Computability and Emergence

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There has been much interest in recent years in the way in which global relations on structures emerge. The mathematics underlying such emergence has intimate connections with iterations of algorithms and complexity related to simple computer programs. Examples of the phenomena involved range from the emergence of natural laws in the physical universe, to patterns governing turbulent environments, to the well-known examples of fractal formation.

We look at the extent to which mathematical definability over appropriate structures can provide a key insight into what is happening. In particular, we examine the extent to which Turing's approach to real-world computability is still relevant today, and point to some fundamental questions facing those with a research interest in computability theory.

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1. An Emergent World around Us

There have always been two very surprising things about the world we live in. The first is the observed high level of *regularity* and *form*, on which the success of the scientific enterprise depends. The surface of the ocean, whatever mysteries it hides, shows us easily discernible patterns which reassure and lull us with their familiar motions. Even the complications of human relationships are navigable working within the social rules and conventions established over time.

But — secondly — we are constantly confronted with the unpredictability, the sheer complexity, with which this regularity and form appears to be awash. It used to be said that only in mathematics could one have absolute certainty, but even here one increasingly has to deal with truth as an emergent phenomenon.

Now there is a third mystery to fathom: *There seems to be an intimate relationship between these first two.* Getting a better understanding of that relationship has already grown to be a massive project extending over many areas of scientific and human activity. The logician's talent for bringing out underlying principles and universalities promises to be an essential ingredient in this.

For the most part, the emergence of *Emergence* as something about which everyone has something to say, has generated more questions than answers, and more excitement than clarity. There are numerous popular books on the topic — one of the earliest being John Holland's [**26**] *Emergence – From Chaos to Order* — which tend to add to (and recycle) the store of striking examples of emergence, and expand the range of carefully constrained situations amenable to some level of analysis, or computer simulation. Steve Strogatz's recent *Sync: The Emerging Science of Spontaneous Order* [**44**] is just one example of how fascinating the topic can be, even when the focus is very specific. But this leaves us a long way from an understanding of the emergence of life on earth, of the formation of extra-galactic structures, of the origins of the laws of nature, of the relationship between the quantum and classical worlds, of the evolution of species, of the nature of consciousness as an emergent phenomenon, and of a whole host of more modest examples.

As Holland [**26**] points out:

> *"Despite its ubiquity and importance, emergence is an enigmatic, recondite topic, more wondered at than analyzed."*

This article is an attempt to take an overview of what is happening, and point to some important mathematical tasks which follow directly from this.

2. Descriptions, Algorithms, and the Breakdown of Inductive Structure

What one is typically confronted with is some particular physical system whose constituents are governed by perfectly well-understood basic rules. These rules are usually *algorithmic*, in that they can be described in terms of functions simulatable on a computer, and their simplest consequences are mathematically predictable. But although the global behavior of the system is *determined* by this algorithmic content, it may not itself be recognizably algorithmic. We certainly encounter this in the mathematics, which may be *nonlinear* and not yield the exact solutions needed to retain predictive control of the system. We may be able to come up with a perfectly precise *description* of the system's development which does not have the predictive — or algorithmic — ramifications the atomic rules would lead us to expect.

If one is just looking for a broad understanding of the system, or for a prediction of selected characteristics, the description may be sufficient. Otherwise, one is faced with the practical problem of extracting some hidden algorithmic content, perhaps via useful approximations, special cases, or computer simulations. Geroch and Hartle [**20**] discuss this problem in their 1986 paper, in which they suggest that "quantum gravity does seem to be a serious candidate for a physical theory for whose application there is no algorithm." (Interestingly, Georg Kreisel — see below — is one of those thanked by the authors for their "helpful advice on a preliminary version of this paper.")

