ENUMERATION REDUCIBILITY, NONDETERMINISTIC COMPUTATIONS AND RELATIVE COMPUTABILITY OF PARTIAL FUNCTIONS[1]

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In a computation using auxiliary informational inputs one can think of the external resource making itself available in different ways. One way is via an oracle as in Turing reducibility, where information is supplied on demand without any time delay. Alternatively the Scott graph model for lambda calculus suggests a situation where new information, only some of it immediately related to the current computation, is constantly being generated (or *enumerated*) over a period of time in an order which is not under the control of the computer. For some purposes, such as in classifying the relative computability of total functions without any time restrictions, it makes no difference whether oracles or enumerations supply auxiliary informational inputs. But this is not generally the case in situations involving partially accessible information or time-bounds, where nondeterministic computations are involved. Clearly both models of computation (based on oracles or enumerations) have wide validity, although much more is known about the former via the rich and extensive theory of Turing computability. The purpose of this article is to survey the existing literature related to the latter with an emphasis on enumeration reducibility and its associated degree structure.

§1. Notions of relative computability for partial functions

In practice a function may have values which are difficult or even impossible to compute according to natural criteria, in which case we are concerned with computability, or perhaps relative computability, of *partial* functions.

There is an immediate extension to partial functions of the Turing notion of relative computability in which unanswered questions to an oracle for a partial function results in an infinite wait and hence in nontermination of the computation. Let $\Phi_i, i \in \omega$, be the partial recursive functional corresponding to the i^{th} Turing machine in some standard listing $\{Z_i\}_{i\in\omega}$ (taking Turing machines to be defined as in Davis [Da58] as consistent finite sets of quadruples), and let $\langle\cdot,\cdot\rangle$ be a standard coding of the pairs of numbers onto the numbers. We define $\Phi_i^g(x)$ for g a partial function as the output obtained from Z_i on input x with oracle $\mathrm{graph}(g) = \{\langle x, y\rangle \mid g(x) = y\}$, where during the computation an applicable quadruple $q_i S_j q_k q_l$ is interpreted as leading to internal state q_k if the number $\langle m, n\rangle$ coded on the tape is in $\mathrm{graph}(g)$, to q_l if $\langle m, n'\rangle$ is in graph(g) for some $n' \neq n$, and nowhere otherwise (the computation gets stuck waiting for an undefined value of g).

DEFINITION 1.1 (Sasso [Sas71], Skordev [Sk72]). We say that f is *T-reducible to g* (written $f \leq_T g$) iff $f = \Phi_i^g$ for some partial recursive functional Φ_i, $i \in \omega$. We write $f \equiv_T g$ iff $f \leq_T g \& g \leq_T f$.

$\equiv_T$ is easily seen to be an equivalence relation. We call the equivalence classes of partial functions under $\equiv_T$ the *T-degrees* and write $\boldsymbol{\mathcal{D}}_T$ for the set of all T-degrees with the ordering $\leq$ induced by $\leq_T$. $\boldsymbol{\mathcal{D}}_T$ is an upper semi-lattice (but by [Sas71] not a lattice, using an adaptation of the Kleene/Post argument [KP54] for $\boldsymbol{\mathcal{D}}$) with least element $\mathbf{0}_T$ = the set of all p.r. functions; and there is an immediate natural embedding of the degrees of unsolvability $\boldsymbol{\mathcal{D}}$ into $\boldsymbol{\mathcal{D}}_T$ (onto $\boldsymbol{\mathcal{T}}$, the T-degrees of total functions). Most of what is known about the structure of the T-degrees is due to Sasso and Casalegno. We give some of the more significant results:

Theorem 1.2 (Sasso [Sas71]). *For any degree $\mathbf{a} \in \boldsymbol{\mathcal{T}}$ there is a degree $\mathbf{b}$ minimal over $\mathbf{a}$.*

The existence of minimal covers in $\boldsymbol{\mathcal{T}}$ for total degrees implies of course the existence of minimal T-degrees. On the other hand Sasso also shows that $\mathbf{0}_T$ is the only total T-degree with a *strong* minimal cover. All the known constructions of minimal covers, minimal T-degrees and initial segments of $\boldsymbol{\mathcal{D}}_T$ depend strongly on the distribution of the *$\mathbf{a}$-semicharacteristic* degrees for $\mathbf{a} \in \boldsymbol{\mathcal{T}}$.

DEFINITION 1.3. If $A \subseteq \omega$, the *semicharacteristic function S_A* for A is the function whose value is 1 on A and undefined elsewhere. A *semicharacteristic* T-degree is one containing a semicharacteristic function. We write $\boldsymbol{\mathcal{S}}$ for the set of all semicharacteristic degrees. $\mathbf{b} \geq \mathbf{a}$ is *$\mathbf{a}$-semicharacteristic* iff $\mathbf{b} = \mathbf{a} \cup \mathbf{c}$ for some $\mathbf{c} \in \boldsymbol{\mathcal{S}}$.

Theorem 1.4 (Casalegno [Cas85]). *Every countable distributive lattice with a least element can be isomorphically embedded as an initial segment of (semicharacteristic) T-degrees.*

Using the Exact Pair Theorem for the T-degrees and the main result of [NS80] Casalegno is able to deduce that the first order theories of $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{D}}_T$ are recursively isomorphic. However:

Theorem 1.5 (Casalegno [Cas85]). *$\langle \boldsymbol{\mathcal{D}}, \leq \rangle$ and $\langle \boldsymbol{\mathcal{D}}_T, \leq \rangle$ are not elementarily equivalent.*

To see this, we need only notice that

(1) All minimal T-degrees are semicharacteristic,
(2) The join of two semicharacteristic degrees is semicharacteristic,
(3) No total degree other than $\mathbf{0}_T$ is semicharacteristic, and
(4) For each $\mathbf{a} \in \boldsymbol{\mathcal{D}}_T$ there is a total $\mathbf{b} \geq \mathbf{a}$.

It follows that there is no upper cone in $\boldsymbol{\mathcal{D}}_T$ of degrees which are the joins of two minimal degrees, in contrast to the situation in $\boldsymbol{\mathcal{D}}$ (see [Co72]). The main result for the T-degrees below $\mathbf{0}'_T$ is the following:

Theorem 1.6 (Sasso [Sas73]). *There is a co-r.e., nonrecursive $A \subseteq \omega$ such that S_A is of minimal T-degree.*

As mentioned above, the construction is simpler than the Spector [Sp56] minimal degree construction for the total degrees in that it suffices to make $\deg_T(S_A)$ minimal in $\boldsymbol{\mathcal{S}}$ ($\boldsymbol{\mathcal{S}}$ is a nontrivial ideal in $\boldsymbol{\mathcal{D}}_T$). This means in effect that the problem is reduced to that of finding a minimal *pc-degree* (see the section on strong enumeration reducibilities below for the definition of *partial conjunctive reducibility*).

The T-degrees have not been extensively studied, and there are a number of outstanding questions worth listing, mainly due to Sasso.

QUESTIONS 1.7.

(1) Since segments above a nontotal T-degree $\mathbf{a}$ cannot in general be considered in terms of segments of $\mathbf{a}$-semicharacteristic degrees, does there exist a minimal cover for a nontotal degree? Do all T-degrees have minimal covers? Are there minimal upper bounds for all countable ideals in $\boldsymbol{\mathcal{D}}_T$? More generally, consider questions of homogeneity.
(2) Characterise the jumps of the minimal T-degrees (cf. [Co73]).
(3) Which total degrees have minimal predecessors (and in particular do all total degrees have minimal predecessors)?
(4) Examine the structure of the degrees of co-r.e. semicharacteristic functions.
(5) Are the total degrees definable in the structure of $\boldsymbol{\mathcal{D}}_T$? Is the jump definable in $\boldsymbol{\mathcal{D}}_T$? (cf. [Co*ta*l])

Let $\{W_i\}_{i\in\omega}$, $\{D_i\}_{i\in\omega}$ be, respectively, standard listings of the recursively enumerable sets and the finite sets of numbers. If $D = D_i$, say, we write $\langle x, D\rangle$ for $\langle x, i\rangle$. A more general reducibility between partial functions is obtained from:

DEFINITION 1.8 (Friedberg and Rogers [FR59]). We say that $\Psi : 2^\omega \to 2^\omega$ is an *enumeration operator* (or *e-operator*) iff for some r.e. set W

$$\Psi(B) = \{x \mid (\exists D)[\langle x, D\rangle \in W \,\&\, D \subseteq B]\},$$

each $B \subseteq \omega$. For any sets A, B define A *is enumeration reducible to* B (or A *is e-reducible to* B, written $A \leq_e B$) by

$$A \leq_e B \Leftrightarrow A = \Psi(B) \text{ for some e-operator } \Psi.$$

This definition formalises the notion of A being reducible to B if and only if there is a procedure such that given any enumeration of the members of B the procedure uniformly provides us with an enumeration of A. We notice that an e-operator can be thought of as being the union of a r.e. set of T-operators.

Writing $\mathrm{graph}(f) = \{\langle x, y\rangle \mid f(x) = y\}$, etc, we can define for partial functions f, g

$$f \leq_e g \Leftrightarrow \mathrm{graph}(f) \leq_e \mathrm{graph}(g).$$

We tend to identify f and $\mathrm{graph}(f)$, for instance by writing Ψ^g for the partial function f (possibly not single-valued) for which $\mathrm{graph}(f) = \Psi^{\mathrm{graph}(g)}$. The notion of *relative partial recursiveness* of partial functions introduced by Kleene in [Kl52] as an extension of the definition by means of systems of equations of the notion of relative recursiveness of total functions is equivalent to $\leq_e$ between graphs of partial functions. As for $\leq_T$ we have that $\leq_e$ is a transitive, reflexive reducibility, and yields a degree structure (the *enumeration* or *e-degrees*, written $\boldsymbol{\mathcal{D}}_e$), with the induced ordering relation $\leq$ under which (see Rogers [Rog67]) $\boldsymbol{\mathcal{D}}_e$ is an upper semi-lattice with a least element ($\mathbf{0}_e =$ the set of all r.e. sets). Using the above definition of $f \leq_e g$, we have a degree structure for the partial functions.

Definition 1.9 (Myhill [My61], Rogers [Rog67]). The *partial degree* $\mathbf{f}$ of f is $\{g \mid f \leq_e g \,\&\, g \leq_e f\}$, where we write $\mathbf{f} \leq \mathbf{g} \Leftrightarrow f \leq_e g$. We write $\boldsymbol{\mathcal{P}}$ for the set of all partial degrees with the ordering $\leq$.

(Sasso, for example in [Sas75], uses the term "partial degrees" interchangably for each of the three main degree structures, $\boldsymbol{\mathcal{D}}_T$, $\boldsymbol{\mathcal{D}}_e$ and $\boldsymbol{\mathcal{D}}_{WT}$ restricted to the partial functions.)

There is an obvious isomorphism between the structure $\boldsymbol{\mathcal{P}}$ of the partial degrees and $\langle \boldsymbol{\mathcal{D}}_e, \leq\rangle$, which depends on the fact that every e-degree contains a *single-valued* set (that is, a set A such that $\langle m, n\rangle, \langle m, n'\rangle \in A \Rightarrow n = n'$, each $m, n, n' \in \omega$). Since $\leq_e$ and $\leq_T$ agree on the total functions, this gives a natural embedding (Myhill [My61]) of $\boldsymbol{\mathcal{D}}$ into $\boldsymbol{\mathcal{D}}_e$, and onto the *total* e-degrees (that is those e-degrees containing a *total* set A, meaning $A = \mathrm{graph}(f)$ for some total function f). If $\mathbf{a}$ is a Turing degree we sometimes write $\mathbf{a}_e$ for the corresponding total e-degree.

The distinction between $\leq_T$ and $\leq_e$ on the partial functions is best described in terms of Turing machines. If we allow the possibility of *inconsistent* quadruples $q_i S x q_k$, $q_i S y q_l$, $xq_k \neq yq_l$, for a Turing machine Z, we obtain the notion of a *nondeterministic oracle machine*. Z uses an oracle for a partial function g as before, but can nondeterministically pursue different computations, thought of as forming the *branches* of a *computation tree*.

Definition 1.10. We say that f is *nondeterministically Turing computable from* g (written $f \leq_{NT} g$) iff there is a nondeterministic oracle-machine Z with oracle g such that

$$\mathrm{graph}(f) = \{\langle m, n\rangle \mid \text{on input } n \text{ some computation branch of } Z \text{ with oracle } g \text{ terminates with output } m\}.$$

Then:

Theorem 1.11 (Cooper, Sasso, and McEvoy [McE84]). *For any partial functions f, g, $f \leq_e g$ if and only if $f \leq_{NT} g$.*

To illustrate the difference between $\leq_T$ and $\leq_e$, we notice (following Myhill [My61]) that if A is any nonrecursive set with characteristic function χ_A then

$$\chi_A \leq_e S_{A\oplus\bar{A}} \quad \text{but} \quad \chi_A \not\leq_T S_{A\oplus\bar{A}}.$$

This is because on the one hand $\chi_A \leq_e S_{A\oplus\bar{A}}$ via the e-operator defined by $\{(\langle 1, n\rangle, \{\langle 1, 2n\rangle\}) \mid n \in \omega\} \cup \{(\langle 0, n\rangle, \{\langle 1, 2n+1\rangle\}) \mid n \in \omega\}$, and on the other hand $\chi_A \leq_T S_{A\oplus\bar{A}}$ would imply that $\chi_A \leq_T 1 =$ the constant function which takes value 1 everywhere, giving χ_A recursive.

Case [Ca71] pointed out an interesting consequence of the proof of Feferman's theorem in [Fe57] which says that every truth-table degree contains a first-order theory. Since the truth-table reduction in the proof is a positive reduction it follows that every e-degree contains a first-order theory. As a theory is axiomatisable if and only if it is effectively enumerable (Craig [Cr53]), this means that the enumeration degrees can be thought of as degrees of unaxiomatisability.

We examine the structure of the enumeration degrees in more detail below, but before that consider a notion of relative computability between partial functions intermediate between $\leq_T$ and $\leq_e$.

Definition 1.12 (Myhill and Shepherdson [MSh55], Rogers [Rog67]). Let $\mathcal{P}$ be the set of all (single-valued) partial functions. We say that $\Theta : \mathcal{P} \to \mathcal{P}$ is a (*partial*) *recursive operator* iff there is an e-operator Ψ such that $\Theta^f = \Psi^f$ for all $f \in \mathcal{P}$ (such that $\Theta^f \downarrow$ or $\Psi^f \in \mathcal{P}$, respectively).

Recursive operators are essentially equivalent to the completely computable or compact functionals of partial functions defined by Davis in [Da58]. Following Sasso, we call the corresponding reducibility *weak Turing reducibility*, and write $f \leq_{WT} g$ if and only if $f = \Theta^g$ for some recursive operator. See Rogers [Rog67], p.281, for a proof of the fact (originally due to Myhill and Shepherdson) that $\leq_{WT}$ is strictly stronger than $\leq_e$. It is easy to see that e and WT computations are distinguished by the way consistency requirements are imposed on the different computational branches, the requirement being relative in the e case and absolute in the WT case. In the former we allow Ψ^g as long as Ψ^g is single-valued, whereas in the latter we must have Ψ^h single-valued for *all* $h \in \mathcal{P}$.

Many results concerning the e-degrees can be carried over to WT-degrees. For instance, it will follow from Gutteridge's theorem (below) that there are no minimal WT-degrees (so neither $\boldsymbol{\mathcal{D}}_{WT}$ nor $\boldsymbol{\mathcal{D}}_e$ are elementarily equivalent to $\boldsymbol{\mathcal{D}}_T$ or $\boldsymbol{\mathcal{D}}$). It is an open question whether $\boldsymbol{\mathcal{D}}_{WT}$ and $\boldsymbol{\mathcal{D}}_e$ are elementarily equivalent. As for the e-degrees the undecidability of the first-order theory of the WT-degrees is not yet known, although in [He79a] Hebeisen states that the "almost all" theory of the WT-degrees is decidable.

Little is known about the structure of the T- or WT-degrees within particular e-degrees.

Definition 1.13. Let $\leq_r$, $\leq_{r'}$ be two reflexive, transitive reducibilities with $\leq_r \ \subseteq \ \leq_{r'}$. We say that an r'-degree $\mathbf{a}$ is *r-contiguous* iff $\mathbf{a}$ contains exactly one r-degree.

Rozinas [Roz74] showed that any nonzero WT-contiguous e-degree is *quasi-minimal* (that is, bounds no nonzero total degrees), and Hebeisen [He79b] demonstrated the abundance of these WT-contiguous degrees.

See [Sas75] for further information comparing $\leq_e$, $\leq_{WT}$ and $\leq_T$.

§2. The Scott graph model for lambda calculus and generalised enumeration operators

In this section we give a brief account of how the enumeration operators can be used to provide a countable version of the graph model for λ-calculus. In so doing, we offer further evidence that enumeration reducibility is *the* fundamental, general concept of relative computability in as much as the nature of the computable universe is intimately bound up with the set of enumeration operators. Most of this material originally appeared in Scott [Sc75a] and [Sc75b] ([Sc75b] is a shorter version of [Sc75a]). See Odifreddi [Od*ta*], Section II.3, for a summary of the basic notions of the λ-calculus and McEvoy [McE84] for a fuller summary of details concerning the graph model.

One problem in finding a model for the (type-free) lambda calculus is that of giving an interpretation of the term $x(x)$. It is not usually possible for a function to be applied to itself as we think of a function as being an object of higher type than its arguments. However, for an e-operator there is no difficulty: an e-operator operates on sets, but an e-operator itself is essentially an r.e. set. The r.e. sets are closed under this definition of application and so the class of computable sets will be the domain of the model. The definition of application is that of evaluating an e-reduction: If x, y are variables in the domain of the model then $x(y) = \{m \mid (\exists D)[\langle m, D\rangle \in x \,\&\, D \subset y]\}$. λ-abstraction describes how we define an e-operator from an r.e. set $\lambda x.\tau = \{\langle m, D\rangle \mid m \in \tau[D/x]\}$ where $\tau[D/x]$ denotes the value of the term τ when every free occurrence of x is replaced by D. We follow Odifreddi [Od*ta*] in summarising the details of the formulation of the model given by Scott.

The two-way correspondence between e-operators and r.e. sets is given by:

Definition 2.1. If A is an r.e. set then Ψ_A is the enumeration operator defined by it, i.e.

$$x \in \Psi_A(B) \Leftrightarrow (\exists u)(\langle x, u\rangle \in A \,\&\, D_u \subseteq B).$$

If Θ is an enumeration operator then G_Θ is a canonical r.e. set defining it, i.e.

$$\langle x, u\rangle \in G_\Theta \Leftrightarrow x \in \Theta(D_u).$$

It is straightforward to verify that $\Psi_{W_i} = \Psi_i$ and $\Psi_{G_\Theta} = \Theta$ (that is, that the e-operator corresponding to W_i is Ψ_i, and that Θ is the e-operator defined by its corresponding canonical set). In later sections we will follow the usual convention of identifying Θ and G_Θ. Under this identification any standard recursive sequence $\{W_i^s\}_{i,s\geq 0}$ of finite approximations to the r.e. sets immediately yields such a sequence $\{\Psi_{i,s}\}_{i,s\geq 0}$ for the e-operators.

We can now define a model of λ-calculus, by associating to every term t an r.e. set $[t]$. If t has n free variables then $[t]$ will be interpreted as an enumeration operator of n set variables (defined by an r.e. expression positive in the set variables). The idea is that closed terms correspond to elements and will be interpreted as r.e. sets, while terms with free variables describe collection of elements, and receive uniform interpretations by interpreting their variables.

Definition 2.2 (Plotkin [Pl72], Scott [Sc75a]). We associate to every variable x of the language of λ-calculus a variable X intended to range over r.e. sets. To every λ-term t with free variables among $x_1, \ldots, x_n$ we inductively associate an arithmetical expression $[t]$ in the variables $X_1, \ldots, X_n$:

(1) $[x] = X$,
(2) $[t_1 \cdot t_2] = \Phi_{[t_1]}([t_2])$,
(3) $[\lambda x.t] = G_{\lambda X.[t]}$.

Intuitively, $[t]$ is interpreted as denoting the value of an e-operator of its set variables, which range over r.e. sets. Thus $[\lambda x.t]$ is inductively interpreted as the function over r.e. sets induced by the expression $[t]$ with respect to the variable corresponding to x. But formally the expression $[t]$ denotes not a function but a set, and so e-operators must be coded by their coded graphs.

It is again straightforward to check, by an induction on the definition of $[t]$ (cf. Plotkin [Pl72], Scott [Sc75a] or Odifreddi [Od*ta*]), that this definition produces what it is intended to:

Proposition 2.3. *For any term t, $[t]$ is well-defined as an enumeration operator of its set variables. In particular, if t is closed then $[t]$ is an r.e. set.*

Looking more closely at the inductive step corresponding to part (3) of Definition 2.2, we have to show that if Ψ denotes the value of an e-operator of the variables $X, Y_1, \ldots, Y_n$ then $G_{\lambda X.\Psi}$ denotes an e-operator of the variables $Y_1, \ldots, Y_n$. But

$$\langle x, u\rangle \in G_{\lambda X.\Psi} \Leftrightarrow x \in \Psi(D_u).$$

By hypothesis, $\Psi(X)$ is an r.e. expression positive in $X, Y_1, \ldots, Y_n$, and it follows that $\Psi(D_u)$ is uniformly r.e. in u and positive in $Y_1, \ldots, Y_n$, as required. A corollary of this part of the proof is that one can iterate the process of taking the graph of an r.e. expression. Thus *any e-operator Ψ of n set variables can be written as the successive composition of e-operators of one variable*, in accordance with the interpretation of functions of many variables in λ-calculus.

We can now verify (again following Plotkin [Pl72] and Scott [Sc75a]) that the interpretation defined above provides a model of λ-calculus:

Proposition 2.4. *The interpretation preserves β-equality, i.e.*

$$t_1 \stackrel{\beta}{=} t_2 \quad \Rightarrow \quad [t_1] = [t_2].$$

Proof. The interpretation preserves applications of the β-rule, since

$$\begin{aligned}
[(\lambda x.t)a] &= \Phi_{[\lambda x.t]}([a]) && \text{by definition of } [t_1 \cdot t_2] \\
&= \Phi_{G_{\lambda X.[t]}}([a]) && \text{by definition of } [\lambda x.t] \\
&= (\lambda X.[t])([a]) && \text{because } \Phi_{G_\Psi} = \Psi \\
&= [t][X/[a]] && \text{by definition of } \lambda X.[t] \\
&= [t[x/a]] && \text{by induction on } t.
\end{aligned}$$

Then, by induction on the number of reduction steps, the interpretation also preserves β-equality. □

We can now use essentially the above construction to describe the graph model for lambda calculus in more generality. This entails identifying which properties of e-operators are actually used in the above construction, and abstracting from this an appropriate notion of *generalised enumeration operator*.

We first define the topological space $\mathcal{P}\omega$. For each finite $D \subset \omega$ we define $U_D \subseteq 2^\omega$ by

$$U_D = \{A \subseteq \omega \mid D \subseteq A\}.$$

The sets U_D are closed under finite intersections, and $\cup\{U_D \mid D \in [\omega]^{<\omega}\} = 2^\omega$, and hence these sets form a countable basis for a topology on 2^ω. Denoting this topological space by $\mathcal{P}\omega$, it follows that the enumeration operators are the computable continuous functions $f : \mathcal{P}\omega \to \mathcal{P}\omega$, and they form the smallest class of functions which is large enough to interpret the lambda calculus.

However, we are now interested in the class of *all* continuous functions $f : \mathcal{P}\omega \to \mathcal{P}\omega$, that is those defined by arbitrary sets $A \in 2^\omega$ with A not necessarily r.e.

