max\{r^X(\beta) \mid \beta \leq \mathrm{loc}(x)\}$, where $\mathrm{loc}(x)$ is the *location* of x, and
(ii) There is a natural number $a \leq m(x)$ such that $a \in A_{s+1} \setminus A_s$.

An $\mathcal{S}$-Strategy. Given an $\mathcal{S}$-requirement, $\mathcal{S}_e$, say, we may introduce several $\mathcal{S}_e$-strategies. The $\mathcal{S}_e$-strategies will compete to define $\Omega(B, W^A; e)$. Suppose that β is an $\mathcal{S}_e$-strategy. Then β will proceed as follows.

1. Let s^- be the greatest stage at which some $(b, \beta, e, 1)$ was enumerated into Ω, if any, and there is a stage v such that $s^- < v \leq s$ such that $\Phi_e(X; e) \uparrow$. Then if there is an axiom $(b, \beta, e, 1) \in \Omega$ with $b \notin B$, then enumerate all these b into B; and if there is no axiom $(b, \beta, e, 0) \in \Omega$ for any $b \notin B$, then let b be a fresh number, and enumerate $(b, \beta, e, 0)$ into Ω.
2. Otherwise, then if there is an axiom $(b, \beta, e, 0) \in \Omega$, then enumerate b into B; if there is no $b \notin B$ such that $(b, \beta, e, 1) \in \Omega$, then let b be fresh, and enumerate $(b, \beta, e, 1)$ into Ω; and define the X-restraint by $r^X(\beta) = \phi_e(e)$.

The X-restraint $r^X(\beta)$ can be injured only finitely often by the γ-markers that are located at nodes $< \beta$.

The unique outcome of β is 1.

The Injury of an $\mathcal{N}$-, or a $\mathcal{Q}$-, or an $\mathcal{S}$-Strategy from an $\mathcal{R}$-Strategy. Suppose that β is an $\mathcal{N}$-, or a $\mathcal{Q}$-, or an $\mathcal{S}$-strategy. β will impose an X-restraint $r^X(\beta)$. However β can be injured by $\mathcal{R}$-strategies $\alpha \leq \beta$ only finitely often. We consider two cases. If $\alpha\hat{\ }\langle 1\rangle \subseteq \beta$, then α injures β only finitely often. If $\alpha\hat{\ }\langle 0\rangle \subseteq \beta$, then at the stages at which β is visited, there is no γ-marker x such that x is located at a node to the right of $\alpha\hat{\ }\langle 0\rangle$ which can be enumerated into X. The idea is that at the stage at which A changed below the W^A-use of α, every $\gamma(e, y) = x > w^A(\alpha)$, if $x > \max\{r^X(\xi) \mid \xi < \alpha\}$ has been enumerated into X. In this case, we say that the $\mathcal{R}$-strategy α *receives attention*.

This idea is crucial for the consistency of Γ. We look at the following situation. Suppose that $\alpha_1 <_{\mathrm{L}} \alpha_2$ are two $\mathcal{R}_e$-strategies. If α_1 is on the true path, we must make sure that there are only finitely many $\gamma(e, y)$ markers which are located at $\alpha_2\hat{\ }\langle i\rangle$ for any $i \leq 1$ permanently. This ensures that for almost every y, $\Gamma(X; e, y)$ will be defined by the $\mathcal{R}_e$-strategy on the true path and are located at the true path.

Updating the Location of a γ-Marker. Since any enumeration of a γ-marker into X must respect some X-restraints. An $\mathcal{N}$- or an $\mathcal{S}$-strategy may update the location of γ-markers as follows.

If an $\mathcal{N}$-, or a $\mathcal{Q}$-, or an $\mathcal{S}$-strategy β increases its X-restraint, then for any $x = \gamma(e, y)$, if $\mathrm{loc}(x) \supset \beta$, then re-locate it at β, by defining $\mathrm{loc}(x) = \beta\hat{\ }\langle 1\rangle$.

3. The Construction

First we introduce some notation which will be useful in the description of the construction and the verification.