For the logician, this is a familiar scenario, for whom something describable in a structure is said to be *definable*. The difference between computability and definability is well-known. For example, if you go to any basic computability text (e.g., Cooper [**5**]) you will find in the *arithmetical hierarchy* a usable metaphor for what is happening here. What the arithmetical hierarchy encapsulates is the smallness of the computable world in relation to what we can describe. And Post's Theorem [**38**] shows us how language can be used to progressively describe increasingly incomputable objects and phenomena within computable structures. An analysis of lower levels of the hierarchy even gives us a clue to the formal role of computable approximations in constraining objects computably beyond our reach.

Of course, there is rather more to it than this extremely schematic picture. Later, we will see how a more detailed analysis of the system resting on some corresponding infinite mathematical structure, as is common in the real world, may lead to a relevant *model*.

Metaphor or model, we would first like to know in general terms what relevance the distinction between definability and computability has for the real world. To do this, we need to look more closely at what it is, in real situations, gives rise to descriptions whose information content is so intrinsically global in character. In the next section I will make more explicit the link between emergence and definability.

The key ingredients of any chaotic environment displaying the sort of emergent relations we are talking about are firstly parallelism — involving three or more component participants — and secondly, interactivity between those participants. Georg Kreisel was brave enough, back in 1970, to propose ([**30**, p. 143]) the possibility of a collision problem related to the 3-body problem which might give "an analog computation of a non-recursive function (by repeating collision experiments sufficiently often)." Turbulence of any kind clearly exhibits these ingredients, echoed by the non-linearity of any mathematical description.

In the biological context, here is how Francisco Varela comments on the significance of his notion of autopoiesis — or self-organization — in Chapter 12 of John Brockman's [**2**] *The Third Culture*:

> *"Regarding the subject of biological identity, the main point is that there is an explicit transition from local interactions to the emergence of the 'global' property that is, the virtual self of the cellular whole, in the case of autopoiesis. It's clear that*

> *molecules interact in very specific ways, giving rise to a unity that is the initiation of the self. There is also the transition from nonlife to life. The nervous system operates in a similar way. Neurons have specific interactions through a loop of sensory surfaces and motor surfaces. This dynamic network is the defining state of a cognitive perception domain. I claim that one could apply the same epistemology to thinking about cognitive phenomena and about the immune system and the body: an underlying circular process gives rise to an emergent coherence, and this emergent coherence is what constitutes the self at that level. In my epistemology, the virtual self is evident because it provides a surface for interaction, but it's not evident if you try to locate it. It's completely delocalized."*

The search for new computational paradigms can help us understand such phenomena. Peter Wegner has particularly focused on the apparent non-algorithmic nature of computations involving these key ingredients — see, for instance, his recent paper [**22**] with Dina Goldin, or his forthcoming edited book [**21**].

It is not at all obvious, however, even in the presence of parallelism and interactivity, that we have something new, something not simulatable by a linear computation. Martin Davis [**13**] has effectively defended the classical model against a number of recently proposed new paradigms (for example, [**8**], [**11**], [**29**]). But new and increasingly convincing tests for the classical model continue to accumulate (for example, in a relativistic context, [**16**]). We obviously need the implied infinities in the underlying mathematical model, otherwise there can be no talk of incomputability. But more than that, it seems we really do need it to be like real science — using real numbers. The model has to be, in some essential way, indiscrete. A well-known feature of the emergence of attractors is their sensitivity to small changes in initial conditions. In fact, it is this feature — that of being far from equilibrium — which becomes itself, for Fritjof Capra in his book [**3**] *The Web of Life*, the third criterion for something new and emergent:

> *"This point is called a 'bifurcation point.' It is a point of instability at which new forms of order may emerge spontaneously, resulting in development and evolution.*
> *Mathematically a bifurcation point represents a dramatic change of the system's trajectory in phase space. A new attractor [fixed point, periodic or strange] may suddenly appear, so that the system's behavior as a whole 'bifurcates,' or branches off, in a new direction. Prigogine's detailed studies of these bifurcation points have revealed some fascinating properties of dissipative structures ... "*