Definition 2.5 (Case [Ca71]). A set $A \in 2^\omega$ defines a *generalised enumeration operator* $\Lambda : 2^\omega \to 2^\omega$ iff for each $B \in 2^\omega$, each $m \in \omega$,

$$m \in \Lambda(B) \Leftrightarrow (\exists D)[\langle x, D\rangle \in A \,\&\, D \subseteq B].$$

The definition of the graph model for λ-calculus will now proceed much as before, working with generalised enumeration operators in place of e-operators, given the following:

Proposition 2.6. *An operator $\Psi : 2^\omega \to 2^\omega$ is a generalised enumeration operator if and only if it is a continuous function $\Psi : \mathcal{P}\omega \to \mathcal{P}\omega$.*

Proof. Let Ψ be a generalised enumeration operator and $O = \cup\{U_D \mid D \in \xi\}$, some $\xi \subseteq [\omega]^{<\omega}$, be an open set in $\mathcal{P}\omega$. Then for any $A \in 2^\omega$ we have

$$\begin{aligned} A \in \Psi^{-1}(O) &\Leftrightarrow \Psi(A) \in O \\ &\Leftrightarrow (\exists D \in \xi)[D \subseteq \Psi(A)] \\ &\Leftrightarrow (\exists D \in \xi)(\exists F \subseteq A)[D \subseteq \Psi(F)]. \end{aligned}$$

So $\Psi^{-1}(O) = \cup\{U_F \mid (\exists D \in \xi)[D \subseteq \Psi(F)]$ and is open.

Conversely, let $\Psi : \mathcal{P}\omega \to \mathcal{P}\omega$ be continuous, and define a set $A \in 2^\omega$ by $\langle m, D\rangle \in A \Leftrightarrow m \in \Psi(D)$. To show that the generalised enumeration operator defined by A is Ψ, it only remains to show that given $m \in \omega$ and $B \in 2^\omega$, $m \in \Psi(B)$ if and only if $(\exists D \subseteq B)[m \in f(D)]$. Let $O = \{X \in 2^\omega \mid m \in X\}$. O is a basis element and so O, and hence $\Psi^{-1}(O)$, is open. It is now easy to use the openness of $\Psi^{-1}(O)$ to verify that $B \in \Psi^{-1}(O)$ if and only if $(\exists D \subseteq B)[D \in \Psi^{-1}(O)]$, from which the result follows. □

Then the definitions corresponding to application and λ-abstraction still apply for arbitrary sets $A \in 2^\omega$, giving the required $\mathcal{P}\omega$ model for the λ-calculus.

Finally, we note that apart from Scott's use of generalised enumeration operators in defining the graph model for the λ-calculus, more general types of e-operators have been studied as interesting objects in themselves. For instance, taking the defining set A in Definition 2.5 to be a member of some standard listing $\{A_i\}_{i\in\omega}$ of the arithmetical sets, we get:

Definition 2.7 (Case [Ca71]). A is *arithmetically enumerable in* B (written $A \leq_{ae} B$) iff there is some arithmetical set A_i such that

$$(\forall x)[x \in A \Leftrightarrow (\exists D)[\langle x, D\rangle \in A_i \,\&\, D \subseteq B].$$

One can define (as in Case [Ca71]) a relation $\equiv_{ae}$ with the usual properties, define a structure $\boldsymbol{\mathcal{D}}_{ae}$ (the *partial arithmetical degrees* or *ae-degrees*) in the usual way, and derive a theory for $\boldsymbol{\mathcal{D}}_{ae}$ which parallels that for $\boldsymbol{\mathcal{D}}_e$ (most of the results of the next section can also be stated for the ae-degrees).

Selman ([Se71] and [Se72]) examines a sequence of positive reducibilities corresponding to the individual levels of the arithmetical hierarchy.

Definition 2.8 (Selman [Se71]). $A \mathfrak{S}_n B \Leftrightarrow (\forall X)[B \in \Sigma_n^X \Rightarrow A \in \Sigma_n^X]$.

For each n, $\mathfrak{S}_n$ is a Σ_n*-reducibility* (that is, a reflexive, transitive subrelation of "Σ_n-in"), and (an alternative characterisation of e-reducibility) $\mathfrak{S}_1 = \leq_e$:

Theorem 2.9 (Selman [Se71])**.**

$$A \leq_e B \Leftrightarrow \forall X\,(B \text{ r.e. in } X \Rightarrow A \text{ r.e. in } X).$$

Proof. The left-to-right implication is immediate. Conversely, assume that $A \not\leq_e B$, and construct $C = \cup_{s \geq 0} C_s$ such that B is r.e. in C but A is not r.e. in C. Satisfy "B r.e. in C" by imposing an overall requirement

$$\exists\,\langle x, y\rangle \in C \Leftrightarrow x \in B \tag{2-1}$$

for each $x \geq 0$. Call a finite $D \supseteq C_s$ *admissible* if it satisfies (2-1) with D in place of C, but with the right-to-left half of (2-1) restricted to $x \leq s$ (so that the admissible D's can be enumerated from an enumeration of B and a *finite* amount of information about $\overline{B}$). Satisfy $A \neq W_s^C$ (at stage $s+1$) by looking for some admissible $D \supseteq C_s$ with $x \in W_s^D - A$. If D exists, choose $C_{s+1} = D$ giving $A \neq W_s^C$. Otherwise, either $x \in A - W_s^D$ for some x, all admissible D (so $A \neq W_s^C$ again), or

$$\forall x\,(x \in A \Leftrightarrow \exists \text{ an admissible } D \text{ such that } x \in W_s^D),$$

giving $A \leq_e B$, a contradiction. □

(The above simplification of Selman's proof is essentially due to Copestake [Cope*ta*3]).

Selman also proves a hierarchy theorem for $\{\mathfrak{S}_n\}_{n \geq 1}$ and shows that $\mathfrak{S}_1$ ($= \leq_e$) is a *maximal* subrelation of "Σ_1-in" (that is, of "r.e. in"). In fact, Case [Ca74] extends these results to show that *every* $\mathfrak{S}_n$, $n \geq 1$, is a maximal Σ_n-reducibility, that there are continuously many such maximal Σ_n-reducibilities for each $n \geq 1$, and that for each $n \geq 1$ $E \in \mathfrak{S}_n$ if and only if

$$AEB \Leftrightarrow (\exists C \in \Sigma_n)(\forall x)[x \in A \Leftrightarrow (\exists D)(\langle x, D\rangle \in C)].$$

Case also extends the proof that $\boldsymbol{\mathcal{D}}_e$ is not a lattice to the degree structures for $\mathfrak{S}_n$, $n \geq 2$ (where the appropriate zero degree is Σ_n in each case) and derives results on *quasi-minimal* $\mathfrak{S}_n$-degrees (that is, degrees with no non-Σ_n total predecessors).

Another generalisation of e-reducibility occurs in Sanchis ([San78] and [San79]) in the form of *hyperenumeration reducibility*, which relates to enumeration reducibility as hyperarithmetic reducibility relates to (total) Turing reducibility.

§3. The total degrees, sets and their complements and the jump operator within the enumeration degrees

We start by looking at the relationship of the enumeration degrees to $\boldsymbol{\mathcal{D}}$ via the embedding of $\boldsymbol{\mathcal{D}}$ onto the total e-degrees. Since any countable partial ordering is embeddable in $\boldsymbol{\mathcal{D}}$ [KP54] we immediately get all such embeddings in $\boldsymbol{\mathcal{D}}_e$. The following simple characterisation of the total e-degrees is due to Case ([Ca71]).

Theorem 3.1. *If* **a** *is an e-degree then the following are equivalent:*

(1) **a** *is total.*
(2) **a** *contains an infinite retraceable set.*
(3) **a** *contains an infinite regressive set.*

Proof. (2) ⇒ (3) is immediate.

For (3) ⇒ (1) let A be an infinite regressive set in **a**. Using the regressing function of A, an enumeration of A in some fixed order $f(0), f(1), \ldots$, given by some function f, can be uniformly effectively obtained from any enumeration of A. Since $f \equiv_e A$ we immediately get **a** total.

(1) ⇒ (2): Given a function $f \in \mathbf{a}$ let A be the set of sequence numbers of the form $\langle f(0), \ldots, f(n) \rangle$, $n \in \omega$. Then $A \equiv_e f$ and is retraceable via the function that removes the final entry from any sequence number. □

Many familiar sets of recursion theory turn out to be of total e-degree. Case [Ca74] notes that for each set A we have $\chi_A \equiv_e A \oplus \overline{A} \equiv_e A'$ (since $A, \overline{A} \leq_1 A'$ and A' is r.e. in A), so that A' is always of total e-degree. Moreover, if $\overline{A} \leq_e A$ we get $\chi_A \equiv_e A \oplus \overline{A} \equiv_e A$, giving $\deg_e(A)$ total ($= \deg_e(A')$). Hence (McEvoy [McE84]) if $\mathbf{a} \geq \mathbf{0}^{(n)}$, $n \geq 0$, and **a** contains a Π^0_{n+1} set then **a** is total. In particular (Gutteridge [Gu71]) the e-degrees of Π^0_1 sets are total, and the Rogers [Rog67] isomorphism between the r.e. Turing degrees and the Π^0_1 e-degrees (depending on the fact that $\chi_A \leq_e \chi_B \Leftrightarrow \chi_A \leq_T \chi_B$) is in fact the isomorphism induced by the natural isomorphism from $\boldsymbol{\mathcal{D}}$ onto the total e-degrees. More generally, we observe that if $D = A - B$ is d-r.e. ($A, B = \cup_{s \geq 0} A^s, \cup_{s \geq 0} B^s$ respectively being r.e.) then if C is the r.e. set

$$C = \{\langle s, x \rangle \mid x \in A^s \,\&\, x \in B^t, \text{ some } t \geq s\},$$

we have $D \leq_e \bar{C}$ and $\bar{C} \leq_e D$, so $D \equiv_e C \oplus \bar{C} \equiv_e \chi_C$. So each d-r.e. e-degree is Π_1 and hence total. A similar argument shows that if D is $(n+1)$-r.e. then $D \equiv_e$ some co-n-r.e. C.

The next result shows that $\boldsymbol{\mathcal{D}}_e$ properly extends the total e-degrees. We first need some notation and terminology for strings. Let $\omega^* = \omega \cup \{\uparrow\}$, where the intended interpretation of "$\varphi(m) = \uparrow$" is "φ is undefined on argument m". $(\omega^*)^\omega$ is then the set $\mathcal{P}$ of partial functions. We use α, β, γ etc. for *binary strings*, mapping from a finite initial segment of ω into $\{0,1\}$, and τ, ρ, σ etc. for ω^**-valued strings*, mapping from a finite initial segment of ω into ω^*. The *length* of string τ is $\ell h(\tau) = \mu x[\tau(x) \uparrow]$. We say τ *strongly extends* σ ($\tau \widetilde{\supset} \sigma$) iff $\forall x < \ell h(\sigma)[\tau(x) = \sigma(x)]$. We say τ *extends* σ ($\tau \supset \sigma$) iff $\forall x < \ell h(\sigma)[\tau(x) = \sigma(x)$ or $\sigma(x) = \uparrow]$. σ is a *beginning* of φ iff $\sigma \widetilde{\subset} \varphi$, and $\varphi \upharpoonright x$ is the beginning of φ of length x. Similarly, α is a *beginning* of A means $\alpha \subset \chi_A$, and $A \upharpoonright x$ is the beginning of A of length x. τ and σ are *compatible* iff one of them strongly extends the other. We write $\tau\hat{\ }\sigma$ for the standard *concatenation* of the two strings τ and σ.

Definition 3.2. A non-r.e. e-degree (or partial degree) **a** is *quasi-minimal* iff there are no total predecessors $\mathbf{b} \leq \mathbf{a}$ other than $\mathbf{0}_e$.

Theorem 3.3 (Medvedev [Me55]). *There exists a quasi-minimal enumeration degree*

Proof. We construct a partial function φ to satisfy the following requirements

$$R_{2i}: \quad \varphi \neq \{i\} \text{ (the } i^{th} \text{ p.r. function)}$$
$$R_{2i+1}: \quad \Psi_i(\varphi) \text{ total } \Rightarrow \Psi_i(\varphi) \text{ p.r.}$$

We build φ by finite initial segments, starting with $\sigma_0 = \phi$ (the empty function). At stage $s+1$, let σ_s be given.

* If $s = 2i$ then we satisfy R_{2i} by choosing the least x such that $\sigma_s(x)$ is not yet defined, and extending σ_s to σ_{s+1} by defining

$$\sigma_{s+1}(x) \neq \{i\}(x).$$

* If $s = 2i+1$ then we try to trivially satisfy R_{2i+1} by making $\Psi_i(\varphi)$ not single-valued. To do this we look for an ω^*-valued string $\sigma \widetilde{\supseteq} \sigma_s$ such that, for some x, y and z:

$$y \neq z \ \& \ \langle x, y\rangle, \langle x, z\rangle \in \Psi_i(\sigma).$$

If such a σ exists (subcase (a)), choose one and define $\sigma_{s+1} = \sigma$. Otherwise, we further look for $\pi, \rho \widetilde{\supset} \sigma_s$ such that for some w $\exists \langle w, u\rangle \in \Psi_i(\pi), \langle w, v\rangle \in \Psi_i(\rho)$, with $u \neq v$. If π, ρ exist (subcase (b)), define

$$\sigma_{s+1} = \sigma_s \cup \{\langle z, 0\rangle\},$$

where $z \geq \max\{z' \mid z' \in \mathrm{dom}(\pi) \cup \mathrm{dom}(\rho)\}$, and otherwise (subcase (c)) let $\sigma_{s+1} = \sigma_s$.

To see that this strategy satisfies R_{2i+1}, we first notice that if subcase (b) applies then we have R_{2i+1} satisfied through $\Psi_i(\varphi)$ not being total (since if $\Psi_i(\varphi)(w)\downarrow$ some beginning of $\varphi \cup \pi$ or $\varphi \cup \rho$ would put us in subcase (a)). We need now only consider subcase (c) in which no such σ exists satisfying (a) or (b) and $\Psi_i(\varphi)$ is total. It is then straightforward to verify that (for $s = 2i+1$)

$$\langle x, y\rangle \in \Psi_i(\varphi) \ \Leftrightarrow \ (\exists \sigma \supseteq \sigma_s)[\langle x, y\rangle \in \Psi_i(\sigma)],$$

so that $\Psi_i(\varphi)$ is p.r. as required. It is important to notice the use of $\Psi_i(\varphi)$ total in proving the right-to-left implication - if $\Psi_i(\varphi)$ is not total, we merely get that $\Psi_i(\varphi)$ has a partial recursive extension. □

If we replace φ with a set A which is not necessarily single valued in the above proof we can dispense with subcase (b) (see Odifreddi [Od89]). The slightly stronger construction is needed for Theorem 4.3 below.

In [My61] Myhill extended Theorem 3.3 by showing that 'most' e-degrees are quasi-minimal in that the set of quasi-minimal degrees is co-meager in $\boldsymbol{\mathcal{D}}_e$. Lagemann [Lag72] showed that the measure of the sets of quasi-minimal degree is one. In the next section we look at how finite extension arguments such as the one above can be presented using a forcing framework.

If we recursively approximate the construction of Theorem 3.3, we can show that there exists a 3-r.e. quasi-minimal e-degree (so that not all 3-r.e. degrees are total). We can construct via a finite injury priority argument a co-d-r.e. set A satisfying

$$R_{2i}: \quad A \neq W_i$$
$$R_{2i+1}: \quad \Psi_i(A) \text{ total } \Rightarrow \Psi_i(A) \text{ p.r.}$$

The R_{2i} requirements are satisfied by choosing some follower x to be extracted from A if x is enumerated into W_i. For R_{2i+1} we wait for some finite D, and some x, y, z, $y \neq z$, with $\langle x, y\rangle, \langle x, z\rangle \in \Psi_i^D$ at stage $s+1$ (say), and if such a D appears, seek at later stages to maintain $D \subset A$. If we avoid ever choosing a follower x for a requirement R_{2i} which has already appeared in such a D, we obtain A co-d-r.e., as required. Incidentally, we notice that we can adapt the construction (see[CLW89]) of a properly n-r.e. Turing degree to show that the hierarchy of sets of n-r.e. e-degrees is proper. Nothing seems to be known about the structure of the n-r.e. e-degrees for $n \geq 3$.

Case [Ca71] gives a number of incomparability results concerning the possible e-degrees of $A, \overline{A}$ (see also [Se71], Theorem 2.9). For instance:

Theorem 3.4 (Jockusch)**.** *Every non-recursive Turing degree contains a set A such that $A \mid_e \overline{A}$.*

Proof. In [Jo68] Jockusch showed that every non-recursive Turing degree contains a semi-recursive set A such that $A, \overline{A}$ are not r.e. The theorem follows by showing that

$$(\forall A)[A \text{ is semi-recursive } \& \ \overline{A} \leq_e A] \Rightarrow \overline{A} \text{ is r.e.}].\square$$

Case notes that the theorem implies the non-existence of total e-degrees which are minimal (cf. section 5 below). This follows from the fact that if f is total then there is a set A such that $A, \overline{A} \leq_e \chi_A \equiv_e f$ and $A \mid_e \overline{A}$. Case also shows that among such sets A with $A \mid_e \overline{A}$, one can have both $A, \overline{A}$ of total e-degree, or both $A, \overline{A}$ of non-total e-degree (either possibilities with $A \leq_T \phi'$). One can also obtain the possibility with $\deg_e(A)$ total and $\deg_e(A)$ non-total (in fact quasi-minimal - see [Gu71], [Mo74] and [Sor88]).

An obvious consequence of Theorem 3.4 is that A and χ_A often have distinct e-degrees. For this reason we will avoid the usual convention in Turing degree theory of identifying a set A with its characteristic function. This will enable us to unambiguously identify Turing degrees with corresponding total e-degrees so long as we write $\deg_T(\chi_A)$ for $\deg_T(A)$.

Selman ([Se71]) showed that for each e-degree there is a larger one by proving that for each set A $\deg_e(A) <_e \deg_e(A'')$, while remarking that $\deg_e(A')$ is not always greater than $\deg_e(A)$ (as pointed out previously, $A' \equiv_e \chi_A$). Given that $A'' \equiv_e \chi_{A'}$, one might hope to extend the jump for the Turing degrees via the Turing double-jump. Unfortunately, the fact that A'' uses negative information about A, so (cf. Theorem 3.1) is dependent on a particular enumeration of A, means that A'' is not invariant with respect to e-equivalence (not even over $\mathbf{0}_e$). Instead:

DEFINITION 3.5 (Cooper [Co84], McEvoy [McE1985]). Let $K_A = \{x \mid x \in \Psi_x(A)\}$. We define the *e-jump* of A by $J(A) = \chi_{K_A}$, and the *jump* of an e-degree $\mathbf{a}$ by $\mathbf{a}' = \deg_e J(A)$, any $A \in \mathbf{a}$.

It is straightforward to verify (see [McE85]) that this jump is well-defined on the e-degrees, that $\mathbf{a} < \mathbf{a}'$ for all $\mathbf{a}$, and that $\chi_{A'} \equiv_e J(\chi_A)$, so that this jump on the e-degrees agrees with the jump on the Turing degrees (under the natural embedding). Inductively defining $J^{(n)}(X)$ by $J^{(n+1)}(X) = J(J^{(n)}(X))$ in the usual way, we can also get an analogue of Post's Theorem [Po44] for the Turing degrees:

Theorem 3.6 (Cooper [Co84]**, McEvoy** [McE85]**).** *For any sets A, B $A \in \Sigma^B_{n+1}$ if and only if $A \leq_e J^{(n)}(\chi_B)$.*

Proof. By induction on n. We just give the case $n = 1$ and leave the inductive step to the reader.

Assuming $A \leq_e J(\chi_B)$, so that $x \in A \Leftrightarrow x \in \Psi_i(J(\chi_B))$ for some i, we get $A \in \Sigma^B_2$ by direct calculation of the quantifier form of $x \in \Psi_i(J(\chi_B))$.

Conversely, if $A \in \Sigma^B_2$ then A is r.e. in $B' = (\chi_B)'$. So $A \leq_e \chi_{(\chi_B)'} \equiv_e J(\chi_B)$ since the jumps agree via the natural embedding. □

Corollary 3.7. *For any set A $A \in \Sigma_{n+1}$ if and only if $A \leq_e J^{(n)}(\phi)$. In particular, $A \in \Sigma_2$ if and only if $A \leq_e J(\phi)$.*

We identify $\mathbf{0}^{(n)}_e = \deg_e(J^{(n)}(\phi))$ in the natural way with $\mathbf{0}^{(n)}$, $n \geq 1$. Then as a corollary of the Friedberg Jump Inversion Theorem it follows that the range of the jump over the e-degrees is exactly the set of total degrees $\geq \mathbf{0}'$. By analogy with the Cooper jump inversion theorem [Co73]:

Theorem 3.8 (McEvoy [McE85])**.** *Given any total e-degree $\mathbf{b} \geq \mathbf{0}'$ there is a quasi-minimal degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{b}$.*

Proof. We add extra stages (analagous to those in Rogers' proof [Ro67] of the Friedberg jump inversion theorem) to the finite extension construction for Theorem 3.3. At these extra stages, we make $J(A) \equiv_e B$ by:

(1) Coding information about $B \in \mathbf{b}$ into A in such a way that we can retrieve the construction from ϕ' and A, giving $B \leq_e A \oplus \phi' \leq_e \chi_{K_A} = J(A)$, and
(2) Trying to ensure $\alpha \subset \chi_A$ for some finite extension $\alpha \supset \alpha_{4s+3}$ (say) with $s \in \Psi_s(\alpha^+)$, $\alpha^+ = \{x \mid \alpha(x) = 1\}$ (that is, we try to make A *force its jump* in an analagous sense to that of [Le83]). Hence, by making the construction recursive in B and ϕ' we get $J(A) = \chi_{K_A} \leq_e B \oplus \phi' \leq_e B$.

□

In addition, as a corollary of the Sacks Jump Inversion Theorem [Sa63] for the r.e. Turing degrees, McEvoy [McE85] verifies that the range of the jump over the Π_1 e-degrees is exactly the set of total Π_2 e-degrees $\geq \mathbf{0}'$.

Finally, we can define (generalised) high/low hierarchies of e-degrees exactly as for the Turing degrees (see [So87] or [Le83]). We will examine the lower levels of these hierarchies in more detail below.

§4. Genericity, basic structure and immunity properties

The most basic structural results for $\boldsymbol{\mathcal{D}}_e$, like those for $\boldsymbol{\mathcal{D}}$ (see for example Lerman [Le83]), are best presented in the context of generic functions and sets, or in some roughly equivalent framework using category, measure or games. At this level (as can be seen from the original such treatments by Case in [Ca71] or by Moore in [Mo74]) the differences in technique needed in the two structures are interesting but undramatic.

Genericity for partial functions can be defined as in Case [Ca71] in terms of Cohen forcing, relative to the language of first order arithmetic augmented in such a way as to allow statements about partial functions to be directly expressed. An equivalent formulation (compare Jockusch [Jo80]) is:

DEFINITION 4.1. φ is *generic* iff for all arithmetical sets S of ω^*-valued strings either

(1) $\exists\sigma\tilde{\subseteq}\varphi$ such that $\sigma \in S$, or
(2) $\exists\sigma\tilde{\subseteq}\varphi$ such that $\forall\tau\tilde{\supseteq}\sigma(\tau \notin S)$.

An e-degree (or partial degree) $\mathbf{a}$ is *generic* iff it contains a generic function φ (that is, contains $graph(\varphi)$ for some generic φ).

We also define in the usual way

DEFINITION 4.2. A is *set generic* iff for all arithmetical sets S of binary strings either

(1) $\exists\alpha \subset \chi_A$ such that $\alpha \in S$, or
(2) $\exists\alpha \subset \chi_A$ such that $\forall\beta \supset \alpha(\beta \notin S)$.

An e-degree $\mathbf{a}$ is *set generic* iff it contains a generic set A.

Case's emphasis on generic functions rather than sets has the advantage that the results below for generic and n-generic e-degrees also hold in $\boldsymbol{\mathcal{P}}$ for generic and n-generic partial degrees (we will see later that the notions do not always agree).