DEFINITION 3.1. (i) Define the *priority ranking of the requirements* by

$$N_0 < \mathcal{Q}_0 < \mathcal{R}_0 < \mathcal{S}_0 < \mathcal{N}_1 < \mathcal{Q}_1 < \mathcal{R}_1 < \mathcal{S}_1 < \cdots$$

(ii) Define the *possible outcomes* of an $\mathcal{R}$-strategy by $0 <_{\mathrm{L}} 1$.
(iii) Define the unique outcome of a $\mathcal{Q}$-, or an $\mathcal{N}$-, or an $\mathcal{S}$-strategy to be 1.

DEFINITION 3.2. We arrange the requirements on nodes of a tree, the *priority tree* T, according to the priority ranking of the requirements, such that at each level there is only one requirement to be located, while each node has immediate successors giving the possible outcomes of the corresponding strategy.

DEFINITION 3.3. (i) We assume that A is enumerated at odd stages, and that at each odd stage there is a unique element which is enumerated into A.

(ii) The tree construction will be implemented at even stages.

Let α be an $\mathcal{R}_e$-strategy for some e. We use $w(\alpha)$ to denote the current use $w^A(e)$, if $e \in W^A$.

DEFINITION 3.4. Let α be an $\mathcal{R}$-strategy, s be an odd stage, and a be the element which is enumerated into A at stage s. We say that α *requires attention at stage* s, if $w(\alpha) \downarrow > a$ at the beginning of stage s.

During the course of the construction, a strategy may be initialized. We interpret the action of initialization as follows.

1. If an $\mathcal{N}$-strategy β is initialized, then f_β is set to be totally undefined, and all restraints of β are cancelled.
2. If a $\mathcal{Q}$-strategy γ is initialized, then the computable partial function g_γ is set to be totally undefined, and the X-restraint $r^X(\gamma)$ is cancelled.
3. If an $\mathcal{S}$-strategy β is initialized, then for any axiom $(b, \beta, e, i) \in \Omega$, if $b \notin B$, then b is enumerated into B.

Now we are ready to describe the construction.

DEFINITION 3.5. The construction will proceed stage by stage as follows.

Stage $s = 0$. Set $X_0 = X_1 = X = \emptyset$, and initialize all strategies.

Stage $s = 2n + 1$. Let a be the number enumerated into A at stage s. There are two phases.

Phase I ($\mathcal{R}$-Strategies receiving attention). Suppose that $\alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_m$ are all $<$-minimal $\mathcal{R}$-strategies which require attention at stage s. In decreasing order, for all $j = m, \cdots, 1$, execute the following:

1. Let $r_j = \max\{r^X(\beta) \mid \beta < \alpha_j\}$.
2. For any γ-marker $\gamma(e, y) = x$, if $\mathrm{loc}(x) \supseteq \alpha_j\hat{}\langle 1\rangle$, and $x > r_j$, then enumerate x into X.
3. For any $\mathcal{N}$-, $\mathcal{Q}$-, or $\mathcal{S}$-strategy β, if $\alpha_j\hat{}\langle 0\rangle <_{\mathrm{L}} \beta$, then β is initialized.

Phase II (Splitting of A). Let β be the $<$-least $\mathcal{N}$-strategy ξ such that $r^{X_i}(\xi) \geq a$, where $i = j \pmod 2$, and j is the index of ξ.

1. If β is defined, then let $i = j + 1 \pmod 2$ for the index j of β, and enumerate a into X_i.
2. Otherwise, enumerate a into X_0.

Stage $s = 2n + 2$. We allow some strategies to act at substages $t < s$. At each substage t, we let some strategy act, and then either close the current stage or specify some lower priority strategy to act at the next substage. Initially we allow the root node λ to act at substage $t = 0$.

Substage t. Let ξ be the strategy which is eligible to act at substage t. If $t = s$, then all nodes ξ' to the right of ξ are initialized, and we close the current stage. If $t < s$, then there are 4 cases.

Case 1. $\xi = \beta$ is an $\mathcal{N}_j$-strategy for some j. Suppose without loss of generality that $j = 2e$ for some e. We build f_β as follows.