Taking a computability-theoretic perspective, Odifreddi ([**34**, p. 110]) discusses incomputability arising from discrete systems, and paraphrases Kreisel from 1965:

> *"**Thesis P (for 'probabilistic') (Kreisel [1965])** Any possible behavior of a discrete physical system (according to present day physical theory) is recursive."*

As Odifreddi comments, the evidence for or against Thesis P is inconclusive. It may well be that as we become better at modelling and analysing interactive computation, building a repertoire of informative theoretical constructs, and hence narrow the gap between what we observe in nature and what we build in computability theory, Kreisel's thesis will eventually need to be hedged around with qualifications — qualifications which essentially express a historical view of what comprises a 'discrete physical system' (tacit in the statement of Thesis P, even).

What is clear from the mathematics is the simplicity of how incomputability arises from computability. We just take an overview of a sufficiently complex computable function, and what we observe (the range of the function) happens to be not computable. Incomputability is what lies 'at the edge of computability' as much as emergence lies 'at the edge of chaos' — where the high degree of interactivity in the physical situation corresponds naturally to the global observation in the mathematical setting. In both cases, the new level of information content is achieved via a breakdown of inductive structure, a kind of phase transition. To pursue this analogy further one needs to mathematically model the way in which emergent forms feed back into the system, becoming part of the process of 'bootstrapping' remarked on by Varela and others.

A special interest in emergence dates back to the beginnings of computability theory. Alan Turing, with his characteristic knack of fixing on scientific questions with hidden significance, was one of the first to try to say something mathematical about the emergence of form in nature, and wrote seminal papers on the morphogenesis — for example [**49**]. According to Odifreddi, '[Gerald] Edelman quotes Turing as a precursor of his work on morphogenesis'.

Turing also had an interest in emergence in the mind, where the emergent forms far outstrip our ability to describe them mathematically. In 1939 he published a paper, little understood at the time, using the constructive ordinals $\mathcal{O}$ of Church and Kleene to inductively extend theories via Gödel-like unprovable sentences. On these opaque technicalities he was able to base some interesting speculations regarding the non-algorithmic nature of intuition. Here is what Turing ([**47**, p. 134–135]) says about the underlying meaning of his paper:

> *"Mathematical reasoning may be regarded . . . as the exercise of a combination of . . . intuition and ingenuity. . . . In pre-Gödel times it was thought by some that all the intuitive judgements of mathematics could be replaced by a finite number of . . . rules. The necessity for intuition would then be entirely eliminated. In our discussions, however, we have gone to the opposite extreme and eliminated not intuition but ingenuity . . . "*

Here he is addressing the familiar mystery of how we often arrive at a mathematical result via what seems like a very unmechanical process, but then promptly retrieve from this a proof which is quite standard and communicable to other mathematicians. Poincaré was also interested in the role of intuition in the mathematician's thinking. A few years after Turing wrote the above passage, Jacques Hadamard [**24**] recounts how Poincaré got stuck on a problem (the content of which is not important):

> *"At first Poincaré attacked [a problem] vainly for a fortnight, attempting to prove there could not be any such function ... [quoting Poincaré:]*
>
> *Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it ... I did not verify the idea ... I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience sake, I verified the result at my leisure."*

Just as we have been, Turing and Poincaré are talking about the apparent breakdown in the algorithmic glue holding our world together. And what they are pointing to is how this is reflected in the way we come by the descriptions of what is happening. As Holland [**26**, p. 9] comments:

> *" ... developing a theoretical construct in science ... is not a matter of deduction. The standard deductive presentation of theoretical constructs in science hides the earlier metaphor-driven models that lead to the constructs."*

To summarize — what we have noticed so far is the ubiquity of emergent phenomena in nature, and the distance between mathematically describing them, and between extracting predictions from those descriptions we can find. We have examined the parallel gap between mathematical definability and computability, and linked the physical situation to well-known hierarchies of incomputable objects. This parallel was reinforced when we noticed the role played by globality — on the one hand via quantification, on the other as attractors arising from local, but highly interactive, complexity — in the emergence of incomputability and of new physical relations. If we understand more about definability in particular structures, the hope is, we may find general and unifying principles governing the way the world works. Mathematical logic (remembering the seminal [**41**] of chaos theory innovator Robert

Shaw) may even have something to say about something as mundane as a dripping tap!