The existence of generic and set generic e-degrees follows in the usual way (as in [Fe65]). We summarise below the more interesting properties of the generic e-degrees.

Theorem 4.3 (Case [Ca71]). *If φ is a generic function then:*

(1) $\mathrm{graph}(\varphi)$ *is infinite and contains no infinite arithmetical subset (so is immune).*
(2) φ *has no partial recursive extension.*
(3) $\deg_e(\varphi)$ *is quasi-minimal.*
(4) *If $\varphi = \varphi_0 \oplus \varphi_1$ then φ_0, φ_1 are generic and $\varphi_0 \mid_e \varphi_1$.*
(5) $\deg_e(\varphi_0), \deg_e(\varphi_1)$ *form a minimal pair (that is, $\deg_e(\varphi_0), \deg_e(\varphi_1)$ are non-zero and share no predecessors other than $\mathbf{0}$) below $\deg_e(\varphi)$.*
(6) *There is a countably infinite e-independent set $\{\varphi_i\}_{i\in\omega} \leq_e \varphi$ (that is, each $\varphi_i \leq_e \varphi$ and no φ_i is $\leq_e$ the recursive join of a finite subset of $\{\varphi_i\}_{i\in\omega}$).*

(The theorem also holds with a generic set A in place of φ and graph(φ)*).*

Proof. With the proof of Theorem 3.3 as background, the proofs of (1) to (4) are fairly straightforward forcing arguments.

To prove (5) let φ be generic. By (4) $\deg_e(\varphi_0), \deg_e(\varphi_1) > \mathbf{0}$. It remains to show that for each $i, j \geq 0$ either (a) $\Psi_i(\varphi_0) \neq \Psi_j(\varphi_1)$, or (b) $\Psi_j(\varphi_1)$ is r.e. Let

$$R = \{\sigma \mid \sigma \Vdash \Psi_i^{\psi_0} \neq \Psi_j^{\psi_1}\}$$

(that is, $R =$ the set of all σ such that there is some x for which $\Psi_i^{\tau_0}(x) \neq \Psi_j^{\tau_1}(x)$ for each $\tau \widetilde{\supseteq} \sigma$). Then R is arithmetical (in fact $\in \Sigma_2$). So since φ is generic either there is some $\sigma \widetilde{\subset} \varphi$ in R (in which case (a) holds), or there is some $\sigma \widetilde{\subset} \varphi$ such that no $\tau \widetilde{\supseteq} \sigma$ is in R. In the latter case we verify that

$$\text{(4-1)} \qquad x \in \Psi_j(\varphi_1) \Leftrightarrow (\exists \tau \widetilde{\supseteq} \sigma)[x \in \Psi_j^{\tau}].$$

The left-to-right implication is immediate. Assume $x \in \Psi_j^{\tau}$, $\tau \widetilde{\supset} \sigma$. Define

$$S_x = \{\pi \widetilde{\supseteq} \tau \mid x \in \Psi_i^{\pi_0}\} \in \Sigma_1.$$

Then since φ_0 is generic (by (4)), and since case (a) does not hold, $x \in \Psi_i^{\varphi_0}$ giving $x \in \Psi_j(\varphi_1)$. It immediately follows from (4-1) that $\Psi_j(\varphi_1)$ is r.e., and (5) is proved.

(6) is obtained by extending the recursive decomposition of φ in (4) to a countably infinite such decomposition. □

We notice that (6) implies, using a similar embedding technique to that of Kleene and Post [KP54] (see also section 5 below), that any countable partial ordering $\leq_e \varphi$ is embeddable in the e-degrees below $\deg_e(\varphi)$. Case also obtains, by constructing a binary branching tree of e-independent generic functions, a continuum of independent quasi-minimal degrees. Also, all the above results are extended to $\leq_{ae}$ and the ae-degrees. Copestake [Cope87] observes that if $\mathbf{a}, \mathbf{b}$ are set generic then (arguing as in [Jo80] for the Turing degrees) the structures $\boldsymbol{\mathcal{D}}_e(\leq_e \mathbf{a})$ and $\boldsymbol{\mathcal{D}}_e(\leq_e \mathbf{b})$ are elementarily equivalent.

Unfortunately, by part (1) of the above theorem generic sets and functions have the disadvantage of being extremely non-constructive. Following Hinman [Hi69] (introducing the notion of *n-genericity*), Posner [Pos77] (formulating 1-genericity in terms of sets of strings) and Jockusch [Jo80] (defining n-generic in this way), Copestake [Cope88] uses the arithmetical hierarchy to refine the notion of genericity for e-degrees:

Definition 4.4. φ is *n-generic* iff for all Σ_n sets S of ω^*-valued strings either (i) $\exists \sigma \widetilde{\subset} \varphi$ such that $\sigma \in S$, or (ii) $\exists \sigma \widetilde{\subset} \varphi$ such that $\forall \tau \widetilde{\supset} \sigma (\tau \notin S)$. An e-degree is *$n$-generic* iff it contains an n-generic function.

Corresponding to Definition 4.2 we have the parallel definitions of *n-generic set* (as in [Jo80] or [Le83]) and *set n-generic e-degree.*

Theorem 4.5 (Copestake [Cope88]).

(1) *There exists an n-generic e-degree below $\mathbf{0}^{(n)}$ for each $n \geq 1$.*
(2) *There is no $(n+1)$-generic e-degree below $\mathbf{0}^{(n)}$, any $n \geq 1$.*
(3) *Every 1-generic e-degree is quasi-minimal.*
(4) *Every 2-generic e-degree bounds a minimal pair of e-degrees.*
(5) *If $\mathbf{a}$ is a 1-generic e-degree then every r.e. partial ordering can be embedded below $\mathbf{a}$ (in the 1-generic degrees below $\mathbf{a}$).*

(All results holding with 'set n-generic' in place of 'n-generic').

Proof. (1) follows from an examination of the standard existence proof for generic functions.

For (2), follow the proof of Theorem 4.3, part (1), in showing that if φ is n-generic then graph(φ) contains no infinite Σ_n subset, and graph(φ) contains no infinite single-valued Σ_n subset. Then the result follows since $\varphi \leq_e \phi^{(n)} \Rightarrow \varphi \in \Sigma_{n+1}$.

An examination of the proof of Theorem 3.3 shows that a 1-generic set suffices to replicate the finite extension argument required, which gives (3).

For (4), we need only notice that the set R of strings in the proof of Theorem 4.3, part (5), is Σ_2.

Finally, considering Theorem 4.3, part (6), we see that the proof can be adapted to show that if φ is n-generic then the recursive decomposition $\{\varphi_i\}_{i\in\omega}$ gives an e-independent set of n-generic functions. Then (5) follows using the case $n = 1$ with the Kleene-Post embedding argument mentioned previously. □

In contrast to part (4) of Theorem 4.5, Copestake [Cope*ta*2] shows, using an infinite injury priority construction, that there is a 1-generic e-degree below $\mathbf{0}'$ which bounds no minimal pair of e-degrees. Jockusch has observed that the direct construction of a minimal pair $\deg_e(A), \deg_e(B)$ is actually more easily achieved when one of the sets A or B is given (so every non-zero e-degree is cappable, despite, as we shall see below, the non-existence of minimal e-degrees). The Jockusch technique occurs in the proof of the Exact Pair Theorem below.

Structural results obtained by the methods of Theorem 4.5 fairly easily relativise to $\mathcal{D}_e(\geq \mathbf{d})$, any given $\mathbf{d}$. This is not possible in general (unlike for the Turing degrees), as the e-reducibility of $D \in \mathbf{d}$ to the sets constructed must be independent of the particular construction used. See Rozinas [Roz78a] for details of the relativisation of the minimal pair construction, and [MC85] for further results using these techniques, such as $(\forall \mathbf{b})(\forall \mathbf{d})[\mathbf{d} < \mathbf{b} \Rightarrow (\exists \mathbf{a} > \mathbf{d})\, \mathbf{a} \cap \mathbf{b} = \mathbf{d}]$ and a correct proof of the:

Exact Pair Theorem 4.6 (Case [Ca71]). *Given a countable ideal $\{\mathbf{b}_0, \mathbf{b}_1, \ldots\}$ of e-degrees, there exist degrees $\mathbf{a}$ and $\mathbf{b}$ such that:*

(i) For every n, $\mathbf{b}_n < \mathbf{a}, \mathbf{b}$, and

(ii) If $\mathbf{d} \leq \mathbf{a}, \mathbf{b}$ then for some n, $\mathbf{d} \leq \mathbf{b}_n$.

Proof. Similar to Spector's proof [Sp56] for the Turing degrees. Let $B_n \in \mathbf{b}_n$ for each n and define B by $\langle k, n\rangle \in B \Leftrightarrow k \in B_n$, each n, k. In the same way we ensure

$B_n \leq_e A$ by coding B_n into the n^{th} row of A at stage $n+1$ of the construction, modulo a finite set of previously defined special values of A concerned with making $\Psi_i(A) = \Psi_j(B) \Rightarrow \Psi_i(A) \leq_e \oplus_{i \leq n} B_i$, each $\langle i,j \rangle \leq n$. The special values defined at stage $n+1$ are intended to make some $x \in \Psi_i(A) - \Psi_j(B)$, $\langle i,j \rangle = n$. If no such special values can be found, we are able to argue that if $\Psi_i(A) = \Psi_j(B)$ then $\Psi_i(A) \leq_e \oplus_{i \leq n} B_i$ as required. □

This shows that infinite ascending sequences of degrees do not have least upper-bounds (little is known about minimal upper-bounds for ascending sequences). We also get:

Corollary 4.7 (Case [Ca71], Selman [Se71]). *$\boldsymbol{\mathcal{D}}_e$ is not a lattice.*

Proof. Apply Theorem 4.7 to any infinite ascending sequence of e-degrees. □

Rozinas [Roz78a] shows that $\boldsymbol{\mathcal{D}}_e$ is not distributive by adapting the minimal pair construction to get sets A, B for which $A \cap B \not\leq B$ and $\deg_e(A) \cap \deg_e(A \cap B) = \mathbf{0}$.

The 1-generic sets and functions are of special interest, as they produce structural results below $\mathbf{0}'$. An examination of the relationship between the (function) 1-generic and the set 1-generic degrees yields some unexpected consequences. We first define:

Definition 4.8 (Copestake [Cope88]). $\mathbf{a} > \mathbf{0}$ is *minimal-like* iff there exists $\varphi \in \mathbf{a}$ such that each $\psi <_e \varphi$ has a partial recursive extension. $\mathbf{a} > \mathbf{0}$ is *strongly minimal-like* iff there exists $A \in \mathbf{a}$ such that each $\psi \leq_e A$ has a partial recursive extension.

This terminology is motivated by a consideration of the usual minimal (Turing) degree construction in the context of $\boldsymbol{\mathcal{P}}$. We say that a binary string α *i-splits* if it has extensions β_1, β_2 such that, for some x, y and z with $y \neq z$,

$$\langle x,y \rangle \in \Psi_i(\beta_1) \text{ and } \langle x,z \rangle \in \Psi_i(\beta_2).$$

Making the obvious modifications in the proof of the Computation Lemma (see for instance [Le83], p.105), we see that if A is on the recursive tree T then: (i) If T is i-splitting we have $\chi_A \leq_e \Psi_i(\chi_A)$, and (ii) If T has no i-splittings then $\Psi_i(\chi_A)$ has a partial recursive extension whenever it is single-valued. Doing the modified minimal degree construction below $\mathbf{0}'$ ([Sa61]), Copestake [Cope87] proves that there is a total minimal-like e-degree $< \mathbf{0}'$. Since it is easy to extend Theorem 4.3, part (2) to show that no 1-generic φ has a p.r. extension, and we know that no 1-generic φ has a non-recursive total predecessor, Copestake immediately gets the corollary that there is an e-degree ($< \mathbf{0}'$) incomparable with all the 1-generic e-degrees.

By Theorem 4.5, part (5), no 1-generic e-degree is minimal-like. On the other hand:

Theorem 4.9 (Copestake [Cope88]). *(i) Every set 1-generic e-degree is strongly minimal-like, and hence no set 1-generic degree has a 1-generic predecessor.*
(ii) A set A is n-generic if and only if it is the domain of some n-generic function φ. Hence there is a set n-generic $\mathbf{b}$ below any given n-generic $\mathbf{a}$, and there is a n-generic $\mathbf{a}$ above an arbitrary set n-generic $\mathbf{b}$.

Proof. (i) Suppose A is 1-generic and $\varphi \leq_e A$, so $\varphi = \Psi_i^A$, some i. Show that some $\alpha \subset \chi_A$ forces each Ψ_i^X, $\chi_X \supset \alpha$, to be single-valued (that is, show that $\exists \alpha \subset \chi_A$

such that no $\beta \supset \alpha$ is in $S = \{\beta \mid \exists x, y, z\, (\langle x, y\rangle \in \Psi_i^{\beta^+} \,\&\, \langle x, z\rangle \in \Psi_i^{\beta^+} \& y \neq z)\}$). Then ψ defined by

$$\psi(a) = b \Leftrightarrow \exists \beta \supset \alpha\, [\langle a, b\rangle \in \Psi_i^{\beta^+}]$$

is a partial recursive function extending φ.

(ii) ($\Leftarrow$) Associate with a given 2-valued set $S \in \Sigma_n$ an ω^*-valued Σ_n set $T = \{\tau \mid \mathrm{d}(\tau) \in S\}$, where $\mathrm{d}(\tau)(x) = 0$ if $\tau(x) = \uparrow$, $= 1$ otherwise, each $x \leq \ell h(\tau)$. If φ is n-generic, either some $\sigma \tilde{\subset} \varphi$ is in T or for some $\sigma \tilde{\subset} \varphi$ no $\tau \tilde{\supset} \sigma$ is in T. Hence, respectively, either $\mathrm{d}(\sigma) \subset \chi_{\mathrm{dom}(\varphi)}$ is in S or $\mathrm{d}(\sigma) \subset \chi_{\mathrm{dom}(\varphi)}$ and no $\alpha \supset \mathrm{d}(\sigma)$ is in S.

($\Rightarrow$) Conversely, let A be n-generic, and define an n-generic $\varphi = \cup_{s \geq 0} \varphi_s$ where $\forall s\, (\mathrm{d}(\varphi_s) \subset \chi_A)$ gives $\mathrm{dom}(\varphi) = A$. At stage $s+1$, assume σ_s constructed, and take care of the n-genericity requirement for the s^{th} Σ_n set S_s of ω^*-valued strings by using the n-genericity of A to find either (i) a $\sigma_{s+1} \tilde{\supset} \sigma_s$ for which $\mathrm{d}(\sigma_{s+1}) \subset \chi_A$ and $\sigma_{s+1} \in S_s$ or (ii) a $\sigma_{s+1} \tilde{\supset} \sigma_s$ for which $\mathrm{d}(\tau) \not\subset \chi_A$, each $\tau \tilde{\supset} \sigma_{s+1}$ with $\tau \in S_s$. The rest of part (ii) follows from the fact that $\mathrm{dom}(\varphi) \leq_e \varphi$, each φ. □

We therefore get the surprising result that not only are the notions of 1-generic and set 1-generic degree distinct, they are mutually exclusive. This reinforces our intuition that, for instance, it is *harder* constructing a quasi-minimal function than a quasi-minimal set (as remarked following the proof of Theorem 3.3), namely because our constructions produce different degrees.

A number of questions arise by analogy with the situation for the 1-generic Turing degrees, such as those concerned with jump and downward closure.

As for the Turing degrees, an e-degree $\mathbf{a}$ is said to be *low*$_n$ (with *low*=low$_1$) if $\mathbf{a}^{(n)} = (\mathbf{a} \cup \mathbf{0}')^{(n-1)}$, and *high*$_n$ (with *high*=high$_1$) if $\mathbf{a}^{(n)} = (\mathbf{a} \cup \mathbf{0}')^{(n)}$. We can characterise the low e-degrees [MC85] as those consisting entirely of Δ_2 sets. Since every 1-generic Turing degree $\leq \mathbf{0}'$ is low (see for instance [Le83], p.80), we have for each 1-generic $A \in \Delta_2$ that $J(A) \leq_e J(\chi_A) \equiv_e \chi'_A \in \mathbf{0}'$, so (Copestake [Cop*eta*1]) every Δ_2 set 1-generic $\mathbf{a}$ is low. On the other hand, there are 1-generic sets of degree $< \mathbf{0}'$ that are not low:

Definition 4.10. An e-degree $\mathbf{a} < \mathbf{0}'$ is *properly-* Σ_2 iff it contains no Δ_2 sets.

Theorem 4.11 (Copestake [Cop*eta*1])**.** *There exists a set* 1*-generic e-degree* $< \mathbf{0}'$ *which is properly* Σ_2*, and hence is not low.*

Proof. Construct a Σ_2 set A by means of a Σ_2 *approximation* $\{A^s\}_{s \geq 0}$ (that is, $\{A^s\}_{s \geq 0}$ is a recursive sequence of finite sets with

$$x \in A \Leftrightarrow \exists t\, \forall s > t\, (x \in A^s)).$$

We need a standard listing $\{B_i\}_{i \geq 0}$ of the Σ_2 sets in which the Δ_2 sets appear with Δ_2 approximations $\{B_i^s\}_{s \geq 0}$, and take $W_i = \cup_{s \geq 0} W_i^s$, $i \geq 0$, to be a standard listing of all r.e. sets of binary strings. The requirements to be satisfied are

$$S_i: \quad \exists \alpha \subset \chi_A\, (\alpha \in W_i) \text{ or } \exists \alpha \subset \chi_A\, \forall \beta \supset \alpha\, (\beta \notin W_i),$$

$$R_i: \quad [A = \Psi_i(B_i) \;\&\; B_i = \Theta_i(A)] \Rightarrow \exists x \lim_s B_i^s(x) \uparrow,$$

where $\{(\Psi_i, \Theta_i)\}_{i\geq 0}$ is a standard listing of all pairs of e-operators.

We satisfy the R_i requirements by looking for a follower (x, D, E) (D, E finite) satisfying the 'set-up' $x \in \Psi_i(D)$ & $D \subseteq \Theta_i(E)$. If some such triple is found we fix $E - \{x\}$ in A and also put x in A. We then wait until $D \subseteq B_i^s$. If this happens we extract x from A and insert it again only if $D \not\subseteq B_i^t$ at a later stage t. We repeat the extraction and insertion of x so that if the set-up returns infinitely often some element of D is forced in and out of B_i, thus preventing it from being Δ_2. (We call this the *'properly Σ_2 strategy'* - note the parallel with the d-r.e. strategy in, for instance, [CLW89]).

We satisfy S_i by looking for a string in W_i^s that is compatible with action taken on higher priority requirements, and making it a beginning of A. Because of the infinite outcome to R_i in which $\lim_s B_i(x)$ does not exist, a tree of strategies is required to provide requirements with sufficient information concerning higher priority outcomes and to help with the coordination of actions on the different requirements. □

By Theorem 4.9, part (ii), we have the corollary that not every 1-generic e-degree below $\mathbf{0}'$ is low. We are left with:

PROBLEM 4.12. Characterise the jumps of the (set) 1-generic e-degrees below $\mathbf{0}'$.

We can also ask (cf. Haught [Ha86]):

QUESTION 4.13. Are the e-degrees of 1-generic sets (below $\mathbf{0}'$) closed downwards?

We have seen that 1-generic sets and the graphs of 1-generic functions are immune. In fact, 1-generic sets ([Jo80]) and the graphs of 1-generic functions ([Cope87]) are hyperimmune (but not cohesive). It follows from the following result that all $\mathbf{b} \geq$ a set 1-generic $\mathbf{a}$ are hyperimmune.

Theorem 4.14 (Rozinas [Roz78b])**.** *(i) The immune e-degrees are closed upwards, and (ii) so are the hyperimmune e-degrees.*

Proof. (i) Say $B \geq_e$ an immune set A. Define an infinite set $C \equiv_e B$ by $C = \{\langle m, n\rangle \mid m \in A \;\&\; n \in B \;\&\; m \geq n\}$. Verify that if W is r.e. and $\subseteq C$, then $X = \{x \mid \exists n\, [\langle x, n\rangle \in W]\}$ is $\subseteq A$ and so finite, giving W finite.
(ii) If A is now hyperimmune, define $C \equiv_e B$ as before. Assume that C is not hyperimmune and let f be a recursive function for which:

(4-2) $\qquad \forall x, y\, [x \neq y \Rightarrow D_{f(x)} \cap D_{f(y)} = \phi]$, and

(4-3) $\qquad \forall x\, [D_{f(x)} \cap C \neq \phi]$.

Define g recursive by $g(0) = f(0)$ and for all x, $g(x+1) = f(n_x)$ where

$$n_x = \mu n \geq x+1\, [\forall z \leq x\, \forall i, j, k\, [\langle i, j\rangle \in D_{g(z)} \Rightarrow \langle i, k\rangle \notin D_{f(n)}]].$$

By (4-3) f is total. Define a recursive h by

$$x \in D_{h(n)} \Leftrightarrow \exists y\, [\langle x, y\rangle \in D_{g(n)}],$$

each x, n. Verify that h witnesses that A is not hyperimmune, a contradiction. □

However, in contrast to the situation in the Turing degrees:

Theorem 4.15 (Rozinas [Roz78b]). *There exist immune-free e-degrees (that is, degrees containing no immune sets).*

Proof. Inductively define approximations A^s, B^s to sets A, B (A^s r.e., B^s finite) satisfying for each s

$$R_s: \quad A^s \cap B^s = \phi,\ A^s, B^s \subset A^{s+1}, B^{s+1} \text{ respectively, and } \bar{A}^s$$
$$\text{contains an infinite r.e. subset } C_s, \text{ and}$$
$$S_s: \quad \Psi_s(\omega - B^s) \text{ is finite, or } \Psi_s(A^s) \text{ contains an infinite r.e. set.}$$

Assume also some steps to make $A \neq W_s$. At stage $s+1$, we look for a finite $D \supset B^s$ with $A^s \cap D = \phi$ and $\Psi_s(\bar{D})$ finite. If such a D exists, defining $B^{s+1} = D$ satisfies S_s.

Otherwise, we are able to inductively define at substages $t \geq 0$ finite extensions $P^t \supset P^t \supset \ldots \supset A^s$, $C^t_{s+1} \supset C^{t-1}_{s+1} \supset \ldots \supset B^s$ (using $C^t_s \supset C^{t-1}_s \supset \ldots \supset B^s$), satisfying

(1) $P^t \cap C^t_{s+1} = \phi$,
(2) $\Psi_s(P^t) \supset \Psi_s(P^{t-1})$ (using the fact that no such D exists), and
(3) $P = \cup_{t \geq 0} P^t$ is r.e.

In this case, defining $A^{s+1} = P$ satisfies S_s. □

It is clear from the above proof that there exist immune-free e-degrees below $\mathbf{0}''$ (while in section 6 below we see that this result is best possible in that below $\mathbf{0}'$ the situation is simpler). See Solon [Sol78] for properties of e-degrees containing hyperimmune retraceable sets. The main gap in our knowledge concerns possible analogues of the Jockusch [Jo73] upward-closure results for the cohesive Turing degrees.

Question 4.16. Are the e-degrees of cohesive sets (and of hyperhyperimmune sets) closed upwards in $\boldsymbol{\mathcal{D}}_e$?

It is not clear yet what role immunity properties play in the theory of the e-degrees, and to what extent such properties naturally relate to structure and jump.

§5. Density in the enumeration degrees

As we saw above, in the context of the e-degrees the Spector minimal degree construction gives a degree which is minimal-like rather than minimal. In fact, Gutteridge [Gu71] was able to show that no minimal e-degrees exist. The proof falls into two parts. The first part, which relativises to show that any e-degree has at most countably many minimal covers, is a demonstration that any set of minimal e-degree must be Δ^0_2. The second is a very different proof, which only relativises above total degrees, of the fact that no Δ^0_2 set is of minimal e-degree.