1. Let x be the least y such that $f_\beta(y) \uparrow$.

2. If $\Phi_e(X_0, X; x) \downarrow = A(x)$, then
   – define $f_\beta(x) = \Phi_e(X_0, X; x)$, define the X_0, and X-restraints by $r^{X_0}(\beta) = \max\{\text{old } r^{X_0}(\beta),\ \phi_e(x)\}$, and define $r^X(\beta) = \max\{\text{old } r^X(\beta),\ \phi_e(x)\}$,
   – for any γ-marker $y = \gamma(e', z')$, if y is located at some node $\xi' \supset \beta$, then relocate it at $\beta\hat{}\langle 1\rangle$,
   – initialize all nodes ξ' with $\xi < \xi'$, and go to stage $s+1$.
3. Otherwise, let $\beta\hat{}\langle 1\rangle$ be eligible to act at substage $t+1$ of stage s.

Case 2. $\xi = \gamma$ is a $\mathcal{Q}_e$-strategy for some e. γ will build a computable partial function g_γ as follows.

1. Let x be the least y such that $g_\gamma(y) \uparrow$.
2. If $\Phi_e(X; x) \downarrow = C(x)$, then define $g_\gamma(x) = \Phi_e(X; x)$, and define the X-restraint by $r^X(\gamma) = \max\{\text{old } r^X(\gamma),\ \phi_e(X; x)\}$; for any γ-marker which is located at some node below γ will be re-located at $\gamma\hat{}\langle 1\rangle$; initialize all nodes ξ' with $\gamma < \xi'$, and go to stage $s+1$.
3. Otherwise, let $\gamma\hat{}\langle 1\rangle$ be eligible to act at substage $t+1$.

Case 3. $\xi = \alpha$ is an $\mathcal{R}_e$-strategy for some e. We run the following

Program α

1. Let s^- be the greatest stage $v < s$ at which $\alpha\hat{}\langle 0\rangle$ was visited.
2. If there is a stage v such that $s^- < v \leq s$ and $e \notin W^A[v]$, then let x be the least y such that $\Gamma(X; e, y)$ is not defined, define $\Gamma(X; e, x) \downarrow = 0$ with $\gamma(e, x)$ fresh, locate the $\gamma(e, x)$ at $\alpha\hat{}\langle 0\rangle$, and let $\alpha\hat{}\langle 0\rangle$ be eligible to act at the next substage.
3. Otherwise, let x be the least y such that $\Gamma(X; e, y)$ is not defined, define $\Gamma(X; e, x) \downarrow = 1$ with $\gamma(e, x)$ fresh, locate $\gamma(e, x)$ at $\alpha\hat{}\langle 1\rangle$, and let $\alpha\hat{}\langle 1\rangle$ be eligible to act next.

Case 4. $\xi = \beta$ is an $\mathcal{S}_e$-strategy for some e. Then implement the following

Program β

1. If there is an axiom $(b, \beta, e, 0) \in \Omega$ such that $b \notin B$ and $\Phi_e(X; e) \downarrow$, then for any axiom $(b, \beta, e, 0) \in \Omega$, if $b \notin B$, enumerate b into B, let b be a fresh number, enumerate $(b, \beta, e, 1)$ into Ω, define the X-restraint by $r^X(\beta) = \phi_e(X; e)$. And every γ-marker $x = \gamma(e, y)$ located at nodes below β is re-located at $\beta\hat{}\langle 1\rangle$, we initialize all nodes ξ' with $\beta < \xi'$, and go to stage $s+1$.
2. If there is no axiom $(b, \beta, e, 0) \in \Omega$, and $\Phi_e(X; e) \uparrow$, then for any axiom $(b, \beta, e, 1) \in \Omega$, if $b \notin B$, we enumerate b into B; let b be fresh, and enumerate $(b, \beta, e, 0)$ into Ω; for any γ-marker $\gamma(e, y) = x$, if x is located at a node below β, we re-locate it at $\beta\hat{}\langle 1\rangle$; and we initialize all nodes ξ' with $\beta < \xi'$, and go to stage $s+1$.
3. Otherwise, let $\beta\hat{}\langle 1\rangle$ be eligible to act at the next substage.

This completes the description of the construction.

4. The Verification

The verification of the construction consists of the following propositions.

PROPOSITION 4.1. $X \leq_T A$.