3. Ontology and Mathematical Structure

We have noticed some basic things about how we extract mathematical descriptions of emergent phenomena. More dramatic in some ways is the observation that descriptions arising from physical structures have themselves a physical reality, whether or not our senses readily confirm that. It is hard to formalize any criterion for distinguishing one particular contingent description as being any more *real* than another. This is nothing to do with old philosophical questions about the intrinsic reality of mathematics. I will keep to areas where a philosophically naïve mathematician can say something useful, and the philosopher can benefit from a mathematical perspective.

We see around us a level of existence, inhabited by us with a reasonable level of success. We are all too aware of other levels, distanced from us by the limitations of reductive science. Our own activities are complex enough to throw up emergent patterns which constrain our lives, but of which we can be imperfectly aware. Here is Hermann Broch, around 1930, on the collective madness of the First World War, and how uncertain a grip on it individual rationality provided (from *Schlafwandler*, translated [**1**] by Willa and Edwin Muir, p.374):

> *"Our common destiny is the sum of our single lives, and each of these single lives is developing quite normally, in accordance, as it were, with its private logicality. We feel the totality to be insane, but for each single life we can easily discover logical guiding motives. Are we, then, insane because we have not gone mad?"*

On the other hand, our everyday physical world which we see around us we now know to be a less-than-solid crust on an ocean of subatomic particles. It is clear that the emergent shapes we live amongst would be as elusive to an observer of subatomic proportions, inhabiting quantum reality, as are the patterns we seek to detect in human history and civilization.

Looking from above, we do believe classical reality to be firmly based on the underlying quantum level, but know better than to try and reduce our everyday problems to this substratum. Analysis at one level is in terms of the relations appropriate to that level, and depends on the algorithmic content of these. We can recognize entities and laws of behavior at the quantum level. But whatever descriptions we can identify relevant to the way the different levels relate, we cannot depend on their predictive content.

Looking from below, even entities and laws are hard to grasp. We certainly cannot "see" them in the way we see our own world. Any school student who has to write an essay on the causes of the First World War is made all too aware of this! The relations by which higher forms are connected are not of our world, and observation of them must be indirect. But they would be entirely real to an inhabitant of that higher world, which is no more surprising than is the sureness with which we move around above our hypothetical subatomic observer. As Holland [**26**] describes it, p. 7:

> *"Persistent patterns at one level of observation can become building blocks for persistent patterns at still more complex levels."*

We now see emergence not just as a producer of unexpected patterns, but as the midwife of different levels of physical reality. And the resulting ontology can be translated into *any* context in which our everyday conceptual framework is stretched or actually fails us. What happens when the familiar laws of nature become invalid, such as near a singularity of a standard model of some aspect of the Universe? How can we say something about the nature of existence itself? If we want mathematical models relevant to such questions, we need to think in very basic terms about existence and its emergent structure.