Theorem 5.1 (Gutteridge [Gu71]). *If* $\deg_e(B)$ *is a minimal cover for* $\deg_e(A)$ *then* $B \in \Delta_2^A$.

Proof. Enumerate an e-operator Θ such that for *any* sets A, B

$$[\Theta^B \leq_e A \text{ or } B \equiv_e A \oplus \Theta^B] \Rightarrow B \in \Delta_2^A.$$

Then given $A <_e B$ we will have $B \in \Delta_2^A$ or $A <_e A \oplus \Theta^B <_e B$, which will prove the theorem. The requirements to satisfy are

$$N_i : \quad B = \Psi_i(A \oplus \Theta^B) \Rightarrow B \leq_e A,$$
$$P_i : \quad \Theta^B = \Psi_i^A \Rightarrow B \in \Delta_2^A,$$

each $i \geq 0$. We recursively enumerate Θ at stages $s \geq 0$ by a combination of ticking and crossing, where to *tick* $\langle n, x\rangle$ (via Θ) means to enumerate $\langle \langle n, x\rangle, \{n\}\rangle$ into Θ, and to *cross* $\langle n, x\rangle$ (via Θ) means to enumerate $\langle \langle n, x\rangle, \phi\rangle$ into Θ. The strategy for P_i is to tick an initial segment of the x^{th} column up to some $\langle n_x, x\rangle$ (each x) in such a way that $\langle n_x, x\rangle$ never gets crossed. Hence if $\Theta^B = \Psi_i^A$ we have

$$x \in B \Leftrightarrow \langle n_x, x\rangle \in \Theta^B \Leftrightarrow \langle n_x, x\rangle \in \Psi_i^A$$

where $\lambda x\,[n_x] \leq_T \phi'$, so $B \in \Delta_2^A$. The strategy for N_i is to ensure that if $x \in \Psi_i(A \oplus \Theta^B)$ then $x \in \Psi_i(A \oplus \Theta^{B \restriction x})$ modulo a finite number of pairs $\langle n, y\rangle$ ticked through $y < i$. This is achieved by crossing numbers $\langle n, y\rangle \in D - [0, x-1]$, $y \geq i$, for some $D \subseteq \Theta^B$ with $x \in \Psi_i(A \oplus D)$, if we get $x \in \Psi_i(A \oplus \Theta^B)$ at some stage. This means that, if we are given an enumeration of A and have $B = \Psi_i(A \oplus \Theta^B)$, then we can inductively get $x \in B$ enumerated in B from an enumeration of $B \restriction x$. The P_i-strategy is allowed to succeed because we only allow crossing of $\langle n, x\rangle$ through N_j if $j \leq x$, and only finitely many numbers are crossed through any given N_j. □

Corollary 5.2 (Gutteridge [Gu71]). *If* $\mathbf{b}$ *is a minimal cover for* $\mathbf{a}$ *then* $\mathbf{b} \leq \mathbf{a}'$. *Hence any e-degree has at most countably many minimal covers.* □

Theorem 5.3 (Gutteridge [Gu71]). *If* $B \in \Delta_2^0$ *is not r.e. then there is an e-operator* Θ *such that* $\phi <_e \Theta^B <_e B$. *Hence there is no minimal e-degree.*

Proof. Let $\{B^s\}_{s\in\omega}$ be a recursive sequence of finite approximations to $B \in \Delta_2^0 - \Sigma_1^0$. We again enumerate Θ by ticking and crossing at stages $s \geq 0$. We satisfy the requirements

$$N_i : \quad B \neq \Psi_i(\Theta^B),$$
$$P_i : \quad \Theta^B \neq W_i.$$

Attend to P_i by monitoring the initial segment of agreement $L(i, s)$ between W_i and Θ^B at stage $s+1$, and as agreement grows tick as yet unmarked numbers $\langle j, i\rangle < L(i, s)$ at stages $s + 1 > i$. Attend to N_i by monitoring the initial segment of agreement $\ell(i, s)$ between B and $\Psi_i(\Theta^B)$ at stage $s + 1$, and try to make the segment of agreement independent of the approximation B^s by crossing numbers $\langle j, i'\rangle \in \Theta^B$

used by $\Psi_i(\Theta^B) \upharpoonright \ell(i,s)$ for which $i' > i$. Inductively verify that the construction is finite injury and each requirement is satisfied. P_i is satisfied, since otherwise $\langle j,i\rangle \in W_i = \Theta^B \Leftrightarrow j \in B$ for all but a finite number of j's, giving B r.e., a contradiction. N_i is satisfied, since otherwise all but a finite set of numbers $\langle j,i'\rangle$ (with $i' \leq i$) are crossed, giving Θ^B r.e. Then use $\langle x,D\rangle \in \Theta \Rightarrow \mid D \mid \leq 1$ to deduce $B = \Psi_i(\Theta^B)$ from the unboundedness of $\ell(i,s)$, $s \geq 0$, again contradicting B not r.e. □

Corollary 5.4 (Gutteridge [Gu71],Lagemann [Lag71]). *If either* **a** *or* **b** *is total, then* **b** *is not a minimal cover for* **a**.

Proof. By Corollary 5.2, we need only show that if $A <_e B$ and either (1) A is total with $B \leq_T A'$, or (2) B is total, then there is an e-operator Θ with $A <_e \Theta^B \oplus A <_e B$.

For (1), just relativise the above proof to get Θ r.e. in A, so $\Theta^B = \Psi_{\Phi(A)}(B)$ for some e-operator Φ, $= \widetilde{\Theta}^B$, say, for some e-operator $\widetilde{\Theta}$, since $A \leq_e B$.

(2) Let Ψ be an e-operator with $A = \Psi^B$. If $B = \{\langle x, b(x)\rangle \mid x \in \omega\}$ where b is total, let $B^s = \{\langle x, b(x)\rangle \mid x < s\}$. Take as requirements: $W_0^{\Psi(B)} \neq \Theta^B$, $B \neq \Psi_0(\Theta^B \oplus \Psi^B), \ldots$, etc., and proceed as in the theorem to get Θ r.e. in B. Then, as in part (1), $\Theta^B = \widetilde{\Theta}^B$ for some e-operator $\widetilde{\Theta}$. □

Having eliminated the possibility of minimal e-degrees (so that $\boldsymbol{\mathcal{D}}_e$ is not elementarily equivalent to most of the standard degree structures, including the Sasso T-degrees), we can now examine the extent to which the above techniques can be extended to characterise the possible initial segments of $\boldsymbol{\mathcal{D}}_e$.

Let $B \in \Delta^0_2$. By combining the ticking and crossing techniques for the P_i- and N_i-strategies (respectively) in Theorem 5.3, Lagemann showed that e-operators Θ_j, Θ_k can be enumerated satisfying incomparability requirements of the form $\Theta_j^B \neq \Psi_i(\Theta_k^B)$. Hence (cf. Theorem 4.5, part (5)):

Corollary 5.5 (Lagemann [Lag71]). *If* **b** *is a non-zero* Δ^0_2 *e-degree, then any r.e. partial ordering can be embedded below* **b**. □

This result relativises (McEvoy [McE84]) along the lines of Corollary 5.4. For further details of the above proofs see [Co82], [MC85] and [Od89].

The total-ness requirements in Corollary 5.4 and the Δ^0_2 restriction in Corollary 5.5 are necessary, as we will see in the following sections. However, below $\mathbf{0}'$ we we can get:

Theorem 5.6 (Cooper [84]). *The structure of the e-degrees below* $\mathbf{0}'$ *is dense*

Proof. Given Σ^0_2 sets A, B where $B \not\leq_e A$, build an e-operator Θ with $A <_e \Theta^{B\oplus A} \oplus A <_e B \oplus A$. Choose Σ_2 approximations $\{A^s\}_{s\in\omega}, \{B^s\}_{s\in\omega}$ to A, B so that $\{A^s \oplus B^s\}_{s\in\omega}$ is a *thin* Σ_2 approximation to $A \oplus B$ (that is, there are infinitely many *stages* s *- called thin* stages - at which $A^s \oplus B^s \subseteq A \oplus B$). The requirements to satisfy are:

$$N_i: \quad B \neq \Psi_i(\Theta^{B\oplus A} \oplus A),$$
$$P_i: \quad \Theta^{B\oplus A} \neq \Psi_i^A,$$

each $i \geq 0$. Monitor N_i, P_i with respective length (of agreement) functions $\ell(i,s)$ and $L(i,s)$. The thin stages ensure that despite our approximations to A, B not being Δ_2, we still have infinitely many stages at which $\ell(i,s)$ and $L(i,s)$ approximate a true eventual outcome for N_i and P_i respectively, and at which our N_i- and P_i-strategies are effectively directed towards satisfying their corresponding requirements (at thin stages we are on the 'true path' of an implicit tree of outcomes).

Attend to P_i at stage $s+1$ by enumerating $\langle\langle x,i\rangle, B^s \oplus A^s\rangle$ into Θ for each $x \in B^s$ with $x \leq L(i,s)$. Attend to N_i at stage $s+1$ by enumerating $\langle z, B^s \restriction \delta \oplus A^s\rangle$ into Θ for each $z = \langle x,j\rangle$ (say) $\in \Theta(B^s \oplus A^s)$, with $j > i$, which is selected to be used in enumerating $\Psi_i(\Theta(B^s \oplus A^s) \oplus A^s) \restriction \ell(i,s)$ at stage $s+1$, where δ is chosen to be the least number we can take without interfering with axioms enumerated in Θ via higher priority requirements N_j, $j < i$. The verification is similar to that of Theorem 5.3 but restricted to the thin stages. Inductively check that we attend to an N_i or P_i at at most finitely many thin stages, and that each requirement is satisfied.

If we attend to P_i infinitely often we get $\Psi_i^A = \Theta^{B\oplus A}$ through the numbers enumerated in Θ at thin stages. Then P_i is satisfied, since otherwise $\langle j,i\rangle \in \Psi_i^A = \Theta^{B\oplus A} \Leftrightarrow j \in B$ for all but a finite number of j's, giving $B \leq_e A$, a contradiction. By looking at thin stages again, if we attend infinitely often to N_i we get $B = \Psi_i(\Theta^{B\oplus A} \oplus A)$. Then N_i is satisfied, since otherwise all but a finite set of numbers $z \in \Theta(B \oplus A)$ selected to be used by $\Psi_i(\Theta^{B\oplus A} \oplus A)$ are made independent of all but a finite part of B, giving $B = \Psi_i(\Theta^{B\oplus A} \oplus A) \leq_e A$, a contradiction. Even though requirements may receive attention at infinitely many non-thin stages, the axioms for Θ so defined will not injure lower priority requirements. □

We look at the structure of the Σ_2^0 e-degrees in more detail in section 7.

§6. Techniques for proving non-density

Whereas A-partial recursive trees (due to Shoenfield [Sh66]) provide the framework for relativisations of the Spector minimal degree construction, for the construction of a minimal cover in the e-degrees we need the following sequence of definitions (see [Co87] and [Co*ta*2]). Let S denote the set $\{0,1\}^{<\omega}$ of all binary strings.

Definition 6.1. A *celling* $\mathcal{C}$ is a partial function $\lambda\alpha C_\alpha$ from binary strings to sets of numbers (where the C_α with $C_\alpha\downarrow$ are the *cells* of $\mathcal{C}$), satisfying for all $\alpha, \beta \in S$ with $C_\alpha, C_\beta\downarrow$:
(a) $\alpha \subseteq \beta$ implies $C_\alpha \subseteq C_\beta$, and (b) If η is a set of strings and $C_\alpha \subseteq \bigcup_{\beta\in\eta} C_\beta$ then $\alpha \subseteq \beta$ some $\beta \in \eta$.

Definition 6.2. The α-*increment* in $\mathcal{C}$ is defined by $I_\phi = C_\phi$ and $I_\alpha = C_\alpha \setminus C_{\alpha^-}$ if $\alpha = (\alpha^-)\hat{\ }i$ for some $i \leq 1$. We write $C_\alpha \in \mathcal{C}$ iff C_α is a cell of $\mathcal{C}$ and $I_\alpha \in \mathcal{I}$ iff I_α is an increment of $\mathcal{C}$. Sometimes we identify $\mathcal{C}$ $(\mathcal{I})$ with $\{\langle x,\alpha\rangle \mid x \in C_\alpha,\ \alpha \in S\}$ $(\{\langle x,\alpha\rangle \mid x \in I_\alpha,\ \alpha \in S\})$.

Definition 6.3. If $C_\alpha \in \mathcal{C}$, we say that C_α *is in* $\mathcal{C}$. If $\alpha \subseteq \beta$, we say that C_β *includes* C_α (in $\mathcal{C}$). $\mathcal{C}$ is *closed* iff for each C_α in $\mathcal{C}$ and each $\beta \subset \alpha$ we have a cell $C_\beta\downarrow$ in $\mathcal{C}$.

Definition 6.4. A set A *weakly respects* the celling $\mathcal{C}$ iff $A = \bigcup_{\alpha\in\xi} C_\alpha$ for some set $\xi \subseteq S$. A *respects* $\mathcal{C}$ iff ξ is a chain in S.

Definition 6.5. We say that $\{C_{\alpha_0}, C_{\alpha_1}, \ldots\}$ is a *skeleton* for a set X of cells of $\mathcal{C}$ iff
(a) $X \subseteq \{C_\beta \mid \ \beta, \alpha_i$ are comparable for some $i = 0, 1, ...\}$,
(b) If $i \neq j$ then $\alpha_i \mid \alpha_j$, and
(c) For each $i = 0, 1, \ldots$ we have $X \cap \{C_\beta \mid \ \beta \supseteq \alpha_i\} \neq \phi$.
We say that the skeleton $\{C_{\alpha_0}, C_{\alpha_1}, \ldots\}$ *omits* A iff $A \mid C_{\alpha_i}$ for each $i = 0, 1, \ldots$. If $\{C_{\alpha_0}\}$ is a skeleton, we say that $\{C_{\alpha_0}\}$ is a *principal skeleton*.

Definition 6.6. We say that C_α is an *outside cell* iff $C_\alpha\downarrow$ but $C_\beta\uparrow$ for each $\beta \supset \alpha$. We say that C_α is a *terminal cell* iff $C_\alpha\downarrow$ and C_α is omitted by a skeleton for $\{C_\beta \mid \ lh(\beta) \geq n\}$ for some $n \geq 0$. (That is, C_α is terminal iff $C_\alpha\downarrow$ and there is some $n \geq 0$ such that for each C_β which includes C_α we have $lh(\beta) < n$.)

Definition 6.7. The *restriction* $\mathcal{C}[\zeta]$ of the celling $\mathcal{C}$ to a set of strings ζ is defined by

$$C[\zeta]_\beta = \begin{cases} C_\beta & \text{if } C_\beta\downarrow \text{ and } \beta \in \zeta \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We write $C[\beta]$ for $\mathcal{C}[\{\alpha \mid \beta \subseteq \alpha\}]$.

Definition 6.8. The *$\mathcal{C}$-enumerating set of D for i following ζ, W* (or, just the *enumerating set of D following ζ, W*), written $ES^{\mathcal{C},i,\zeta,W}(D)$ (or just $ES^{\zeta,W}(D)$), is defined by

$$ES^{\mathcal{C},i,\zeta,W}(D) = \{C_\beta \mid \ \beta \in \zeta \ \& \ D \subseteq \Psi_i^{W\cup C_\beta}\}.$$

We write $ES^{\alpha,W}(D)$ for $ES^{\{\beta \ \mid \ \alpha\subseteq\beta\},W}(D)$, $ES^{\alpha,W}(x)$ for $ES^{\alpha,W}(\{x\})$, $ES^{\alpha}(D)$ for $ES^{\alpha,\phi}(D)$ and $ES(D)$ for $ES^{\phi,\phi}(D)$.
More generally, we define the *$\mathcal{C}$-weak enumerating set of D for i following ζ, W* by

$$WES^{\mathcal{C},i,\zeta,W}(D) = \{\{C_{\beta_0}, C_{\beta_1}, \ldots, C_{\beta_m}\} \in [\mathcal{C}]^{<\omega} \mid \\ \{\beta_0, \beta_1, \ldots, \beta_m\} \subseteq \zeta \,\&\, D \subseteq \Psi_i(W \cup C_{\beta_0} \cup C_{\beta_1} \cup \ldots \cup C_{\beta_m})\}.$$

Definition 6.9. We say that $\{C_{\alpha_0}, C_{\alpha_1}, \ldots\}$ is an *i-skeleton* for D (beyond ζ, W) iff $\{C_{\alpha_0}, C_{\alpha_1}, \ldots\}$ is a skeleton for $ES(D)$ ($ES^{\zeta,W}(D)$ respectively).

Definition 6.10. We say that C_α is *i, D-distinguished* (beyond ζ, W) iff $D \subseteq \Psi_i^{C_\alpha}$ ($D \subseteq \Psi_i^{W\cup C_\alpha}$, respectively) and $\{C_\alpha\}$ is a principal i-skeleton for D (beyond ζ, W).

Definition 6.11. We say that C_α is *strongly i, D-distinguished* (*beyond ζ, W*) iff C_α is i, D-distinguished (beyond ζ, W respectively) and for each A weakly respecting $\mathcal{C}$ ($\mathcal{C}[\zeta]$) we have $D \subseteq \Psi_i^A \Leftrightarrow C_\alpha \subseteq A$ ($D \subseteq \Psi_i^{W\cup A} \Leftrightarrow C_\alpha \subseteq A$).

DEFINITION 6.12. $\mathcal{C}$ is (*strongly*) *i-distinguished with A* (weakly, respectively) respecting $\mathcal{C}$ iff there is a function $f : dom\mathcal{C} \to [\omega]^{<\omega}$, called a (*strong*, respectively) *i-labelling* of $\mathcal{C}$ *with* A, and a cofinite set $\zeta \subseteq S$, such that for each $\alpha \in \zeta$ we have that

$$C_\alpha \subseteq A \Leftrightarrow \left[(\forall \beta \subset \alpha) C_\beta \subseteq A \& f(\alpha) \subseteq \Psi_i\Big(C_\alpha \cup \Big(A \cap \Big(\bigcup_{\beta \notin \zeta} C_\beta\Big)\Big)\Big)\right].$$

DEFINITION 6.13. Let A be a set respecting $\mathcal{C}$. Then A is *i-undistinguished* in $\mathcal{C}$ iff A is omitted by no nonempty i-skeleton in $\mathcal{C}$. (This corrects Definition 3.16 of [Co87].)

DEFINITION 6.14. A *maximally i-enumerates over* $\mathcal{C}$ ($\mathcal{C}[\zeta]$, some $\zeta \subseteq S$) iff A weakly respects $\mathcal{C}$ and $\Psi_i\left(\bigcup_{\alpha \in S} C_\alpha\right) = \Psi_i^A$ ($\Psi_i\left(A \cup \bigcup_{\alpha \in \zeta} C_\alpha\right) \subseteq \Psi_i^A$, respectively).

DEFINITION 6.15. We say $f : \omega \times dom\mathcal{C} \to [\omega]^{<\omega}$ is a *strong labelling* (or just *labelling*) *of* $\mathcal{C}$ *with* A (weakly respecting $\mathcal{C}$) iff for each $i \geq 0$ there is some cofinite $\zeta \subseteq S$ such that either:
(i) A maximally i-enumerates over $\mathcal{C}[\zeta]$, or
(ii) $\lambda\alpha f(i,\alpha)$ is a strong i-labelling of $\mathcal{C}$ (with corresponding set ζ in Definition 6.12).

We note that when discussing the *degree* of $\mathcal{C}$, or placing $\mathcal{C}$ in the arithmetical hierarchy, we refer to the enumeration degree or quantifier form of the set $\{\langle x, \alpha\rangle \mid x \in C_\alpha\}$. We identify a labelling f with the set $\{\langle D, \alpha, i\rangle \mid D = f_i(\alpha)\}$.

The basic lemmas corresponding to the two parts of the usual Computation Lemma are:

Lemma 6.16. *Let $\mathcal{C}$ be a strongly i-distinguished celling with A with strong i-labelling f. Let A be a set weakly respecting the celling $\mathcal{C}$. Then $A \leq_e f \times \mathcal{C} \times \Psi_i^A$.*

Proof. It is straightforward to verify that the following describes an algorithm for enumerating A from enumerations of Ψ_i^A, f and $\mathcal{C}$: Start by setting up an enumeration of Ψ_i^A, $\{(x,\alpha) \mid x \in C_\alpha, \alpha \in S\}$ and the graph of f. Say $(\alpha, f(\alpha))$ is enumerated in the graph of f and the members of $f(\alpha)$ are enumerated in Ψ_i^A. Then enumerate the members of C_α into A. □

Lemma 6.17. *Let A be a set weakly respecting the celling $\mathcal{C}$, and let A maximally i-enumerate over $\mathcal{C}$. Then $\Psi_i^A \leq_e \mathcal{C}$.*

Proof. If A satisfies the conditions of the Lemma then $\Psi_i^A = \Psi_i(\cup_{\alpha \in S} C_\alpha)$, giving $\Psi_i^A \leq_e \cup_{\alpha \in S} C_\alpha \leq_e \mathcal{C}$. □

We include in this section a definition from [Co87] which will be useful in the discussion below:

Definition 6.18. For each $i, x \geq 0$

$$\begin{aligned} \varepsilon_x^i &= \{D \in [\omega]^{<\omega} \mid \langle x, D\rangle \in \Psi_i\}, \\ \uparrow \varepsilon_x^i &= \{K \in [\omega]^{<\omega} \mid (\forall D \in \varepsilon_x^i)(K \cap D \neq \phi)\}, \\ \downarrow \varepsilon_x^i &= \{W \in [\omega]^{<\omega} \mid (\forall K \in \uparrow \varepsilon_x^i)(W \cap K \neq \phi)\}. \end{aligned}$$

We often write ε_x for ε_x^i. More generally, given a pair W, K of finite sets of numbers, we define

$$\begin{aligned} \varepsilon_x^i(W,K) &= \{D \in [\omega]^{<\omega} \mid (D \cap K = \phi) \,\&\, (x \in \Psi_i^{D \cup W})\}, \\ \uparrow \varepsilon_x^i(W,K) &= \{K' \in [\omega]^{<\omega} \mid K' \cap W = \phi \,\&\, K' \cup K \in \uparrow \varepsilon_x^i\}, \\ \downarrow \varepsilon_x^i(W,K) &= \{W' \in [\omega]^{<\omega} \mid W' \cap K = \phi \,\&\, W' \cup W \in \downarrow \varepsilon_x^i\}. \end{aligned}$$

As expected, the existence of minimal covers can now be reduced to the problem of finding appropriate cellings satisfying Lemmas 6.16 or 6.17. But unlike the analagous use of splitting and anti-splitting trees for getting minimal total degrees it is not possible to nest distinct cellings corresponding to individual requirements.

Theorem 6.19 (Cooper [Co*ta*2]). *There is a celling $\mathcal{C}$ with labelling f, and a set A weakly respecting $\mathcal{C}$, such that*

(i) $f, \mathcal{C} \leq_e A$,

(ii) $A \not\leq_e f \oplus \mathcal{C}$, *and*

(iii) For each $i \geq 0$, *either* $\Psi_i^A \leq_e \mathcal{C}$ *or* $A \leq_e \Psi_i^A \oplus f \oplus \mathcal{C}$.

Corollary 6.20. *The enumeration degrees are not dense.*

Proof. Take $\mathbf{a} = deg_e(A)$, $\mathbf{b} = deg_e(f \oplus \mathcal{C})$ in Theorem 6.19. □

Proof of Theorem 6.19. Construct a triple $\mathcal{C}, f, P$, where $\mathcal{C}$ is a celling with labelling f, and a set A weakly respecting $\mathcal{C}$ satisfying the requirements

$\mathcal{R}_i$: $A \neq \Psi_i^{f \oplus \mathcal{C}}$

$\mathcal{S}_i$: $\exists \alpha^* \in dom(\mathcal{C})$ such that either A maximally i-enumerates over $\mathcal{C}[\alpha^*]$ or $\mathcal{C}[\alpha^*]$ is a strongly i-distinguished celling with A with strong i-labelling $\lambda\alpha\, f(i,\alpha)$ (written f_i),

where P is the *prohibiting function* $dom(\mathcal{C}) \to [\omega]^{<\omega} \cup \{\omega\}$ concerned with policing the construction of $\mathcal{C}$ on behalf of a successful outcome to the definition of the labelling f.