Proof. A number x can be enumerated into X, only if x is defined to be $\gamma(e, y)$ at some stage v for some e, y, and v.

For any $s \geq v$, if $x \notin X[s]$, then the *permitting marker* $m(x)$ is the maximum of $w(\alpha)$ over all $\mathcal{R}$-strategies $\subset$ loc (x).

Let $C(x)[s]$ be the set of all $\mathcal{R}$-strategies $\subset$ loc (x) during stage s. By the construction, we have that

$$(1) \qquad C(x)[v] \supseteq C(x)[v+1] \supseteq C(x)[s].$$

We prove that for any $s > v$,

$$(2) \qquad m(x)[s] = \max\{w(\alpha)[v] \mid \alpha \in C(x)[s]\}.$$

To get a contradiction, assume that s is the least stage $> v$ at which A changes below $m(x)[v]$. Let α be the longest $\mathcal{R}$-strategy which requires attention at stage s. By the choice of s, loc $(x) \supset \alpha$. Therefore there is no $\mathcal{N}$-, $\mathcal{Q}$-, or $\mathcal{S}$-strategy $\beta \subset \alpha$ which has increased its X-restraint (otherwise, the location of x at stage s is above α). Therefore, the maximum X-restraints by strategies $\subset \alpha$ at stage s is no more than that observed at stage v. Since x is chosen to be fresh at stage v, x is not restrained from entering X at stage s. (2) follows.

By combining (1) and (2), the permitting marker $m(x)$ is decreasing (to be precise, non-increasing) before it is enumerated into X.

On the other hand, at the stage s at which x is enumerated into X, there is an $a \leq m(x)$ which enters A.

The direct permitting method ensures that X is computable in A. Proposition 4.1 follows. □

PROPOSITION 4.2. $X_0 \cap X_1 = \emptyset$, and $X_0 \cup X_1 = A$.

Proof. By the construction. □

DEFINITION 4.3. (i) Let δ_s be the longest strategy visited at an even stage s.

(ii) Define the *true path* TP of the construction by

$$TP = \liminf_{s=2n} \delta_s.$$

We examine the existence of the true path TP.

PROPOSITION 4.4. Let $A \not\leq_{\mathrm{T}} \emptyset$, and $C \not\leq_{\mathrm{T}} \emptyset$. For any $\alpha \in TP$, there is an outcome o such that

(i) $\alpha\hat{}\langle o\rangle$ is initialized only finitely often.

(ii) $\alpha\hat{}\langle o\rangle \in TP$ is visited infinitely often.

Proof. This is straightforward from the construction. □

PROPOSITION 4.5 (The True Path Properties Proposition). Let $A \not\leq_{\mathrm{T}} \emptyset$, and $C \not\leq_{\mathrm{T}} \emptyset$. Given $\xi \in TP$:

(i) If $\xi = \beta$ is an $\mathcal{N}_j$-strategy for some j, then

– if f_β is built infinitely often, then f_β is total, and $f_\beta =^* A$.

(ii) If $\xi = \gamma$ is a $\mathcal{Q}$-strategy, then

– if g_γ is built infinitely often, then g_γ is total and $g_\gamma =^* C$.

(iii) If $\xi = \alpha$ is an $\mathcal{R}_e$-strategy for some e, then

(3a) For all y, $\Gamma(X; e, y)$ is defined eventually and permanently.

(3b) $\lim_y \Gamma(X; e, y) \downarrow = W^A(e)$.

(iv) If $\xi = \delta$ is an $\mathcal{S}_e$-strategy for some e, then

(4a) If there is an axiom $(b, \beta, e, i) \in \Omega$ for some $b \notin B$ and some i, then there is no axiom $(b', \beta, e, 1-i) \in \Omega$ for any $b' \notin B$.

(4b) There is a fixed quadruple (b, β, e, i) such that it is kept in Ω permanently for some $b \notin B$ and some i.

(4c) $X^{'}(e) = i$ if and only if there is an axiom $(b, \beta, e, i) \in \Omega$ for some $b \notin B$.