Definition 6.21. We say $\mathcal{D}, g, Q$ is a *finite triple* iff $\mathcal{D}$ is a closed finite celling, $g : \omega \times dom(\mathcal{D}) \to [\omega]^{<\omega}$ and $Q : dom(\mathcal{D}) \to [\omega]^{<\omega} \cup \{\omega\}$, satisfying:

(a) $dom(\mathcal{D}) = dom(Q)$ and $(\forall \alpha, \beta \in dom(\mathcal{D}))(\alpha \subset \beta \Rightarrow D_\beta \cap Q(\alpha) = \phi)$, and

(b) $(\forall i \geq 0)(dom(\mathcal{D}) \supseteq dom(g_i))$ and $(\forall \alpha, \beta \in dom(\mathcal{D}))(\alpha \subseteq \beta \Rightarrow (g_i(\alpha) \downarrow \Rightarrow g_i(\beta) \downarrow))$, and

(c) $(\forall \alpha, \beta \in dom(\mathcal{D}))(\alpha \subseteq \beta \Rightarrow Q(\alpha) \subseteq Q(\beta))$.

Definition 6.22. If $\mathcal{D}, g, Q$ is a finite triple and D_β (say) $\subseteq$ some boundary cell $D_\alpha \in \mathcal{D}$ with $Q(\alpha) \neq \omega$, we say that D_β is *potentially nonterminal*, or just *nonterminal* when there is no ambiguity.

We say that $\mathcal{D}, g, Q$ is *potentially nonterminal*, or just *nonterminal*, (*beyond* ζ) iff (for each $\alpha \in \zeta$, respectively) there is some potentially nonterminal cell $D_\beta \in \mathcal{D}$ ($\in \mathcal{D}[\alpha]$, respectively).

We say that $\mathcal{D}, g, Q$ is *uniquely nonterminal* (*beyond* ζ) iff (for each $\alpha \in \zeta$, respectively) there is exactly one such boundary cell $D_\beta \in \mathcal{D}$.

Definition 6.23. We say $\mathcal{C}', f', P'$ is a *finite extension* of $\mathcal{D}, g, Q$, and write $\mathcal{C}', f', P' \succ \mathcal{D}, g, Q$ iff $\mathcal{C}', f', P'$ is a finite triple with $\mathcal{C}'$, f' and P' extensions of $\mathcal{D}$, g, Q respectively.

During the construction construct an infinite nest (with respect to $\succ$) of finite triples whose union will be the required triple $\mathcal{C}, f, P$. In order to get $f, \mathcal{C} \leq_e A$, progressively code f and $\mathcal{C}$ into every infinite A respecting $\mathcal{C}$ using a uniformly recursive family $\{\xi(x, \alpha, E, i) \mid i, x \geq 0,\, \alpha \in S,\, E \in [\omega]^{<\omega}\}$ of disjoint infinite sets $\xi(x, \alpha, E, i) \subset \omega$. The aim will be to have for each infinite A weakly respecting $\mathcal{C}$, and each (x, α, E, i) that

$$A \cap \xi(x, \alpha, E, i) \neq \phi \Leftrightarrow x \in C_\alpha \,\&\, E = f_i(\alpha).$$

Assume as usual that each $z \in \xi(x, \alpha, E, i)$ is greater than $x, i, max\{y \in E\}$ and $lh(\alpha)$.

Definition 6.24. We say that $\mathcal{C}', f', P'$ is *alright codewise* iff for each A respecting $\mathcal{C}'$ we have

$$A \cap \xi(x, \alpha, E, i) \neq \phi \Rightarrow x \in C'_\alpha \,\&\, E = f'_i(\alpha).$$

Since the $\mathcal{R}_i$-strategy (a straightforward diagonalisation) requires the building into the construction of a certain amount of choice in defining A weakly respecting $\mathcal{C}$, the coding process presents difficulties for the $\mathcal{S}_i$-strategy. When constructing at stage $s+1$, say, certain extensions $C_{\beta\hat{}\gamma_0}, C_{\beta\hat{}\gamma_1}$ for C_β, we will want to do this as far as possible so as to have $C_{\beta\hat{}\gamma_j}$ strongly $i, \widetilde{f}_i(\beta\hat{}\gamma_j)$-distinguished with A for each $j \leq 1$, each $i \leq s$. But numbers in an appropriate $I_{\beta\hat{}\gamma_j}$ may be codes z for quadruples (x, α, E, i) with C_α as yet undefined. As long as $x \notin P(\beta)$, this will not in general prevent the definition of $I_{\beta\hat{}\gamma_j}$ with $z \in I_{\beta\hat{}\gamma_j}$. Merely extend the (so far defined part of) $\mathcal{C}, f, P$ (call it $\widehat{\mathcal{C}}, \widehat{f}, \widehat{P}$) to an alright codewise finite extension $\mathcal{C}', f', P'$ with, in particular, $z \in I'_{\beta\hat{}\gamma_j}$, $x \in C'_\alpha$ and $E = f'_i$.

The problem then is that no precautions can be taken to make the whole of $\mathcal{C}', f', P'$ suitably strongly i-distinguished. This means we only want $\mathcal{C}', f', P'$ to give us, in effect, the extensions $C_{\beta\hat{}\gamma_0}, C_{\beta\hat{}\gamma_1}$. To do this ensure during the rest of the construction that A is only allowed a nontrivial choice (in satisfying the $\mathcal{R}_i$ requirements) between designated *unique* nonterminal boundary cells C_{α_j} beyond $C_{\beta\hat{}\gamma_j}$, $j \leq 1$. This means that the strong $i, \widetilde{f}_i(\beta\hat{}\gamma_j)$-distinguishing of each $C_{\beta\hat{}\gamma_j}$, $j \leq 1$, will be sufficient to get us to the next pair of designated extensions beyond C_β, proceeding within the A which weakly respects $\mathcal{C}$, given an enumeration of $\mathcal{C}, f_i$ and Ψ_i^A.

However, in order to obtain a detailed enumeration of exactly how much of $\cup\{C'_\alpha \mid \alpha \in dom(\mathcal{C}') - dom(\widehat{\mathcal{C}})\}$ is in A, some care needs to be taken in defining the labelling over the parts of $\mathcal{C}' - \widehat{\mathcal{C}}$ which we are forced to include because they are coded by some $z \in C_{\beta\hat{\ }\gamma_j}$, some $j \leq 1$. Care also needs to be taken in defining $A \cap [\cup\{C'_\alpha \mid \alpha \in dom(\mathcal{C}') - dom(\widehat{\mathcal{C}})\}]$.

Achieve the former by asking that the labelling f satisfies (roughly speaking) $f_i(\alpha) \subseteq \Psi_i^{C_\alpha}$ for each $\alpha \in dom(f_i)$. Then satisfy the latter need by defining some $C_{\beta\hat{\ }\gamma_j} \subset A$, and then taking (again roughly speaking)

$$A \cap \left\{\cup_{\alpha \in dom(\mathcal{C}') - dom(\widehat{\mathcal{C}})} C'_\alpha\right\} = \cup_{\alpha \notin \mathcal{C}'[\beta\hat{\ }\gamma_{(1-j)}]} C'_\alpha.$$

There is another problem, arising from the way in which the $\mathcal{S}_i$-requirements need to be satisfied when there are not sufficient strong i, D-distinguishings available. In this case one cannot use i-undistinguished cellings (as for the compact reducibilities discussed in section 8 below), but must construct A to be maximally i-enumerating.

In this situation there are many α's with $\cup_{\alpha' \supseteq \alpha} C_{\alpha'}$ maximally i-enumerating as far as those numbers x for which $\uparrow \varepsilon_x^i(C_\beta, P_\beta) \neq \phi$ goes. Deal with the other numbers by including in the construction some simple actions to ensure that (again roughly speaking) modulo the coding constraints if $\uparrow \varepsilon_x^i(C_\beta, P_\beta) = \phi$ then for some $B \subseteq A$ weakly respecting $\mathcal{C}$ we have $x \in \Psi_i^B$. Then, choosing such an α which does not interfere with the steps to satisfy the $\mathcal{R}_i$-requirements, obtain A maximally i-enumerating by making $\cup_{\alpha' \supseteq \alpha} C_{\alpha'} \subset A$. To prevent such interference, it is necessary to include precautions to ensure that if $C_{\beta\hat{\ }\gamma_j} \subset A$ is chosen, as above, then no B respecting $\mathcal{C}[\beta\hat{\ }\gamma_{(1-j)}]$ can be maximally i-enumerating over $\mathcal{C}$. Having done this, take $\cup_{\alpha' \supseteq \alpha} C_{\alpha'} \subset A$ for *all* such α's (in a trivial way).

A problem then is that the weak respecting by A of $\mathcal{C}$ becomes nontrivial. And that means that in defining strongly i'-distinguished extensions for C_β, say, in $\mathcal{C}$ $(i' \neq i)$ it is necessary to know about $A \cap \{\cup_{\alpha \not\subseteq \alpha'} C_{\alpha'}\}$. This requires that A is constructed at the same time as $\mathcal{C}, f, P$ is constructed, using the knowledge of that part of $A \cap \{\cup_{\alpha \not\subseteq \alpha'} C_{\alpha'}\}$ which has already been defined in defining extensions for C_β, and then using the prohibiting function P to protect the strong i'-distinguished extensions from injury in defining the rest of $A \cap \{\cup_{\alpha \not\subseteq \alpha'} C_{\alpha'}\}$. (We notice that that the interdependence of $\mathcal{C}$ and A during the construction is the main impediment to constructing a continuum of minimal covers of the form $deg_e(A)$ for $deg_e(\mathcal{C} \oplus f)$.)

To summarise: Assume that by stage $s+1$ of the construction a finite triple $\mathcal{C}^s, f^s, P^s$ and a finite set A^s weakly respecting $\mathcal{C}^s$ have been defined . Also assume given a boundary cell $C^s_{\beta^*} \subseteq A^s$ for $\mathcal{C}^s$ which has been designated A^{s+1}-*burgeoning* at the end of stage s. Then:

(1) First examine what freedom the actions on requirements $\mathcal{S}_i$, $i < s$, have given for satisfying $\mathcal{S}_i$ through the first part of $\mathcal{S}_i$.
(2) If there is no such freedom, decide best how to make A maximally s-enumerate while still providing room to satisfy lower priority requirements.
(3) Mop up numbers x we want to be in Ψ_i^A, $i \leq s$.
(4) Look for appropriately distinguished extensions of the A^{s+1}-burgeoning cell C^s_β within the context of a finite triple $\mathcal{C}^{s+1}, f^{s+1}, P^{s+1} \succ \mathcal{C}^s, f^s, P^s$. The main

concern here is the satisfaction of the $\mathcal{S}_i$-requirements for $i \leq s$ in such a way as to allow the satisfying of the requirement $\mathcal{R}_s$ when A^{s+1} is defined.

(5) Take advantage of the choice of distinguished cellings to satisfy $\mathcal{R}_s$.

(6) Code up most of what has been defined in $\mathcal{C}^{s+1}, f^{s+1}$ within A^{s+1} while choosing A^{s+1} to weakly respect $\mathcal{C}^{s+1}$.

To complete the proof verify that: (a) A is an infinite set weakly respecting the celling $\mathcal{C}$, (b) $f \oplus \mathcal{C} \leq_e A$ (so the coding works), (c) for each $i \geq 0$ there is a $\mathcal{C}[\alpha(i)]$ such that either A maximally i-enumerates over $\mathcal{C}[\alpha(i)]$ or $\mathcal{C}[\alpha(i)]$ is a strongly i-distinguished celling with A with a strong i-labelling f_i (=that part of f relating to Ψ_i), so that each $\mathcal{S}_i$ is satisfied, and (d) $\mathcal{R}_i$ is satisfied for all $i \geq 0$. □

A direct calculation of the quantifier form of A in Theorem 6.19 yields a Σ_7^0 relation, so that A is r.e. in $\phi^{(6)}$, giving $A \leq_e \phi^{(6)}$. Hence:

Corollary 6.24. *The enumeration degrees below* $\mathbf{0}^{(6)}$ *are not dense.* □

The gap between Theorem 5.6 and Corollary 6.24 leads to some obvious but important questions:

Question 6.25. What is the smallest n, $2 \leq n \leq 6$, for which we can prove nondensity of $\boldsymbol{\mathcal{D}}_e(\leq \mathbf{0}^{(n)})$?

Question 6.26. Are the enumeration degrees below $\mathbf{0}_e''$ dense? (We conjecture that at least the e-degrees of the Π_2 sets are dense).

Question 6.27. Are all finite distributive lattices embeddable as (non-initial) segments of the e-degrees?

Question 6.28. Can the techniques of Theorem 6.19 be adapted to provide minimal upper bounds in the e-degrees?

Little seems to be known (along the lines of the classic Spector result [Sp56] for ascending sequences of Turing degrees) concerning the existence of minimal upper bounds in the e-degrees, although every degree $\leq \mathbf{0}'$ *is* a minimal upper bound for some ascending sequence (using the density result).

Corollary 6.20 provides the only result so far concerning the global theory of the e-degrees: $\boldsymbol{\mathcal{D}}_e$ is not homogeneous. Slaman and Woodin [SW*ta*] have used their coding technique to obtain results concerning definability in the e-degrees. Otherwise everything is still open. For example:

Question 6.29. Are the total degrees definable in $\boldsymbol{\mathcal{D}}_e$? Is the jump definable in $\boldsymbol{\mathcal{D}}_e$?

Question 6.30. Classify the automorphisms of $\boldsymbol{\mathcal{D}}_e$.

Rogers [Rog67] asks:

QUESTION 6.31. Is the collection of total degrees order-theoretic, i.e., invariant under all automorphisms $\boldsymbol{\mathcal{D}}_e$?

§7. The structure of the enumeration degrees below $\mathbf{0}'$

The e-degrees below $\mathbf{0}'$ are of special interest, being the degrees of the Σ^0_2 sets. In this section we examine structure below $\mathbf{0}'$, in particular in relation to the high/low hierarchy and the classes Δ^0_2, Π^0_1. We assume below that all standard listings $\{B_i\}_{i\geq 0}$ of the Σ^0_2 sets come with thin Σ_2 approximations $\{B^s_i\}_{s\geq 0}$ (see Jockusch [Jo68] for a proof that this can be done), where every Δ^0_2 set appears somewhere in the list with a Δ_2 approximation. We sometimes use the notation $\mathcal{A}[s]$ to denote expression $\mathcal{A}$ evaluated at stage s.

DEFINITION 7.1 (McEvoy [McE84], McEvoy and Cooper [MC85]). If $\{A^s\}_{s\geq 0}$ is a Σ_2 approximation to a set A, define a *computation function* C_A for A by

$$C_A(x) = \mu s > x\,[A^s \restriction x \subset A],$$

each $x \geq 0$. A Σ_2*-high approximation* to A is a Σ_2 approximation to A for which C_A is total and dominates every recursive function. A set is Σ_2*-high* iff it has a Σ_2 approximation, and a degree is Σ_2*-high* iff it contains a Σ_2-high set.

It is easy to see that on the total degrees the notion of Σ_2-high coincides with the usual notion of high for the Turing degrees (under the natural embedding). To see that the Σ_2-high degrees contain the high Turing degrees (cf. [McE85]), we just notice that if $\chi_A \in \mathbf{a}$ with A high, then $A \oplus \overline{A} \in \mathbf{a}_e$ is Σ_2-high. For the converse, observe that $A \leq_e C_A$, and if A is a total set (so $\overline{A} \leq_e A$) then $A \equiv_e C_A$. The result follows from the fact (cf. Robinson [Rob68]) that any $A \in$ a Σ_2-high degree $\mathbf{a}$ has a Σ_2-high approximation, so $\mathbf{a}$ contains a C_A dominating every recursive function. McEvoy [McE85] showed that the Σ_2-high degrees properly extend the high Turing degrees by constructing a quasi-minimal Σ_2-high degree. In fact:

Theorem 7.2 (Cooper and Copestake [CC88])**.** *There exists a Σ_2-high properly Σ_2 degree.*

Proof. Construct A with Σ_2 approximation $\{A^s\}_{s\geq 0}$. Get C_A to dominate $\{i\}$ total by delaying extraction from A of members of the i^{th} column. Reconcile this with the properly Σ_2 strategy (see Theorem 4.11) by means of a tree of outcomes. □

However:

QUESTION 7.3. Characterise the jumps of the Σ_2-high e-degrees.

From Shoenfield's construction [Sh59] of a non-r.e. Turing degree below $\mathbf{0}'$, we know that the Δ^0_2 e-degrees properly extend the class of Π^0_1 e-degrees. Yates' construction [Ya65] of a Turing degree below $\mathbf{0}'$ which is incomparable with all the r.e. Turing degrees other than $\mathbf{0}$ and $\mathbf{0}'$ has an immediate corollary in the e-degrees:

Proposition 7.4. *There is a Δ^0_2 e-degree incomparable with all Π^0_1 e-degrees other than $\mathbf{0}$ and $\mathbf{0}'$.* □

Unlike the total degrees below $\mathbf{0}'$, the e-degrees below $\mathbf{0}'$ properly extend the degrees of Δ^0_2 sets (by Cooper and Copestake [CC88], there exist properly Σ_2 e-degrees). In fact there is the following analogue of the Yates result:

Theorem 7.5 (Cooper and Copestake [CC88])**.** *Let $\mathbf{h}$ be a Σ_2-high degree. Then there exists an e-degree $\mathbf{a}$ below $\mathbf{h}$ incomparable with all the Δ^0_2 e-degrees below $\mathbf{h}$ (other than $\mathbf{0}$ and $\mathbf{h}$).*

Proof. Build $A \in \Sigma^0_2$ by recursive approximation to satisfy the requirements:

E_i: If $A = \Psi_i(B_i)$, then either (a) there exists some x such that $\lim_s B^s_i(x)$ does not exist, or (b) $H \leq_e B_i$,

F_i: If $B_i = \Psi_i(A)$, then either (a) there exists some x such that $\lim_s B^s_i(x)$ does not exist, or (b) B_i is r.e.

The basic strategy for F_i is to try and make B_i r.e. by fixing D in A at stages $t > s$ whenever we get some $x \in B_i \cap \Psi_i(D)[s]$. But to leave room for the E_i-strategy in the presence of this infinitary outcome, split F_i into subrequirements $F_{i,x}$, $x \geq 0$, where $F_{i,x}$ will be satisfied if either (a$'$) $\lim_s B^s_i(x)$ fails to exist, or (b$'$) if $x \in B_i \cap \Psi_i(D)$ for some D, then there exists such a D which does not interfere with any E_j, $j \leq \max\{i, x\}$, and we fix D in A. Seek to produce such a D for x by working with the equation $B_i = \Psi_i(A)$ as in the properly Σ_2 strategy, using 'temporary' negative restraints for $F_{i,x}$ which are taken in turn via a 'line' which will serve the purpose of isolating the true path in the tree of outcomes from the 'Σ_2 noise' produced by the temporary restraints.

The E_i requirements conflict with the F_i requirements in that they will require extraction of numbers from A. In satisfying E_i define values of a one-to-one coding $\alpha_i : \omega \rightarrow \omega$. For each x aim to have

(7-1) $\qquad \alpha_i(x) \in \Psi_i(D)$ with $D \subset B_i \Leftrightarrow (x \in H[\max\{y \in D\}] \Rightarrow x \in H)$,

resulting in $H \leq_e B_i$. Work with the equation $A = \Psi_i(B_i)$ as in the properly Σ_2 strategy using the highness of H, either getting a y for which $\lim_s B^s_i(y)$ does not exist, or getting all the B_i-extractions needed for stage-by-stage rectification of the coding in (7-1), giving $H \leq_e B_i$. □

Little is known about jump restricted interpolation in the Σ^0_2 e-degrees, although we can extend the proof (Theorem 7 of [MC85]) that every Δ^0_2 $\mathbf{b} > \mathbf{0}$ has a low predecessor $\mathbf{a} > \mathbf{0}$ to get:

Theorem 7.6. *If* $\mathbf{h}$ *is* Σ_2*-high then there is a non-zero low* $\mathbf{a} < \mathbf{h}$*.*

Proof. Construct $A = \Theta(H)$, H Σ_2-high, satisfying

$$\begin{aligned} N_i: &\quad \Theta(H) \neq W_i, \\ P_i: &\quad \lim_s \Psi_i^s(\Theta^s(H^s))(i) \text{ exists}, \end{aligned}$$

each $i \geq 0$. Satisfy N_i in the usual way, but use the Σ_2-highness of H to get a follower $x \notin \Theta(H)[s]$ if $x \in W_i^s$. Satisfy P_i by enumerating $\langle y, H^t \restriction y\rangle$ into Θ at all stages $t > s$ when we get $i \in \Psi_i^s(D)$ and $D \subset \Theta^s(H^s)$, unless in conflict with a higher priority N_j. □

We now look at minimal pairs below $\mathbf{0}'$. Using the previously mentioned observation of Jockusch that capping in the e-degrees is often easier than constructing minimal pairs, we show that we can always cap low degrees below Σ_2-high degrees (and, in particular, below $\mathbf{0}'$).

Theorem 7.7 (McEvoy and Cooper [MC85])**.** *If* $\mathbf{0} < \mathbf{a} < \mathbf{h}$ *with* $\mathbf{a}$ *low and* $\mathbf{h}$ Σ_2*-high, then there is a degree* $\mathbf{c} < \mathbf{h}$ *such that* $\mathbf{a} \cap \mathbf{c} = \mathbf{0}$*.*

Proof. Carry out a Jockusch-style construction of a minimal pair $\mathbf{a}, \mathbf{c}$ by recursive approximation. Use the lowness of $\mathbf{a}$ to make the construction finite injury, and use Σ_2-highness to get it permitted below $\mathbf{h}$. □

Theorems 7.6 and 7.7 combine to give (compare [Co74]) the following extension of Corollary 7.1 of [MC85]:

Corollary 7.8. *If* $\mathbf{h}$ *is* Σ_2*-high then there is a minimal pair of degrees below* $\mathbf{h}$*.* □

Every low minimal pair of r.e. Turing degrees is a minimal pair in the e-degrees (under the natural embedding):

Theorem 7.9 (McEvoy and Cooper [MC85])**.** *If* $\mathbf{a}, \mathbf{b}$*, with* $\mathbf{a}$ *low, is a minimal pair in the* Π_1^0 *e-degrees, then* $\mathbf{a}, \mathbf{b}$ *is a minimal pair in the e-degrees.*

This follows immediately from:

Proposition 7.10. *If* $\mathbf{g} \leq$ Π_1^0 *e-degrees* $\mathbf{a}, \mathbf{b}$*,* $\mathbf{a}$ *low, then there is a* Π_1^0 $\mathbf{e}$ *with* $\mathbf{g} \leq \mathbf{e} \leq \mathbf{a}, \mathbf{b}$*.*

Proof of the Proposition. Let $\{A^s\}_{s\geq 0}$ be a low Π_1 approximation to $A \in \mathbf{a}$ and let $\{B^s\}_{s\geq 0}$ be a Π_1 approximation to $B \in \mathbf{b}$. Let G, i and j be such that $G \in \mathbf{g}$ and $G = \Psi_i(A) = \Psi_j(B)$. Define $E \in \mathbf{e}$ by:

$$\langle x, t\rangle \in E \Leftrightarrow (\forall s \geq t)[x \in \Psi_i^s(A^s) \vee x \in \Psi_j^s(B^s)]. \square$$

This means that any lattice embedding in the low r.e. degrees is a lattice embedding in the e-degrees. McEvoy and Cooper give some immediate consequences of the Lachlan/Yates construction of a minimal pair of (low) r.e. degrees, and of Lachlan's embedding results [La72] for the r.e. degrees:

Corollary 7.11. *(i)* (Gutteridge [Gu71]) *There exists a minimal pair of* Π^0_1 *e-degrees.*

(ii) Any countable distributive lattice can be embedded in the e-degrees, and the two five-element nondistributive lattices can be embedded in the e-degrees. Hence the upper semi-lattice of the e-degrees below $\mathbf{0}'$ *is not distributive.* □

McEvoy and Cooper show that the lowness condition in Theorem 7.9 is necessary.

Theorem 7.12. *There is a minimal pair in the* Π^0_1 *degrees which is not a minimal pair in the e-degrees.*

Proof. Adapt the Lachlan [La66] construction of a high minimal pair of r.e. degrees. Construct co-r.e. $A, B \in \mathbf{a}, \mathbf{b}$ respectively, with $\mathbf{a}, \mathbf{b}$ a minimal pair in $\boldsymbol{\mathcal{D}}$. Introduce enough changes in membership of A, B to enable us to get e-operators Γ, Λ for which $C = \Gamma^A = \Lambda^B$ with $C \neq W_i$ each i. Permit extractions of followers from C using non-simultaneous extractions of elements of columns from A and B which do not (through the timing of the extractions) injure the minimal pair strategy. This works because we do not need, as in the Turing case, Δ_2 'use functions' for Γ and Λ. □

The lowness condition in Theorem 7.7 is also necessary:

Definition 7.13. $\mathbf{a} < \mathbf{c}$ is *noncappable below* $\mathbf{c}$ iff for no nonzero $\mathbf{b} < \mathbf{c}$ do we have $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$, and $\mathbf{a}$ is *noncappable* iff $\mathbf{a}$ is noncappable below **0'**.

We say there is a *breakdown in capping below* $\mathbf{c}$ iff there is an $\mathbf{a}$ noncappable below $\mathbf{c}$.

McEvoy and Cooper show that for each low $\mathbf{b} > \mathbf{0}$ there is a breakdown in capping below some nonzero $\mathbf{a} \leq \mathbf{c}$. Further development of these techniques yields:

Theorem 7.14. *There exists a noncappable e-degree* $\mathbf{a}$*.*

Proof. Build $A \in \Sigma^0_2$ by recursive approximation satisfying:

$$P_i: \quad \overline{K} = \Psi^A_i \Rightarrow \Psi^A_i \text{ r.e.},$$
$$N_{\langle i,j \rangle}: \quad B_i \text{ r.e.} \vee (\exists G_i)[G_i \leq_e A \,\&\, G_i \leq_e B_i \,\&\, G_i \neq W_j],$$

each $i, j \geq 0$. Satisfy P_i by an analogue of Sacks restraints. To get $G_i \leq_e A$ and B_i build e-operators Γ_i and Λ_i with $G_i = \Gamma^A_i = \Lambda^{B_i}_i$. Appoint followers x of $N_{\langle i,j \rangle}$ which are held in G_i via Γ_i and Λ_i as long as $x \notin W_j$. If eventually $x \in W_j$, apply the following easily proved lemma:

If $A \in \Sigma^0_2$ *and the computation function* C_A *is dominated by a recursive function then* A *is r.e.*

to get B_i-permission (along with A-permission) to extract (in the limit) some x from G_i if B_i is not r.e. Use a tree of outcomes to enable the P_i-strategy to live with (for instance) the infinitary outcome when B_i is r.e. □

We have already mentioned the result of Copestake [Cope*ta*2] that there is a 1-generic e-degree below $\mathbf{0}'$ which bounds no minimal pair. The existence of nonbounding e-degrees (cf. [La79]) also follows (independently) from:

Theorem 7.15 (Cooper and Sorbi [CS*ta*]). *There is a linearly ordered initial segment of the Σ_2^0 e-degrees.*

Proof. Let $\{\Psi_i, \Theta_i\}_{i\geq 0}$ be a standard listing of all pairs of e-operators. Construct $A \in \Sigma_2^0$ and e-operators Γ_i, Λ_i, $i \geq 0$, satisfying:

$$N_i: \quad A \neq W_i,$$
$$P_i: \quad \forall i\,[\Theta_i^A = \Gamma_i(\Psi_i^A) \vee \Psi_i^A = \Lambda_i(\Theta_i^A),$$

each $i \geq 0$. Pursue the usual N_i-strategy involving extraction of followers x when $x \searrow W_i$ ('x enters W_i'). To satisfy P_i we need to rectify (in the limit) Γ_i or Λ_i. Very roughly speaking, we play off a Γ_i-strategy of fixing certain $D \subset A$ to keep Θ_i^A against a Λ_i-strategy of extracting certain a z from Θ_i^A via some $y \in E \subset A$ to rectify Λ_i. These strategies contrast in that the former requires many D's, the latter few E's. They allow room for the N_j-strategies by the rectification of Γ_i, Λ_i for x 'favouring' requirements N_j, $j > x$. Uncertainty as to the true path on the tree of outcomes results in N_i possibly requiring attention infinitely often, giving $A \notin \Delta_2^0$. □

By Lagemann's embedding results (Corollary 5.5 above) all the degrees in the constructed linear ordering must be properly Σ_2, and by the density of the Σ_2^0 degrees it must have order type that of the rationals between 0 and 1. A corollary of Theorem 7.15 is that $\boldsymbol{\mathcal{D}}_e(\leq \mathbf{0}')$ is not elementarily equivalent to the class of Δ_2^0 e-degrees.

QUESTION 7.16. Characterise the possible order-types of the initial segments of $\boldsymbol{\mathcal{D}}_e$ (or of $\boldsymbol{\mathcal{D}}_e(\leq \mathbf{0}')$.

It also follows from Theorem 7.15 that there are degrees below $\mathbf{0}'$ which cannot be nontrivially split. By the Sacks Splitting Theorem (p.124 of [So87]) each Π_1^0 e-degree can be split, but Ahmad [Ah89] has independently announced that there is a Δ_2^0 (in fact, low) e-degree which cannot be split.

Cooper and Sorbi [CS*ta*] have the following positive result:

Theorem 7.17. *Every Σ_2-high degree can be split.*

Proof. Given H Σ_2-high, define low sets $A = \Gamma^H$, $B = \Lambda^H$ with $H = \Omega^{A\oplus B}$ by using the highness of H to ensure that we can always get H-permission via Γ or Λ to rectify Ω via A or B respectively with regard to the higher priority lowness requirements. □

Other interesting results are announced by Ahmad in [Ah89], including a generalisation (Theorem 1) of her analagous result to the diamond theorem ([Co72]) for the Δ_2^0 degrees:

Theorem 7.18 (Ahmad [Ah*ta*]). *There exist incomparable e-degrees* $\mathbf{a}, \mathbf{b}$ *with* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$ *and* $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$.

Proof. Construct low sets A and B such that $\overline{K} \leq_e A \oplus B$ and $\forall i, j\,[\Psi_i^A = \Psi_j^B \Rightarrow \Psi_i^A$ is r.e.$]$. Get $\overline{K} \leq_e A \oplus B$ by ensuring $\forall x\,[x \in A \cap B \Leftrightarrow x \in \overline{K}]$. Carry out the minimal pair construction by recursive approximation (analagous to the minimal r.e. Turing pair construction in its alternation between A- and B-restraints), while favouring higher priority lowness and minimal pair restraints in deciding on the extraction of an A- or B-trace for $x \in K$. As usual, use a tree to identify 'windows' correctly, using the non-r.e.-ness of A and B to recover from actions taken off the true path. $\square$

By the Lachlan Nondiamond Theorem ([So87], p.162), Theorem 7.18 shows that the e-degrees below $\mathbf{0}'$ are not elementarily equivalent to the structure of the r.e. degrees (as does Theorem 7.15). Many questions concerning the e-degrees below $\mathbf{0}'$, including a number suggested by the Turing case, remain to be answered, including:

QUESTION 7.19. Characterise the degree of the first-order theory of $\boldsymbol{\mathcal{D}}_e(\leq \mathbf{0}')$.

It would also be interesting to see results exploring the context of the total e-degrees within $\boldsymbol{\mathcal{D}}_e(\leq \mathbf{0}')$.

§8. Strong enumeration reducibilities

The need for a theoretical counterpart to real computational situations in which only restricted information is available has motivated various areas of recursion theory, including that of strong Turing reducibilities (see [Rog67] or [Od89]) such as many-one, truth-table, weak truth-table and bounded truth-table reducibilities. We now look at strong reducibilities which use only positive information (in providing positive information), that is those which can be considered as restrictions of $\leq_e$. Most of the material in this section can be found in a fuller form in [Co87].

A number of truth-table reducibilities can be defined as restrictions of $\leq_e$. Recall (dropping the i in Definition 6.18) that $\varepsilon_x = \{D \in [\omega]^\omega \mid \langle x, D\rangle \in \Psi\}$, and define:

DEFINITION 8.1. The *norm* $\|\varepsilon_x\|$ of ε_x is defined by $\|\varepsilon_x\| = \sup\{\, |\, D\,| \;\mid D \in \varepsilon_x\}$.

Then:

DEFINITION 8.2.

(1) (*Many-one reducibility*, Post [Po44]). Ψ is a *many-one operator* iff $(\forall x)\ |\cup \varepsilon_x\,| = 1$. A is *many-one reducible to* B ($A \leq_m B$) iff $A \leq_e B$ via a many-one Ψ. The *one-one reducibility* $\leq_1$ is the special case where the ε_x's are mutually disjoint.

(2) (*Conjunctive reducibility*, Jockusch [Jo66]). Ψ is a *c-operator* iff $(\forall x)\ |\,\varepsilon_x\,| = 1$. $A \leq_c B$ iff $A \leq_e B$ via a c-operator.

(3) (*Bounded conjunctive reducibility*, Jockusch [Jo66]). Ψ is a *bc-operator* iff it is a c-operator and $(\exists n)(\forall x)\,\|\varepsilon_x\| \leq n$. $A \leq_{bc} B$ iff $A \leq_e B$ via a bc-operator.

(4) (*q-reducibility*, Friedberg and Rogers [FR59]). Ψ is a *q-operator* iff $\lambda x\, \varepsilon_x$ is recursive and $(\forall x)\,\|\varepsilon_x\| = 1$. $A \leq_q B$ iff $A \leq_e B$ via a q-operator. ($\leq_q$ is also called *disjunctive reducibility* and written $\leq_d$.)

(5) (*Bounded q-reducibility*, Jockusch [Jo66] and Lachlan [La65]). Ψ is a *bq-operator* iff $(\exists n)(\forall x)(\|\varepsilon_x\| = 1\,\&\ |\,\varepsilon_x\,| = n)$. $A \leq_{bq} B$ iff $A \leq_e B$ via a bq-operator.

(6) (*Positive reducibility*, Jockusch [Jo66]). Ψ is a *p-operator* iff $\lambda x\, \varepsilon_x$ is recursive. $A \leq_p B$ iff $A \leq_e B$ via a p-operator.

(7) (*Bounded p-reducibility*, Jockusch [Jo66] and Lachlan [La65]). Ψ is a *bp-operator* iff Ψ is a p-operator and $(\exists n)(\forall x)\ |\cup \varepsilon_x\,| \leq n$.

We say that a reducibility E is a *truth-table e-reducibility* iff E is a subreducibility of $\leq_p$: that is, iff $\lambda x\, \varepsilon_x$ is a recursive function for each E-operator Ψ.

Remembering the close relationship between e-reducibility and nondeterministic relative computability of partial functions, p-reducibility and q-reducibility can be thought of as nondeterministic versions of c-reducibility and m-reducibility respectively (see section 9 on polynomial time-bounded e-reducibilities).

In [PR79] Polyakov and Rozinas consider a generalisation of the notion of a truth-table e-reducibility: if $\leq_E$ is some e-reducibility, denote by rE the restriction of E (=the set of E-operators) to operators $\Psi \in E$ which are recursive. Unfortunately, if $\leq_E$ is not already a truth-table reducibility, $\leq_{rE}$ will, in general be non-transitive. Moreover, for most natural E we have $A \leq_E B \Leftrightarrow A \leq_{rE} B \oplus \omega$, where $B \equiv_{rE} B \oplus \omega$.

Polyakov and Rozinas [PR77] make the useful distinction between *decision reducibilities* (such as $\leq_T$ and $\leq_{tt}$) and *enumeration reducibilities* (such as $\leq_e$ and $\leq_s$). Each of the above truth-table reducibilities has a corresponding degree structure which can be looked at *either* as that for a decision reducibility *or* as that for an enumeration reducibility. Since for each reducibility $\leq_R$ above $\mathbf{0}_R$ consists of recursive sets only, the former view seems more appropriate.

Definition 8.3. We say that an e-reducibility $\leq_R$ is a *proper enumeration reducibility* iff $\mathbf{0}_R$ includes all r.e. sets.

If we allow *partial* versions of the truth-table e-reducibilities, where we only ask for $\lambda x\, \varepsilon_x$ to be partial recursive, we do obtain proper e-reducibilities. In doing this, we will of course want the resulting reducibility $\leq_{pR}$ to be transitive. Satisfactory existing notions are:

Definition 8.4.

(1) (*Partial many-one reducibility*, Rogers [Rog67]). Ψ is a *pm-operator* iff $(\forall x)\ |\cup \varepsilon_x| \leq 1$. $A \leq_{pm} B$ iff $A \leq_e B$ via a pm-operator. (Rogers also defines *partial one-one reducibility* $\leq_i$ in the natural way.)

(2) (*pc-reducibility*). Ψ is a *pc-operator* iff $|\varepsilon_x| \leq 1$. $A \leq_{pc} B$ iff $A \leq_{pc} B$ via a pc-operator. (The notation $\leq_{pc}$ is due to Polyakov and Rozinas [PR77], although the notion appears earlier in Skordev [Sk72] and in Sasso [Sas73] via his $\leq_T$ restricted to the semi-characteristic functions.)

Polyakov and Rozinas [PR77] note that transitivity fails for the partial counterparts $\leq_{pp}$, $\leq_{pq}$ for $\leq_p$ and $\leq_q$. There is no problem in giving partial counterparts $\leq_{pbp}$, $\leq_{pbc}$ and $\leq_{pbq}$ for the bounded reducibilities $\leq_{bp}$, $\leq_{bc}$ and $\leq_{bq}$.

Referring again to the trees of nondeterministic computations which underly a given enumeration operator, we can identify three natural classifications of strong e-reducibilities.

DEFINITION 8.5 (Cooper [Co87]).

(1) (*Finite e-reducibility*). Ψ is *branch finite* iff $(\forall x)(\varepsilon_x$ is finite). $A \leq_{fe} B$ iff $A \leq_e B$ via a branch finite Ψ. If $\leq_r$ is an e-reducibility, we sometimes denote by $\widehat{r}$ the restriction of r to branch finite e-operators.

(2) (*Bounded e-reducibility*). Ψ is *norm bounded* iff $(\exists n)(\forall x)(\|\varepsilon_x\| \leq n)$. $A \leq_{be} B$ iff $A \leq_e B$ via a norm bounded Ψ.

(3) (*btt-like e-reducibility*). Ψ is *btt-like* iff $(\exists n)(\forall x)(\mid \cup \varepsilon_x \mid \ \leq n$. $A \leq_{btte} B$ iff $A \leq_e B$ via a btt-like Ψ.

Apart from $\leq_{be}$, all the reducibilities considered so far have been branch finite. A particularly important subreducibility of $\leq_{be}$ is the branch infinite counterpart for $\leq_{pm}$:

DEFINITION 8.6 (*Singleton e-reducibility*, Friedberg and Rogers [FR59]). Ψ is an *s-operator* iff $(\forall x)(\|\varepsilon_x\| \leq 1)$. $A \leq_s B$ iff $A \leq_e B$ via an s-operator. (The branch finite version $\leq_{\hat{s}}$ of $\leq_s$ appears in Odifreddi [Od81] as $\leq_Q$.)

The definition given here is actually that of McEvoy's $\leq_{se}$ in [McE84], Friedberg and Rogers originally defining: $A \leq_s B$ if and only if there is a recursive f such that

$$(\forall x)(x \in A \Leftrightarrow W_{f(x)} \cap B \neq \phi).$$

Polyakov and Rozinas [PR77] showed that the requirement for f to be total could be dropped. In fact (see [Co87]) all three definitions are equivalent. s-operators occur commonly in constructions in the e-degrees (cf. section §5 above). In fact, Watson [Wa*ta*] notices that the following transitive subreducibilities of $\leq_{\hat{s}}$ are those most often used:

(*Nearly m-reducibility/nearly pm-reducibility*). $A \leq_{nm} B \,/\, A \leq_{npm} B$ iff $A = \Psi^B \cup W$ for some r.e. W, some m-operator/pm-operator (respectively) Ψ.

We notice that even though $\leq_{be}$ and $\leq_s$ are branch infinite, combinatorially the be- and s-operators are not that far removed from the fe-operators (Cooper [Co87]): If Ψ is norm bounded then Ψ is $\downarrow$branch finite, where we say Ψ is $\downarrow$*branch finite* iff

$$(\forall x)(\exists D_{i_x})(\forall D)[D \in\downarrow \varepsilon_x \Rightarrow D \cap D_{i_x} \in\downarrow \varepsilon_x).$$

All the reducibilities so far defined are transitive, so that we can refer to their corresponding degree structures. We can summarise the known implications between the various proper e-reducibilities in the following diagram:

		$\leq_s$	$\leq_{be}$	
$\leq_i$	$\leq_{pm}$	$\leq_{btte}$	$\leq_{fe}$	$\leq_e$
		$\leq_{pc}$		

Most of the non-implications are easy to prove (see [PR77], [PR79] and [Co87]). See [Co87] for a summary how the truth-table e-reducibilities fit into this picture.

We now examine the degree structures for the proper e-reducibilities, with special reference to the s-degrees and their close relationship to the e-degrees. Following Vuckovic [Vu74] we define:

DEFINITION 8.7. A is *strong* iff $A \equiv_{pm} [A]^{<\omega}$. If $\leq_r$ is some transitive reflexive e-reducibility, then we say the r-degree $\mathbf{a}_r$ is *strong* iff there is some strong $A \in \mathbf{a}_r$.

Since $A \equiv_e [A]^{<\omega} \equiv_{pm} [[A]^{<\omega}]^{<\omega}$, every e-degree $\mathbf{a}$ contains a strong r-degree $\mathbf{a}_r$, say, and if $\leq_r$ includes $\leq_s$ then this strong $\mathbf{a}_r$ is of largest r-degree in $\mathbf{a}$ (cf. Vuckovic [Vu74]). In particular:

Every e-degree contains a largest s-degree.

Since $[\overline{K}]^{<\omega}$ is Π^0_1, so $\equiv_m \overline{K}$, $\overline{K}$ is an example of a non-r.e. strong set. If $A\,(\neq\omega)$ is r.e., then $\chi_{\overline{A}} \equiv_{pm} \overline{A}$, so if $\leq_r$ includes $\leq_{pm}$, then $\overline{K}$ is of strong total r-degree.

We easily adapt Definition 3.5 to get a jump on the r-degrees:

DEFINITION 8.8. $\mathbf{a}'_r = \deg_r J(A)$.

Hence, for any $\leq_r$ including $\leq_{pm}$ we have that $\mathbf{0}'_r = \deg_r(\overline{K})$ is a strong total Π^0_1 r-degree, and if $\leq_r$ includes $\leq_s$, then $\mathbf{0}'_r$ is of largest r-degree in $\mathbf{0}'$.

DEFINITION 8.9. Given two reducibilities $\leq_y$ and $\leq_z$ with $\leq_y$ included in $\leq_z$, we say $\mathbf{a}_z \neq \mathbf{0}_z$ is a *y-contiguous z-degree* iff $\forall A, B \in \mathbf{a}_z\,[A \equiv_y B]$: that is, $\mathbf{a}_z$ consists of a single y-degree.

The non-existence of s-contiguous e-degrees follows from:

Theorem 8.10 (Zakharov [Za84])**.** *Every non-zero e-degree contains at least two distinct s-degrees.*

Proof. Given A_0 not r.e., define

$$A = \{\langle x, y\rangle \mid x \in \Psi_y(A_0)\}, \quad B = S \cup [\bigcup_{x \in A} D_{f(x)}],$$

where S is hypersimple and f is a recursive function such that $D_{f(x)} = \{2^x - 1, \ldots, 2^{x+1} - 2\}$. Verify that $A_0 \equiv_e A \equiv_e B$ but that $A \not\leq_s B$. □

The above proof is not sufficient to resolve:

QUESTION 8.11. Does every non-zero e-degree contain infinitely many distinct s-degrees?

Zakharov [Za84] showed that if B is as in Theorem 8.10, then $B <_m B \times B <_m \ldots <_m [B]^n <_m \ldots$, so every e-degree contains an infinite ascending sequence of m-degrees. Using an analogue of Beigel, Gasarch and Owings' [BGO87] notion of *terse set*, Copestake [Cope*ta*1] showed that every 1-generic e-degree $\mathbf{a}$ contains an infinite ascending chain of s-degrees, namely the s-degrees got from the sequence

$$A <_s [A]^2 <_s \ldots <_s [A]^n <_s \ldots,$$

A a terse set in $\mathbf{a}$. Watson [Wa*ta*] gave a constructive version of Zakharov's theorem for the Σ^0_2 degrees, and proved:

Theorem 8.12 (Watson [Wa*ta*]). *If $\mathbf{a} > \mathbf{0}$ is Δ^0_2 or Σ_2-high, then $\mathbf{a}$ contains no s-degree minimal in $\mathbf{a}$. Hence every $\mathbf{a} > \mathbf{0}$ which is Δ^0_2 or Σ_2-high contains an infinite descending sequence of s-degrees.*

Proof. Let $\widehat{\Psi}_i$ be a standard listing of all s-operators. Given a non-r.e. set $A \in \Delta^0_2$ construct a $B \in \Delta^0_2$ satisfying $R_i : A \neq \widehat{\Psi}_i^B$, each $i \geq 0$, $P : A = \Gamma^B$ (Γ an e-operator to be constructed) and $Q : B = \Lambda^A$ (Λ an s-operator to be constructed). Monitor the initial segment of agreement $\ell(i,s)$ for R_i, and as this grows seek to preserve it (working against the pseudo-outcome $A = \widehat{\Psi}_i^B$ r.e.) by restraining a corresponding singleton y in B for each $x \in A$, $x < \ell(i,s)$.

Ensure the rectification of Γ in P eventually leaves room for the R_i-strategy by using large sets D in $\langle x, D\rangle \in \Gamma$, so that if $x \nearrow A$ ('x leaves A') we can get $x \nearrow \Gamma^B$ by extracting some $y \in D$ from B while respecting the R_i-restraints corresponding to numbers $y \leq x < \ell(i,s)$. Achieve Q by making $\langle y, \{x\}\rangle \in \Lambda$ if $\langle x, D\rangle \in \Gamma$ with $y \in D$, and using crossing for rectifying Γ when $x \nearrow A^t$ and $y \in B^t$. In the A Σ_2-high part of the theorem the strategies are similar. □

It is not known whether there exist e-degrees containing minimal or least s-degrees, or which are closed under formation of meets of pairs of s-degrees. Watson [Wa88] conjectures that there exists a minimal pair of s-degrees contained in a single e-degree. More generally Watson [Wa*ta*] asks:

Question 8.13. Do there exist (distinct) proper e-reducibilities $\leq_y$ and $\leq_z$, with $\leq_y$ contained in $\leq_z$, for which there exists a y-contiguous z-degree?

On the positive side, Watson [Wa*ta*] is able to show: There exists a Π^0_1 pc-degree $\mathbf{a}$, $\mathbf{0} < \mathbf{a} < \mathbf{0}'$, such that if A and B are Π^0_1 sets in $\mathbf{a}$ then $A \equiv_s B$.