Proof. Suppose by induction that the proposition holds for all $\xi' \subset \xi$. Let s_0 be minimal such that for any $s > s_0$,

(1) If $\xi' \subset \xi$ is an $\mathcal{N}$-, or a $\mathcal{Q}$-, or an $\mathcal{S}$-strategy, then ξ' does not act at stage s.
(2) ξ will not be initialized at stage s.
(3) If $\alpha \subset \alpha\hat{}\langle 1\rangle \subseteq \xi$ is an $\mathcal{R}_e$-strategy for some e, then $e \in W^A[s]$.
We prove the proposition for ξ by considering the following cases.

Case 1. $\xi = \beta$ is an $\mathcal{N}_j$-strategy for some j.

Suppose without loss of generality that $j = 2e$ for some e. By induction we can choose a minimal stage $s_1 > s_0$ with the following properties,

(a) For each $i \leq 1$, the maximum X_i-restraints which are imposed by strategies $\xi' < \beta$ has reached to its limit by stage s_1, we use r to denote the limit of this maximum restraints.

(b) A will never change below r after stage s_1.

(c) The maximum of the X-restraints imposed by strategies $< \beta$ has reached to its limit by stage s_1. We use r^X to denote this limit.

Suppose that $\alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_m$ are all $\mathcal{R}$-strategies α with $\alpha\hat{}\langle 0\rangle \subset \beta$.

Let $s > s_1$ be a stage at which β is visited. We prove that:

($*$) For any j, any γ-marker $x = \gamma(e', y')$, if loc $(x)[s] \supseteq \alpha_j\hat{}\langle 1\rangle$, and $x > r^X$, then x is in X during stage s (before β is visited at stage s).

Suppose that a γ-marker $x = \gamma(e', y')$ was created and located at a node $\supseteq \alpha_j\hat{}\langle 1\rangle$ at a stage $v < s$ ($s_1 < v$). Then $\alpha_j\hat{}\langle 1\rangle$ was visited at stage v, so that the permitting marker of the γ-marker x is greater than the use of $w(\alpha_j)[v]$. Since β is visited at stage s there is a stage s' such that $v < s' < s$ at which A changes below $w(\alpha_j)[v]$. Let s'' be the least such s'. Then x has been enumerated into X at stage s''. This argument applies to all j. So ($*$) is established.

Let $s_2 > s_1$ be minimal after which there is no γ-marker located at any node $\supseteq \alpha_j\hat{}\langle 1\rangle$ for any j, which can be enumerated into X.

By ($*$) and by the choice of s_2, the X-restraint of β after stage s_2, will not be injured by any strategy ξ' with $\beta \not\subseteq \xi'$.

Given an x, if $f_\beta(x)$ is created at a stage $s > s_2$, then the X-restraint $r^X(\beta)[s]$ ensures that $\Phi_e(X_0, X; x)[s] \downarrow$ will be X-correct at any stage $s' \geq s$. On the other hand, by the choice of s_1, s_2, the X_0-restraint $r^{X_0}(\beta)[s]$ will never be injured by the splitting of A at odd stages $s' > s$. Therefore, $f_\beta(x) = \Phi_e(X_0, X; x)[s]$ will be preserved forever.

So if f_β is built infinitely often, then f_β is total, and $f_\beta =^* A$.

By the assumption that $A \not\leq_{\mathrm{T}} \emptyset$, f_β is built only finitely often, the length function $l(A, \Phi_e(X_0, X))[s]$ will be bounded during the course of the construction.

So $\mathcal{N}_{2e}$ is satisfied, and (i) follows.

Case 2. $\xi = \gamma$ is a $\mathcal{Q}$-strategy. The proof of (ii) is similar to that of (i).

Case 3. $\xi = \alpha$ is an $\mathcal{R}_e$-strategy for some e.

Suppose that α is an $\mathcal{R}_e$-strategy on the true path TP. First we prove that:

($**$) There are only finitely many y such that $\Gamma(X; e, y)$ is defined permanently by $\mathcal{R}_e$-strategies $\alpha' \neq \alpha$.

Since α is on the true path, there are only finitely many stages at which some $\mathcal{R}_e$-strategy $\alpha' <_{\mathrm{T}} \alpha$ is visited. We only consider an $\mathcal{R}_e$-strategy α' to the right of α. By the construction of the tree, there is an $\mathcal{R}$-strategy α_0 such that $\alpha_0\hat{}\langle 0\rangle \subseteq \alpha$, and $\alpha_0\hat{}\langle 1\rangle \subseteq \alpha'$. By induction, we can choose r_0 to be the limit of maximum of all X-restraints imposed by nodes $< \alpha_0$.