Turning to general structural questions, [Co87] gives an in-depth analysis of density and non-density in a wide range of strong e-degree structures. We are unable to do more here than summarise the main results, and hint at the techniques involved. Fundamental to this analysis are notions of *compactness*, trivial for the Turing case but of crucial importance in deriving existence results for suitably i-distinguished cellings.

DEFINITION 8.14. (1) Let ζ be a set of numbers. We say that ζ is a Ψ*-tower* iff there is a pair W, K such that

$$(\forall x)[x \in \zeta \Leftrightarrow W \notin \downarrow \varepsilon_x \& K \notin \uparrow \varepsilon_x] \tag{8-1}$$

and

$$(\forall x, y \in \zeta)[\downarrow \varepsilon_x(W,K) \subseteq \downarrow \varepsilon_y(W,K) \vee \downarrow \varepsilon_y(W,K) \subseteq \downarrow \varepsilon_x(W,K)]. \tag{8-2}$$

(2) We say that W^*, K^* *decides* the Ψ-tower ζ iff $W^* \supseteq_{\text{fin}} W$ and $K^* \supseteq K$ and for all x, $W^* \in \downarrow \varepsilon_x$ or $K^* \in \uparrow \varepsilon_x$.
(3) We say that Ψ is *compact* iff every Ψ-tower can be decided by some pair W^*, K^*. We say that $\leq_E$ is *compact* iff each $\Psi \in E$ is compact.

Proposition 8.15. *Each of the reducibilities* $\leq_m$, $\leq_1$, $\leq_c$, $\leq_{bc}$, $\leq_q$, $\leq_{bq}$, $\leq_{bp}$, $\leq_i$, $\leq_{pm}$, $\leq_{pc}$, $\leq_s$ $\leq_{\hat{s}}$ $\leq_{btte}$ $\leq_{be}$ *or* $\leq_T$ *(via its corresponding e-reducibility) is compact.*

Proof. Observe that each of these reducibilities is norm bounded or $\uparrow$*norm bounded* (that is, for each Ψ $(\exists n)(\forall x)(\exists K \in \uparrow \varepsilon_x)[\mid K \mid \leq n]$), and show that all such reducibilities are compact. □

It will follow that for most known proper e-reducibilities, compactness characterises those reducibilities $\leq_r$ for which we can find degrees with uncountably many minimal covers in the r-degrees. As remarked in [Co87], the truth-table reducibilities are approached differently, within the context of the decision reducibilities (see for example Odifreddi [Od81] or Lachlan [La70]). For compact operators one can derive i-distinguished and i-undistinguished cellings (instead of just i-maximally enumerating cellings as in the proof of Theorem 6.19) in such a way that they can be nested without too much loss of effectiveness. Then Lemmas 6.16 and 6.17 can be adapted (in 6.17 replacing i-maximally enumerating with i-undistinguished), to provide the satisfaction of a wide range of minimality and minimal cover requirements. The nesting depends technically on the notion (analagous to that of subtree in the Turing case) of *a continuation* $\mathcal{C}'$ of a celling $\mathcal{C}$.

Theorem 8.16 (Cooper [Co87]). *Let* $\leq_r$ *be a partial truth-table reducibility. Then, if* $\leq_r$ *is a proper e-reducibility, there exists a continuum of minimal r-degrees.*

In particular, there exists a continuum of minimal pm-degrees, pc-degrees and of T-degrees.

Proof. The construction is formally the same as that for a continuum of minimal Turing degrees (see [Le83]). For each $\alpha \in 2^\omega$ construct a distinct nest $\{\mathcal{C}_i(\alpha)\}_{i\in\omega}$ of appropriate cellings where for each $i \in \omega$ $\mathcal{C}_i(\alpha) \in \Sigma^0_1$, $\mathcal{C}_{i+1}(\alpha)$ is a continuation of $\mathcal{C}_i(\alpha)$, and (roughly speaking) either $\mathcal{C}_{i+1}(\alpha)$ is i-undistinguished or $\mathcal{C}_{i+1}(\alpha)$ is i-distinguished with Σ^0_1 i-labelling. Define A_α to be the unique set respecting all the cellings $\mathcal{C}_i(\alpha)$, $i \in \omega$, in the usual way, and get the minimality of $\deg_r(A_\alpha)$ by applying suitable variants of Lemmas 6.16 and 6.17.

The result for the T-degrees follows from the earlier observation (derived from Sasso's construction [Sas73] of a minimal T-degree $< \mathbf{0}'$) that sets of minimal pc-degree yield unique semicharacteristic functions of minimal T-degree. □

It is clear from the proof of Gutteridge's Theorem (5.3) that there do not exist minimal r-degrees for any e-reducibility $\leq_r$ which includes $\hat{s}$-reducibility. We now see that for most compact proper e-reducibilities relativisation of the Gutteridge result breaks down completely.

DEFINITION 8.17. (1) If f is a (possibly partial) function, then the f-contraction $f\Psi$ of Ψ is defined by $\langle x, D\rangle \in f\Psi \Leftrightarrow \langle x, f^{-1}D\rangle \in \Psi$.
(2) We say a reducibility $\leq_r$ is *combinatorially compact* iff whenever $\Psi \in r$ we have that each f-contraction $f\Psi$ is compact.

Since the proof of Proposition 8.15 only depends upon norm boundedness or $\uparrow$norm boundedness, all the reducibilities is actually seen to be combinatorially compact.

Theorem 8.18 (Cooper [Co87]). *Let $\leq_r$ be a combinatorially compact e-reducibility. Then, if $\leq_r$ includes $\leq_s$, there exists a continuum of minimal covers for $\mathbf{0}'_r$.*

In particular, there exists a continuum of minimal covers for $\mathbf{0}'_s$ in the s-degrees, and of $\mathbf{0}'_{be}$ in the be-degrees.

Proof. The proof is similar to that for Theorem 8.16, except that we now get $\mathcal{C}_i(\alpha)$ and its corresponding increment $\mathcal{I} \in \Sigma^0_2$ for each $i \in \omega$, and similarly any i-labelling $f^{\mathcal{I}} \in \Sigma^0_2$. Again applying suitable variants of Lemmas 6.16 and 6.17, we get that for each i either $A_\alpha \leq_r f^{\mathcal{I}} \times \mathcal{I} \times \Psi_i^{A_\alpha}$ or $\Psi_i^{A_\alpha} \leq_r \mathcal{C}$. Since $\overline{K}$ is r-complete in Σ^0_2, we get $A_\alpha \leq_r \overline{K} \times \Psi_i^{A_\alpha}$ or $\Psi_i^{A_\alpha} \leq_r \overline{K}$. To obtain $\deg_r(A_r)$ minimal over $\mathbf{0}'_r$ make $\overline{K} \leq_r A_\alpha$ by coding $\overline{K}$ into $\mathcal{C}_0(\alpha)$ in a trivial way. □

Further detailed consideration confirms the existence of continuously many minimal covers for $\mathbf{0}'_{\hat{s}}$ in the ŝ-degrees, but the problem for the T-degrees and the btte-degrees remains open.

We briefly return to the non-compact case.

DEFINITION 8.19. We say that a branch finite Ψ is *anti-compact* iff there is a Ψ-tower (with $W, K = \phi$) and for any two sets A, B we have that $\Psi^A = \Psi^B$. (It follows from the definition that no compact Ψ can be anti-compact.)

Proposition 8.20. *There exists an anti-compact p-operator.*

Proof. Define the *Slaman operator* $\widehat{\Psi}$ as follows: Associate with each D_x a rational number

$$a_x = \tfrac{1}{2} + (-1)^{D_x(0)+1}(\tfrac{1}{2})^2 + \ldots + (-1)^{D_x(k)+1}(\tfrac{1}{2})^{k+2}$$

where $k = \max\{z \in D_x\}$. Let $D_x \in \widehat{\varepsilon}_x$ if and only if $a_x < a_y$ and

$$\mu w\, [D_x(w) \neq D_y(w) \& w \leq \max\{z \in D_x\} + 1 \& w \leq \max\{z \in D_y\}] \quad \text{exists} = \max\{z \in D_y\}.$$

The following generalises the theorem of Gutteridge (Corollary 5.2) on minimal covers in the e-degrees.

Theorem 8.21 (Cooper [Co87]). *(i) If an e-reducibility $\leq_r$ contains an anti-compact Ψ, then there are at most countably many minimal covers for any $\deg_r(B)$, where each such minimal cover $\leq_T B'$. In particular, there are at most countably many minimal covers for a given degree in each of the e-degrees, the fe-degrees and the p-degrees.*

(ii) If a truth-table e-reducibility $\leq_r$ contains an anti-compact Ψ, then any minimal cover for $\deg_r(A)$ in the r-degrees is r.e. in A. In particular, every minimal p-degree is r.e.

Proof. For (i), show that if Ψ is an anti-compact e-operator then for any sets A, B $\Psi^A \leq_e B \Rightarrow A \leq_T B'$. Define a new anti-compact Θ by relabelling the outputs for Ψ on ω to allow us, given $A = \Psi_i(\Theta^A \oplus B)$, to uniformly enumerate $x+1$ from $\Psi^{A \restriction x} \oplus B)$ just in the case $x+1 \in A$, giving $A \leq_e \Theta^A \oplus B \Rightarrow A \leq_e B$. Proposition 8.20 gives the particular cases.

(ii) is similar. Show that if Ψ is an anti-compact truth-table operator, then $\Psi^A \leq_p B \Rightarrow A \leq_p B$, and that we can modify Ψ to get an anti-compact truth-table operator Θ for which $A \leq_p \Theta^A \oplus B \Rightarrow A \leq_p B$. □

On the other hand, in contrast to Gutteridge's Theorem for the e-degrees, one can construct a (r.e.) minimal p-degree (see [Co*ta*2]).

There are many open questions (see §5 of [Co87]), including:

Question 8.22. Are there techniques for producing upper cones with required properties in the e- or s-degrees?

Question 8.23. Are the structures of the pm- or pc-degrees above $\mathbf{0}'_{pm}$ or $\mathbf{0}'_{pc}$ respectively dense?

Polyakov and Rozinas [PR79] have proved some basic results concerning s-, pm- and pc-degrees. Also:

Theorem 8.24 (Zakharov [Za84]). *The elementary theories of the r-degrees and s-degrees are different for r$\in$ {e,pc,p,c}.*

Proof. The proposition

$$(\exists \mathbf{a} \neq \mathbf{0})(\forall \mathbf{b})(\forall \mathbf{c})[\mathbf{a} \leq \mathbf{b} \cup \mathbf{c} \;\Rightarrow\; \mathbf{a} \leq \mathbf{b} \vee \mathbf{a} \leq \mathbf{c}] \tag{8-3}$$

is true in $\boldsymbol{\mathcal{D}}_s$ since if A is a retraceable set we have

$$(\forall B)(\forall C)[A \leq_s B \oplus C \;\Rightarrow\; A \leq_s B \vee A \leq_s C].$$

However, direct construction shows that (8-3) is not true in $\boldsymbol{\mathcal{D}}_r$ each r$\in \{\mathrm{e, pc, p, c}\}$. (In the e case Zakharov shows that one can join above $\mathbf{a}$ with quasi-minimal e-degrees $\mathbf{b}, \mathbf{c}$.) □

Using Theorem 7.14 above, we can show that $\boldsymbol{\mathcal{D}}_s(\leq \mathbf{0}'_s)$ is not elementarily equivalent to $\boldsymbol{\mathcal{D}}_s(\leq \mathbf{0}'_s)$:

Theorem 8.25. *If* $\mathbf{a}$ *is an s-degree below* $\mathbf{0}'_s$ *then* $\mathbf{a}$ *is cappable in the s-degrees below* $\mathbf{0}'_s$.

Proof. Let $\{(\widehat{\Psi}_i, \widehat{\Theta}_i)\}_{i\geq 0}$ be a standard listing of all pairs of s-operators. Given $A \in \Sigma^0_2$, $\overline{K} \not\leq_s A$, construct $B \in \Sigma^0_2$ satisfying:

$$N_i: \quad B \neq W_i,$$
$$P_i: \quad \widehat{\Psi}^A_i = \widehat{\Theta}^B_i \Rightarrow \widehat{\Theta}^B_i \text{ r.e.},$$

each $i \geq 0$. The N_i-strategy is as usual. For P_i monitor the initial segment of agreement, of length $\ell(i,s)$. If almost all $y < \ell(i,s)$ at some stage s which are in $\widehat{\Psi}^A_i[s]$ are enumerated into $\widehat{\Theta}^B_i[s]$ via more than one singleton $\{y\} \subset B[s]$, then we can protect $\ell(i,s)$ via restraints which still allow room for the N_j's. Otherwise, we can use infinitely many members of $\widehat{\Theta}^B_i$ to code $\overline{K}$, when $x \nearrow \overline{K}$ using a B extraction to achieve an extraction of the x-trace from Θ^B_i, forcing $\liminf_s \ell(i,s)$ to be finite. □

QUESTION 8.26. Prove density of the s- and be- degrees $\leq \mathbf{0}'$. Characterise the degrees of the first-order theories of these structures.

§9. Polynomial time-bounded enumeration reducibilities

All the reducibilities introduced above give rise to computation bounded reducibilities. These take various forms, mainly dependent on the type of bound on time or space allowed for the reduction and on the presence or otherwise of nondeterminism in the reduction procedures. Polynomial time bounds are closely related to practical limits on computations, and so far this is the area in which notions are well developed. The connection between relative e-reducibility of sets and relative nondeterministic computability of functions gives polynomial bounded enumeration reducibilities an important role in clarifying the nature of polynomial time reducibility between functions.

Before giving definitions, the bounding process requires us to be more precise in describing computations. For convenience, we take the multi-tape Turing machine as our model of computation, inputs being words in a finite alphabet (we choose $\{0,1\}$). Numbers are identified with their binary presentations, and we write $|\,x\,|$ for the length of $x \in \{0,1\}^*$ (so $|\,x\,| \simeq \log x$, the binary logarithm of x). A Turing machine M (either deterministic or nondeterministic) runs in polynomial time if there is a polynomial p such that on any input of length n, any computation of M halts in at most $p(n)$ steps. M *accepts* on input x, and *recognises* a set A, have their usual meanings. Turing machines equipped with distinguished output tapes for computing functions are called *transducers*. In general, nondeterministic transducers will be permitted to compute many-valued functions. For relative computations we use oracle Turing machines with a designated oracle tape. An oracle Turing machine M runs in polynomial time if there is a polynomial p such that on any input of length n, any computation of M halts in at most $p(n)$ steps, independently of the oracle used. As usual, $\mathcal{P}$ denotes the class of all subsets of $\{0,1\}^*$ (deterministically) computable in polynomial time and $\mathcal{NP}$ is the class of all such sets nondeterministically computable in polynomial time.

Definition 9.1. (1) (*Polynomial time Turing reducibility*, Cook [Coo71]/ *Nondeterministic polynomial time Turing reducibility*, Meyer and Stockmeyer [MS72]). $A \leq_T^{\mathcal{P}} B$ / $A \leq_T^{\mathcal{NP}} B$ iff there is a deterministic/ nondeterministic (respectively) oracle Turing machine M which runs in polynomial time such that $x \in A$ iff some computation sequence of M on input x with oracle B accepts.
(2) (*Polynomial time many-one reducibility*, Karp [Ka72]/ *Nondeterministic polynomial time many-one reducibility*, Ladner, Lynch and Selman [LLS75]). $A \leq_m^{\mathcal{P}} B$ / $A \leq_m^{\mathcal{NP}} B$ iff there is a deterministic/ nondeterministic transducer M which runs in polynomial time such that $x \in A$ iff M, on input x, produces some output $y \in B$.

Both deterministic reducibilities are reflexive, transitive relations giving rise to degree structures with least degree $= \mathcal{P}$ (ignoring ϕ and $\{0,1\}$ in the former case). $\leq_m^{\mathcal{NP}}$ (but not $\leq_T^{\mathcal{NP}}$) is transitive with least degree $= \mathcal{NP}$. Ladner, Lynch and Selman [LLS75] added further polynomial time bounded positive reducibilities to $\leq_m^{\mathcal{P}}$ and $\leq_m^{\mathcal{NP}}$. For technical reasons, these need to be approached via truth table reducibility, where an abstract approach is taken in order to make the bounds independent of the particular presentation of the Boolean functions giving the truth tables.

Let Δ be a fixed finite alphabet (for example $\{\wedge, \vee, \neg\}$) for encoding Boolean functions and let $c \notin \Delta \cup \{0,1\}^*$.

Definition 9.2 (Ladner, Lynch and Selman [LLS75]). (i) A *tt-condition* is a member of $\Delta^* c(c\{0,1\}^*)^*$.

(ii) A *tt-condition generator*/ *nondeterministic tt-condition generator* is a deterministic/ nondeterministic (respectively) transducer M such that each computation sequence of M produces a tt-condition.

(iii) A *tt-condition evaluator* is a recursive mapping of $\Delta^* c\{0,1\}^*$ into $\{0,1\}$.

(iv) Let e be a tt-condition evaluator. A tt-condition $\alpha c c y_1 \ldots c y_k$ is e-satisfied by B iff $e(\alpha c \chi_B(y_1) \ldots \chi_B(y_k)) = 1$.

(v) (*Polynomial time truth table reducibility*/ *Nondeterministic polynomial time truth table reducibility*). $A \leq_{tt}^{\mathcal{P}} B$ / $A \leq_{tt}^{\mathcal{NP}} B$ iff there is a deterministic/ nondeterministic polynomial time computable generator M and a polynomial time computable evaluator e such that $x \in A$ iff M, on input x, computes some tt-condition which is e-satisfied by B.

Ladner, Lynch and Selman look at various alternative formulations of these definitions (such as allowing nondeterminism in the evaluator in $\leq_{tt}^{\mathcal{NP}}$) and show them to lead to the same notions.

Definition 9.3 (Ladner, Lynch and Selman [LLS75]). (i) (*Polynomial time positive reducibility/ Nondeterministic polynomial time positive reducibility*). $A \leq_p^{\mathcal{P}} B$/ $A \leq_p^{\mathcal{NP}} B$ iff $A \leq_{tt}^{\mathcal{P}} B$/ $A \leq_{tt}^{\mathcal{NP}} B$ (respectively) with evaluator e having the property that if $e(\alpha c \sigma_1 \ldots \sigma_k) = 1$ and $\sigma_i = 1$ implies $\tau_i = 1$ for $1 \leq i \leq k$, then $e(\alpha c \tau_1 \ldots \tau_k) = 1$.

(ii) (*Polynomial time conjunctive reducibility/ Nondeterministic polynomial time conjunctive reducibility*). $A \leq_c^{\mathcal{P}} B$/ $A \leq_c^{\mathcal{NP}} B$ iff $A \leq_{tt}^{\mathcal{P}} B$/ $A \leq_{tt}^{\mathcal{NP}} B$ with evaluator e having the property that $e(\alpha c \sigma_1 \ldots \sigma_k) = 1$ iff $\sigma_i = 1$ for $1 \leq i \leq k$.

(iii) (*Polynomial time disjunctive reducibility/ Nondeterministic polynomial time disjunctive reducibility*). $A \leq_d^{\mathcal{P}} B$/ $A \leq_d^{\mathcal{NP}} B$ iff $A \leq_{tt}^{\mathcal{P}} B$/ $A \leq_{tt}^{\mathcal{NP}} B$ with evaluator e having the property that $e(\alpha c \sigma_1 \ldots \sigma_k) = 0$ iff $\sigma_i = 0$ for $1 \leq i \leq k$.

(Bounded versions of these reducibilities can be defined by adding suitable restrictions on the generator.)

In general, following Baker, Gill and Solovay [BGS75] for the Turing case, the nondeterministic reducibilities (in contrast to the situation without oracles) can be shown to properly include the deterministic notions. Ladner, Lynch and Selman [LLS75] summarise the implications and non-implications (these often, but not always, proved on the sets computable in 2^n time) between the various reducibilities in the following diagram (showing a collapse of $\leq_d^{\mathcal{NP}}$ and $\leq_p^{\mathcal{NP}}$):

$$\begin{array}{cccccc} & \leq_d^{\mathcal{P}} & \leq_m^{\mathcal{NP}} = \leq_d^{\mathcal{NP}} & & \leq_c^{\mathcal{NP}} = \leq_p^{\mathcal{NP}} & \\ \leq_m^{\mathcal{P}} & & & & & \\ & \leq_c^{\mathcal{P}} & \leq_p^{\mathcal{P}} & & \leq_{tt}^{\mathcal{P}} & \leq_T^{\mathcal{P}} \end{array}$$

The transitivity of $\leq_c^{\mathcal{NP}}$ with corresponding least degree $= \mathcal{NP}$ leads Ladner, Lynch and Selman to suggest $\leq_c^{\mathcal{NP}}$ as a polynomial time bounded analogue of enumeration reducibility. There are many questions concerning the polynomial time bounded degree structures, often intimately connected with the question of $\mathcal{P} =?\mathcal{NP}$.

Following from Adleman and Manders [AM77] (for the $\leq_m$ case), Long [Lo82] introduces the notion of *strong* nondeterministic polynomial time reducibility in which there is a consistency condition imposed on the outputs of each nondeterministic Turing machine M for any input x, and shows the strong reducibilities to lie between the deterministic and unrestricted nondeterministic notions. Other related reducibilities include the log space reducibilities of Meyer and Stockmeyer [MS73] and Jones [Jon73].

We now consider notions of polynomial time bounded e-reducibility (referring the reader to Selman [Se78] and Copestake [Cope*ta*3] for a fuller discussion). Arguing by analogy with Theorem 2.9 above, Selman [Se78] defines:

Definition 9.4. A is polynomial time enumeration reducible to B ($A \leq_{pe} B$) iff

$$\forall x \, [B \leq_T^{\mathcal{NP}} X \Rightarrow A \leq_T^{\mathcal{NP}} X].$$

This is intended to capture the notion that A is polynomial time enumeration reducible to B just in the case that every 'polynomial-enumeration' of B (relative to any set X) yields some polynomial-enumeration of A (relative to X). Then, following Selman's proof [Se71] that $\leq_e$ is a maximal transitive subrelation of 'r.e. in', we have:

Theorem 9.5 (Selman [Se78]). *$\leq_{pe}$ is a maximal transitive subrelation of $\leq_T^{\mathcal{NP}}$.*

The definition of $\leq_{pe}$ can be presented so as to look more like the usual definition of enumeration reducibility.

Theorem 9.6 (Selman [Se78]). *$A \leq_{pe} B$ iff for every polynomial q, there is a set W in $\mathcal{NP}$, a polynomial p, and a constant k such that*

$$\begin{aligned}\forall x\,[x \in A \Leftrightarrow \exists \alpha\,[\,|\,\alpha\,|\; \leq p(\,|\,x\,|\,)\,\&\, xc\alpha \in W \\ &\,\&\, \alpha = cy_1 \ldots cy_n ccz_1 \ldots cz_m \\ &\,\&\, y_1, \ldots, y_n, z_1, \ldots, z_m \in \{0,1\}^* \\ &\,\&\, \forall j \leq m\;\; q(\,|\,z_j\,|\,) \leq k \log |\,x\,| \\ &\,\&\, \{y_1, \ldots, y_n\} \subseteq B \,\&\, \{z_1, \ldots, z_m\} \subseteq \overline{B}]].\end{aligned}$$

The unexpected negative information about B appearing in this technically more useful definition of $\leq_{pe}$ is due to the fact that a polynomial-enumeration of B will give us usable information about $\overline{B}$. Selman shows that $\leq_{pe}$ lies strictly between $\leq_c^{\mathcal{NP}}$ and $\leq_T^{\mathcal{NP}}$, but that on TIME($2^{\mathcal{P}}$) (=the class of sets recognised by deterministic Turing machines which run in time $2^{p(n)}$, $p(n)$ a polynomial) $\leq_{pe}$ and $\leq_c^{\mathcal{NP}}$ coincide. Partially justifying the definition of $\leq_{pe}$ Selman verifies that for polynomial bounded functions we have:

$$f \leq_T^{\mathcal{NP}} g \text{ if and only if } \operatorname{graph}(f) \leq_{pe} \operatorname{graph}(g).$$

In contrast to the situation with the deterministic polynomial time reducibilities (see for example [Am*ta*] or [SS*ta*]), little is known about the degree structures for the nondeterministic positive reducibilities. Silver [Si*ta*] examines various polynomial time analogues of singleton reducibility ($\leq_s$), including $\leq_m^{\mathcal{NP}}$, Long's $\leq_m^{SN}$ (originally appearing in Adleman and Manders [AM77] as γ-reducibility) and a singleton version of $\leq_{pe}$.