By the construction at odd stages, every γ-marker defined by α' which is greater than r_0 will be eventually enumerated into X before α is visited next time.

Therefore there are only finitely many γ-markers defined by strategies to the right of α which will be kept out of X permanently.

Hence (**) is established.

By (**), for almost every y, $\Gamma(X; e, y)$ will eventually be defined by α.
By observing α, we have that
(1) for every y, $\Gamma(X; e, y)$ is eventually defined, and
(2) $\lim_y \Gamma(X; e, y) \downarrow = W^A(e)$.

Case 4. $\xi = \delta$ is an $\mathcal{S}_e$-strategy for some e.

By the proof for case 1, the X-restraint $r^X(\delta)$ will ensure that there are only finitely many axioms of the form (b, δ, e, i) which are enumerated into Ω.

By observing δ, it is easy to see that

(1) If there is an axiom $(b, \delta, e, i) \in \Omega$ with $b \notin B$, then there is no axiom $(b', \delta, e, 1-i) \in \Omega$ for any $b' \notin B$.

(2) $X^{'}(e) = i$ if and only if there is an axiom $(b, \delta, e, i) \in \Omega$ such that $b \notin B$.

So (iv) holds.

This completes the inductive proof of the proposition. □

PROPOSITION 4.5. $TP \leq_{\mathrm{T}} W^A$.

Proof. The only non-trivial case is that for an $\mathcal{R}_e$-strategy $\alpha \in TP$. By definition of the tree, $\alpha\hat{\ }\langle W^A(e)\rangle \in TP$. The proposition follows. □

PROPOSITION 4.6. If $A \not\leq_{\mathrm{T}} \emptyset$, and $C \not\leq_{\mathrm{T}} \emptyset$, then all $\mathcal{N}$-, $\mathcal{Q}$-, and $\mathcal{R}$-requirements are satisfied.

Proof. This follows from proposition 4.4. □

PROPOSITION 4.7. All $\mathcal{S}_e$ are satisfied unless either A or C is computable.

Proof. Given e, by proposition 4.4, $X^{'}(e) = i$ if and only if there is an axiom $(b, \delta, e, i) \in \Omega$ such that $b \notin B$, and $\delta \in TP$.

Therefore, $X^{'} \leq_{\mathrm{T}} B \oplus TP \leq_{\mathrm{T}} B \oplus W^A \leq_{\mathrm{T}} S$. All $\mathcal{S}$-requirements are satisfied.
The proposition follows. □

This completes the proof of the theorem.

References

1. S. B. Cooper [2004], *Computability Theory,* Chapman & Hall/CRC, Boca Raton, London, New York, Washington, D.C., 2004.
2. S. B. Cooper and A. Li [2002], Splitting and nonsplitting, II: A low$_2$ c.e. degree above which $\mathbf{0}'$ is not splittable, The Journal of S. Logic, Vol. 67, No. 4, 2002, pp 1391 – 1430.
3. A. H. Lachlan [1975], A recursively enumerable degree which will not split over all lesser ones, Ann. Math. Logic, 9 (1975), 307–365.
4. R. W. Robinson [1971a], Interpolation and embedding in the recursively enumerable degrees, Ann. of Math., (2) 93 (1971), 285–314.
5. R. W. Robinson [1971b], Jump restricted interpolation in the recursively enumerable degrees, Ann. of Math., (2) 93 (1971), 586–596.
6. G. E. Sacks [1963a], On the degrees less than $\mathbf{0}'$, Ann. of Math., (2) 77 (1963), 211–231.
7. G. E. Sacks [1963b], Recursive enumerability and the jump operator, Tran. Amer. Math. Soc., 108 (1963), 223 – 239.

8. G. E. Sacks [1964], The recursively enumerable degrees are dense, Ann. of Math., (2) 80 (1964), 300–312.
9. R. I. Soare [1987], *Recursively enumerable sets and degrees*, Springer, 1987.

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