There is a problem with $\leq_{pe}$ arising from the nonconstructiveness of its definition (there are no accessible pe-operators). Copestake [Cope*ta*3] convincingly argues that what is needed is a polynomial bounded analogue of the *full* description of $\leq_e$, namely: $A \leq_e B$ iff *there is an algorithm* which given any enumeration of B will provide us with an enumeration of A. See [Cope*ta*3] for an alternative definition of polynomial time bounded e-reducibility.

§10. The Medvedev lattice and its extension to partial functions

Enumeration operators underly Medvedev's [Me55] definition of the lattice of *degrees of difficulty*, derived from a natural way of comparing relative computability of classes of functions (or *mass problems*). Of special interest is the *Dyment lattice* [Dy76] which generalises the concept of degree of difficulty to the classification of classes of partial functions. We briefly review the basic definitions (see [Rog67] or [Sor*ta*1]).

Definition 10.1. $\mathcal{A}$ is a *mass problem* iff $\mathcal{A}$ is a collection of total functions. $\mathcal{A}$ is a *mass problem of partial functions* iff $\mathcal{A}$ is a collection of partial functions.

Rogers gives a number of examples of natural mass problems (consisting of sets of functions with something in common). In order to formalise the idea that $\mathcal{A}$ (say) is reducible to $\mathcal{B}$ iff from any 'solution to' (that is member of) $\mathcal{B}$ we can effectively get a solution to $\mathcal{A}$, we recall the definition (1.12 above) of recursive operator.

Then every e-operator Ψ_i defines a partial recursive operator $\Theta : \mathcal{P} \to \mathcal{P}$. The Fundamental Operator Theorem (Rogers [Rog67], Theorem XXIII) says that although not every Ψ_i^f, $f \in \mathcal{P}$ is necessarily a function (so Θ is not necessarily a recursive operator), we can uniformly define $\Psi_{\sigma(i)}$ so that it defines a recursive operator which agrees with Θ on the total functions. This provides us with an effective enumeration $\{\Theta_i\}_{i\in\omega}$ of the recursive operators on $\mathcal{F}$ (the set of total functions), where Θ_i is the recursive operator defined by $\Psi_{\sigma(i)}$.

Definition 10.2 (Medvedev [Me55]). Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems.

(i) $\mathcal{A}$ is *reducible* to $\mathcal{B}$ ($\mathcal{A} \leq \mathcal{B}$) iff $\exists i\,[\Theta_i(\mathcal{B}) \subseteq \mathcal{A}]$.

(ii) $\mathcal{A}$ is *equivalent* to $\mathcal{B}$ ($\mathcal{A} \equiv \mathcal{B}$) iff $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$ (where $\equiv$ is easily seen to be an equivalence relation).

(iii) The resulting equivalence classes are called the *degrees of difficulty* (denoted by M) and have a *reducibility ordering* $\leq$ induced by the reducibility ordering. We call $\boldsymbol{\mathcal{M}} = \langle M, \leq \rangle$ the *Medvedev lattice*.

If $\mathcal{A}$ and $\mathcal{B}$ are mass problems of partial functions, define $\mathcal{A} \oplus \mathcal{B} = \{\varphi \oplus \psi \mid \varphi \in \mathcal{A} \,\&\, \psi \in \mathcal{B}\}$ and, for each $x \in \omega$, $x \circledast \mathcal{A} = \{x \circledast \varphi \mid \varphi \in \mathcal{A}\}$ where $x \circledast \varphi$ is the partial function θ such that $\theta(0) = x$, and $\theta(y) = \varphi(y-1)$ if $y > 0$. Then, denoting the degree of difficulty of $\mathcal{A}$ by $[\mathcal{A}]$, we can define a *meet* $\bigvee$ and *join* $\bigwedge$ on $\boldsymbol{\mathcal{M}}$ by

$$[\mathcal{A}] \bigwedge [\mathcal{B}] = [\mathcal{A} \oplus \mathcal{B}] \quad \text{and} \quad [\mathcal{A}] \bigvee [\mathcal{B}] = [0 \circledast \mathcal{A} \cup 1 \circledast \mathcal{B}].$$

Medvedev [Me55] showed that $\boldsymbol{\mathcal{M}}$ is in fact a distributive lattice with these operations (see [Rog67]), with a least degree **0** consisting of those mass problems containing a recursive function, and a greatest degree **1** (the degree of the empty mass problem). Also, a special source of interest in $\boldsymbol{\mathcal{M}}$, there is a natural monomorphism, preserving least element, of $\boldsymbol{\mathcal{D}}$ into $\boldsymbol{\mathcal{M}}$ (and onto the *degrees of solvability*) got by mapping $\deg_T(A)$ to $[\{\chi_A\}]$. Similarly, $\boldsymbol{\mathcal{D}}_e$ is isomorphically embeddable in $\boldsymbol{\mathcal{M}}$ (onto the *degrees of enumerability*) via the mapping taking $\deg_e(A)$ to $[\{f \mid \operatorname{range} f = A\}]$.

We can extend the above definitions to obtain a classification of mass problems of partial functions:

Definition 10.3 (Dyment [Dy76]). Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems of partial functions. $\mathcal{A} \leq_e \mathcal{B}$ iff

$$(\exists \text{ partial recursive operator } \Theta)(\forall \varphi \in \mathcal{B})[\varphi \in \operatorname{dom}(\Theta) \,\&\, \Theta(\varphi) \in \mathcal{A}].$$

Let $\mathcal{A} \equiv_e \mathcal{B}$ iff $\mathcal{A} \leq_e \mathcal{B}$ and $\mathcal{B} \leq_e \mathcal{A}$.

As before $\equiv_e$ is an equivalence relation and, defining meet and join operations on M_e (the set of equivalence classes under $\equiv_e$, or *degrees of difficulty*) in a similar way as before, we obtain the *Dyment lattice* $\boldsymbol{\mathcal{M}}_e = \langle M_e, \leq_e \rangle$ with least element $\mathbf{0}_e = [\{\varphi \mid \varphi \text{ is p.r.}\}]_e$ and greatest element $\mathbf{1}_e = [\phi]_e$. There is a natural embedding of $\boldsymbol{\mathcal{D}}_e$ onto the *degrees of enumerability in* $\boldsymbol{\mathcal{M}}_e$ (those degrees having the form $[\{\varphi\}]$ for some partial function φ) given by $V_e(\deg_e(\varphi)) = [\{\varphi\}]_e$.

Both structures have many attractive and interesting features, and relate closely to $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{D}}_e$. We can only refer briefly to some of the more recent work in this area. Dyment [Dy76] examines basic questions of incomparability and interpolation. Answering a question of Rogers [Rog67], it is shown that the property of being a degree of solvability in $\boldsymbol{\mathcal{M}}$ is lattice-theoretic: a degree is a degree of solvability if and only if there is a smallest degree among all the degrees greater than it. In [Dy80], Dyment examines analogues of Spector's work on upper bounds for countable ideals in $\boldsymbol{\mathcal{D}}$. One of the original motivations for the study of $\boldsymbol{\mathcal{M}}$ was a relationship with certain nonstandard logics ($\boldsymbol{\mathcal{M}}$ is a Brouwer algebra), which is followed through in a sequence of three papers [Sor*ta*1], [Sor*ta*2] and [Sor*ta*3], in which Sorbi carries through an intensive study of the algebraic properties of $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{M}}_e$. It is shown in [Sor*ta*1] that $\boldsymbol{\mathcal{M}}$ is not a Heyting algebra. In [Sor*ta*2] the author investigates some non-principal filters and ideals of $\boldsymbol{\mathcal{M}}$, as well as the corresponding quotient lattices, and in [Sor*ta*3] studies in more detail the structure of $\boldsymbol{\mathcal{M}}$ as a Brouwer algebra.

There are still many questions concerning the Medvedev and Dyment lattices, including two originally asked by Rogers [Rog67]:

QUESTION 10.4. Is the property of being a degree of enumerability lattice-theoretic in $\boldsymbol{\mathcal{M}}$?

QUESTION 10.5. Is the jump operation on degrees of solvability lattice-theoretic or lattice-theoretic with respect to the degrees of solvability (that is, invariant under all automorphisms of the lattice that carry the degrees of solvability onto the degrees of solvability)?

REFERENCES

Ah89. S. Ahmad, *Some results on the structure of the* Σ_2 *enumeration degrees*, Recursive Function Theory Newsletter **38** (July, 1989), item 373.

Ah*ta*. S. Ahmad, *Embedding the diamond in the* Σ_2 *enumeration degrees*, to appear.

Am*ta*. K. Ambos-Spies, *Minimal pairs for polynomial time reducibilities*, to appear.

AM77. L. Adleman and K. Manders, *Reducibility, randomness, and intractibility*, Proc. 9th ACM STOC (1977), 151-163.

BGS75.T. Baker, J. Gill and R. Solovay, *Relativizations of the P=NP? question*, SIAM J. Comput. **4** (1975), 431-442.

BGO87R. Beigel, W. Gasarch and J. Owings, *Terse sets and verbose sets*, Recursive Function Theory Newsletter **36** (1987), item 367.

Cas85. P. Casalegno, *On the T-degrees of partial functions*, J. Symbolic Logic **50** (1985), 580-588.

Ca71. J. Case, *Enumeration reducibility and partial degrees*, Ann. Math. Log. **2** (1971), 419-439.

Ca74. J. Case, *Maximal arithmetical reducibilities*, Z. Math. Logik Grundlag. Math. **20** (1974), 261-270.

Coo71. S. Cook, *The complexity of theorem proving procedures*, Proc. 3rd ACM STOC (1971), 151-158.

Co72. S.B. Cooper, *Degrees of unsolvability complementary between recursively enumerable degrees, Part I*, Ann. Math. Logic **4** (1972), 31-73.

Co73. S.B. Cooper, *Minimal degrees and the jump operator*, J. Symbolic Logic **38** (1973), 249-271.

Co74. S.B. Cooper, *Minimal pairs and high recursively enumerable degrees*, J. Symbolic Logic **39** (1974), 655-660.

Co82. S.B. Cooper, *Partial degrees and the density problem*, J. Symbolic Logic **47** (1982), 854-859.

Co84. S.B. Cooper, *Partial degrees and the density problem. Part 2: The enumeration degrees of the Σ_2 sets are dense*, J. Symbolic Logic **49** (1984), 503-513.

Co87. S.B. Cooper, *Enumeration reducibility using bounded information: Counting minimal covers*, Z. Math. Logik Grundlag. Math. **33** (1987), 537-560.

Co*ta*1. S.B. Cooper, *The jump is definable in the structure of the degrees of unsolvability*, to appear Bull. Amer. Math. Soc. (1990).

Co*ta*2. S.B. Cooper, *The enumeration degrees are not dense*, to appear.

CC88. S.B. Cooper and C.S. Copestake, *Properly Σ_2 enumeration degrees*, Z. Math. Logik Grundlag. Math. **34** (1988), 491-522.

CLW89 S.B. Cooper, S. Lempp and P. Watson, *On the degrees of d-r.e. sets*, Israel J. Math. **67** (1989), 137-152.

CS*ta*. S.B. Cooper and A. Sorbi, *Initial segments of the enumeration degrees*, to appear.

Cope87 C.S. Copestake, *The Enumeration Degrees of Σ_2 Sets*, Ph.D. Thesis, Leeds University, 1987.

Cope88 C.S. Copestake, *1-genericity in the enumeration degrees*, J. Symbolic Logic **53** (1988), 878-887.

Cope*ta*1 C.S. Copestake, *1-generic enumeration degrees below $\mathbf{0}'_e$*, to appear in the proceedings of Heyting '88, Bulgaria, 1988 (Plenum Press).

Cope*ta*2 C.S. Copestake, *A 1-generic enumeration degree which bounds no minimal pair*, to appear.

Cope*ta*3 C.S. Copestake, *Nondeterminacy, enumeration reducibility and polynomial bounds*, to appear.

Cr53. W. Craig, *On axiomatizability within a system*, J. Symbolic Logic **18** (1953), 30-32.

Da58. M. Davis, *Computability and Unsolvability*, McGraw-Hill, New York, 1958.

Dy76. E.Z. Dyment, *Certain properties of the Medvedev lattice*, Mat. Sb. (new series) **101 (143)** (1976), 360-379 (Russian).

Dy80. E.Z. Dyment, *Exact bounds of denumerable collections of degrees of difficulty*, Mat. Zametki **28** (1980), 899-910 (Russian).

Fe57. S. Feferman, *Degrees of unsolvability associated with classes of formalized theories*, J. Symbolic Logic **22** (1957), 161-175.

Fe65. S. Feferman, *Some applications of the notions of forcing and generic sets*, Fund. Math. **56** (1965), 325-345.

FR59. R.M. Friedberg and H. Rogers, Jr., *Reducibility and completeness for sets of integers*, Z. Math. Logik Grundlag. Math. **5** (1959), 117-125.

Gu71. L. Gutteridge, *Some Results on Enumeration Reducibility*, Ph.D. Dissertation, Simon Frazer University, 1971..

Ha86. C.A. Haught, *The degrees below a 1-generic degree and less than $\mathbf{0}'$*, J. Symbolic Logic **51** (1986), 770-777.

He79a. F. Hebeisen, *Masstheoretische Ergebnisse fur WT-Grade*, Z. Math. Logik Grundlag. Math. **25** (1979), 33-36.

He79b. F. Hebeisen, *Uber Halbordnungen von WT-Graden in e-Graden*, Z. Math. Logik Grundlag. Math. **25** (1979), 209-212.

Hi69. P.G. Hinman, *Some applications of forcing to hierarchy problems in arithmetic*, Z. Math. Logik Grundlag. Math. **15** (1969), 341-352.

Jo66. C.G. Jockusch Jr., *Reducibilities in Recursive Function Theory*, Ph.D. Dissertation, MIT, Cambridge, Mass.,1966.

Jo68. C.G. Jockusch Jr., *Semirecursive sets and positive reducibility*, Trans. Amer. Math. Soc. **131** (1968), 420-436.

Jo73. C.G. Jockusch Jr., *Upward closure and cohesive degrees*, Israel J. Math. **15** (1973), 332-335.

Jo80. C.G. Jockusch Jr., *Degrees of generic sets*, in F.R. Drake and S.S. Wainer, eds., Recursion Theory: its Generalisations and Applications, Proc. of Logic Colloquium, Leeds, 1979 (Cambridge University Press, 1980), 110-139.

Jon73. N.D. Jones, *Reducibility among combinatorial problems in log n space*, in Proc. of Seventh Annual Princeton Conference on Information Sciences and Systems, 1973.

Ka72. R. Karp, *Reducibility among combinatorial problems*, in Miller and Thatcher, eds., Complexity of Computer Computations (Plenum Press, New York, 1972), 85-103.

Kl52. S.C. Kleene, *Introduction to Metamathematics*, Van Nostrand, New York, 1952.

KP54. S.C. Kleene and E.L. Post, *The upper semi-lattice of degrees of recursive unsolvability*, Ann. of Math. (2) **59** (1954), 379-407.

La65. A.H. Lachlan, *Some notions of reducibility and productiveness*, Z. Math. Logik Grundlag. Math. **11** (1965), 17-44.

La66. A.H. Lachlan, *Lower bounds for pairs of recursively enumerable degrees*, Proc. London Math. Soc. **16** (1966), 537-569.

La70. A.H. Lachlan, *Initial segments of many-one degrees*, Canad. J. Math. **22** (1970), 75-85.

La72. A.H. Lachlan, *Embedding nondistributive lattices in the recursively enumerable degrees*, in W. Hodges, ed., Conference in Mathematical Logic, London, 1970 (Springer-Verlag, Berlin, 1972), 149-177.

La79. A.H. Lachlan, *Bounding minimal pairs*, J. Symbolic Logic **44** (1979), 626-642.

LLS75. R. Ladner, N. Lynch and A. Selman, *A comparison of polynomial time reducibilities*, Theoret. Comput. Sci. **1** (1975), 103-123.

Lag72. J. Lagemann, *Embedding Theorems in the Reducibility Ordering of the Partial Degrees*, Ph.D. Dissertation, MIT, 1972.

Le83. M. Lerman, *Degrees of Unsolvability*, Perspectives in Mathematical Logic, Omega Series, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

Lo82. T.J. Long, *Strong nondeterministic polynomial-time reducibilities*, Theoret. Comput. Sci. **21** (1982), 1-25.

McE84.K. McEvoy, *The Structure of the Enumeration Degrees*, Ph.D. Thesis, Leeds University, 1984.

McE85.K. McEvoy, *Jumps of quasi-minimal enumeration degrees*, J. Symbolic Logic **50** (1985), 839-848.

MC85. K. McEvoy and S.B. Cooper, *On minimal pairs of enumeration degrees*, J. Symbolic Logic **50** (1985), 983-1001.

Me55. Yu.T. Medvedev, *Degrees of difficulty of the mass problem*, Dokl. Akad. Nauk SSSR, N.S. **104** (1955), 501-504 (Russian).

MS72. A.R. Meyer and L.J. Stockmeyer, *The equivalence of regular expressions with squaring requires exponential space*, in Proc. Thirteenth Annual IEEE Symposium on Switching and Automata Theory (1972), 125-129.

MS73. A.R. Meyer and L.J. Stockmeyer, *Word problems requiring exponential time*, in Proc. Fifth Annual Symposium on Theory of Computing (1973).

Mo74. B.B. Moore, *Structure of the Degrees of Enumeration Reducibility*, Ph.D. Dissertation, Syracuse University, 1974.

My61. J. Myhill, *A note on degrees of partial functions*, Proc. Amer. Math. Soc. **12** (1961), 519-521.

MSh55.J. Myhill and J.C. Shepherdson, *Effective operations on partial recursive functions*, Z. Math. Logik Grundlag. Math. **1** (1955), 310-317.

NS80. A. Nerode and R.A. Shore, *Second order logic and first order theories of reducibility orderings*, in J. Barwise et al, eds., The Kleene symposium (North-Holland, Amsterdam, 1980), 181-200.

Od81. P. Odifreddi, *Strong reducibilities*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), 37-86.

Od89. P. Odifreddi, *Classical Recursion Theory*, North-Holland, Amsterdam, New York, Oxford, 1989.

Od*ta*. P. Odifreddi, *Classical Recursion Theory, Vol. II*, North-Holland, Amsterdam, New York, Oxford, to appear.

Pl72. G.D. Plotkin, *A set-theoretical definition of application*, Memo. MIP-R-95, School of Artificial Intelligence, University of Edinburgh, 1972.

PR77. E.A. Polyakov and M.G. Rozinas, *Enumeration reducibilities*, Sib. Math. J. **18** (1977), 594-599.

PR79. E.A. Polyakov and M.G. Rozinas, *Relationships between different forms of relative computability*, Math. USSR-Sbornik **35** (1979), 425-436.

Pos77. D. Posner, *High Degrees*, Ph.D. Dissertation, University of California, Berkeley, 1977.

Po44. E.L. Post, *Recursively enumerable sets of positive integers and their decision problems*, Bull. Amer. Math. Soc. **50** (1944), 304-337.

Rob68. R.W. Robinson, *A dichotomy of the recursively enumerable sets*, Z. Math. Logik Grundlag. Math. **14** (1968), 339-356.

Rog67. H. Rogers Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York, 1967.

Roz74. M.G. Rozinas, *Partial degrees and r-degrees*, Siberian Math. J. **15** (1974), 935-941.

Roz78aM.G. Rozinas, *The semilattice of e-degrees*, Ivanov. Gos. Univ., Ivanovo (1978), 71-84 (Russian).

Roz78bM.G. Rozinas, *Partial degrees of immune and hyperimmune sets*, Siberian Math. J. **19** (1978), 613-616.

Sa61. G.E. Sacks, *A minimal degree less than* $\mathbf{0}'$, Bull. Amer. Math. Soc. **67** (1961), 416-419.

Sa63. G.E. Sacks, *Recursive enumerability and the jump operator*, Trans. Amer. Math. Soc. **108** (1963), 223-239.

San78. L.E. Sanchis, *Hyperenumeration reducibility*, Notre Dame J. Formal Logic **19** (1978), 405-415.

San79. L.E. Sanchis, *Reducibilities in two models for combinatory logic*, J. Symbolic Logic **44** (1979), 221-233.

Sas71. L.P. Sasso Jr., *Degrees of Unsolvability of Partial Functions*, Ph.D. Dissertation, University of California, Berkeley, 1971.

Sas73. L.P. Sasso Jr., *A minimal partial degree* $\leq \mathbf{0}'$, Proc. Amer. Math. Soc. **38** (1973), 388-392.

Sas75. L.P. Sasso Jr., *A survey of partial degrees*, J. Symbolic Logic **40** (1975), 130-140.

Sc75a. D. Scott, *Lambda calculus and recursion theory*, in Kanger, ed., Proc. Third Scandinavian Logic Sympos. (North-Holland, Amsterdam, 1975), 154-193.

Sc75b. D. Scott, *Data types as lattices*, in Proc. Logic Conf., Kiel, SVLNM **499** (1975), 579-651.

Se71. A.L. Selman, *Arithmetical reducibilities I*, Z. Math. Logik Grundlag. Math. **17** (1971), 335-350.

Se72. A.L. Selman, *Arithmetical reducibilities II*, Z. Math. Logik Grundlag. Math. **18** (1972), 83-92.

Se78. A.L. Selman, *Polynomial time enumeration reducibility*, SIAM J. Comput. **7** (1978), 440-457.

SS*ta*. J. Shinoda and T.A. Slaman, *On the theory of the polynomial degrees of the recursive sets*, to appear.

Sh59. J.R. Shoenfield, *On degrees of unsolvability*, Ann. of Math. (2) **69** (1959), 644-653.

Sh66. J.R. Shoenfield, *A theorem on minimal degrees*, J. Symbolic Logic **31** (1966), 539-544.

Si*ta*. A. Silver, *Polynomial time singleton enumeration reducibilities*, to appear.

Sk72. D.G. Skordev, *On partial conjunctive reducibility*, in Second All-Union Conference on Math. Logic, Inst. Prikl. Mat. (1972), 43-44.

SW*ta*. T.A. Slaman and W.H. Woodin, *Definability in the degrees*, to appear.

Sol78. B.Ya. Solon, *e-powers of hyperimmune retraceable sets*, Siberian Math. J. **19** (1978), 122-127.

So87. R.I. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, Berlin, New York, London, Tokyo, 1987.

Sor88. A. Sorbi, *On quasi-minimal e-degrees and total e-degrees*, Proc. Amer. Math. Soc. **102** (1988), 1005-1008.

Sor*ta*1. A. Sorbi, *Some remarks on the algebraic structure of the Medvedev lattice*, to appear.

Sor*ta*2. A. Sorbi, *On some filters and ideals of the Medvedev lattice*, to appear.

Sor*ta*3. A. Sorbi, *Embedding Brouwer algebras in the Medvedev lattice*, to appear.

Sp56. C. Spector, *On degrees of recursive unsolvability*, Ann. of Math. (2) **64** (1956), 581-592.

Vu74. V. Vuckovic, *Almost recursivity and partial degrees*, Z. Math. Logik Grundlag. Math. **20** (1974), 419-426.

Wa*ta*. P.Watson, *On restricted forms of enumeration reducibility*, to appear.

Ya65. C.E.M. Yates, *Three theorems on the degree of recursively enumerable sets*, Duke Math. J. **32** (1965), 461-468.

Za84. S.D. Zakharov, *e- and s-degrees*, Algebra and Logic **23** (1984), 273-281.