4. Where Does It All Start?

It is hard to say anything new, unless it comes out of some relatively new piece of knowledge — as in this article, some mathematical notions which have not been widely thought about. Let us begin with the *principle of sufficient reason*, which in the words of Gottfried Wilhelm Leibniz (see [**32**, Secs. 31, 32]) says that:

> *" ... there can be found no fact that is true or existent, or any true proposition, without there being a sufficient reason for its being so and not otherwise, although we cannot know these reasons in most cases."*

The mystery of why anything exists at all is beyond us, of course. This is the ultimate failure of reductionism, something which finds its echo in so many aspects of science and human knowledge in general. Hermann Broch [**1**] describes the role of God here in terms of the non-Euclidean point of intersection of two parallel lines, reducing our changing view to presentational 'style' (p. 426):

> *" ... the First Cause has been moved beyond the 'finite' infinity of a God that still remained anthropomorphic, into a real infinity of abstraction; the lines of inquiry no longer converge on this idea of God ... , cosmogony no longer bases itself on God but on the eternal continuance of inquiry, on the consciousness that there is no point at which one can stop, that questions can forever be advanced, that there is neither a First Substance or a First Cause discoverable, that behind every system of logic there is still a meta-logic, that every solution is merely a temporary solution, and that nothing remains but the act of questioning itself ... "*

But existence itself — of facts or other entities — must take a form inductively determined by the mathematics of its emerging specifics. It must materialize according to sufficient reason, and that mathematics we know tells us that mathematical entities can exist which are not uniquely determined by their context. From which observation, we might expect to encounter realities which are not fully determined, and so materialize according to a range of possibilities. What happens when a particle's existence is guaranteed by 'sufficient reason', but its history is not? Clearly, if there is no pressing reason for a unique history, but there is a pressing reason for history, then it must be non-unique history for which there is sufficient reason. And then, from this perspective, there is nothing mysterious in itself about the parallelism encountered at the quantum level, for instance — so long as we have a convincing mathematical model within which such failures of 'sufficient reason' are expected to occur.

So what do we mean in mathematical terms by 'sufficient reason'? How does a structure determine facts or entities which are not obviously part of the basic knowledge we have of that structure. Clearly, if we can uniquely describe some object or relation in the structure, then, according to what we said in the previous section, it has an existence. And again from what we said before, the corresponding mathematical concept is that of *definability* in an appropriate mathematical model. However, in mathematics language is known to have limited descriptive powers, and our experience of everyday life suggests a similar situation there. In mathematics one can expand ones language, but in human communication there are obvious limitations on the usefulness of this. This does not stop us seeking out extended notions of 'description' to help us make sense of the origins of perceived reality.

There is another way of making mathematically precise what we mean by an aspect of a structure being determined by sufficient reason. Let us first consider a simple example of an organization in which the members each fill their own individual roles. It may be possible to reorganize the membership in such a way that the organization continues to function just as it did before the reorganization - a number of people now do different jobs from previously, but observers of the workings of the organization only notice that certain operatives have different names. It may be though that however one reorganizes, certain people necessarily have to retain their original allocated positions, due to particular personal qualities or expertise, or because of *the relationships of these to the* *other members of the organization.* These people are *invariant* under any such reorganization, so we can reasonably say that that constitutes sufficient reason for them having their designated jobs. The mathematician will instantly recognize in such a reorganization of a structure an *automorphism* — that is, a one-to-one mapping of the structure onto itself which retains all the basic relations between members of the structure. And the notion of invariance need not only be applied to the individuals of the structure, but to any relation on it. For instance, there may be an invariant *set* of members of the structure, which is not moved by any automorphism (although there may be movement within the set). Of course, it may be that a structure has no automorphisms which move anything (all the automorphisms are *trivial*), in which case the structure is said to be *rigid*. Back in the real world, a rigid Universe would be one in which there was no quantum ambiguity, in which every history was uniquely determined by sufficient reason of things necessarily being that way. One can see from this that if we succeed in finding an appropriate mathematical model of the real world, then an understanding of its automorphism group might clarify many mysteries facing scientists.

What we have so far is a mathematical framework within which to set a basic model, but no model as yet. Given a model, the framework promises to go some way towards helping it play a fundamental role in explaining emergence and the familiar fragmentation of scientific knowledge. To get any further, we need to think constructively.

How do we start out from nothing, and end up with a whole Universe? We cannot escape from the mystery of existence, which is not in keeping with our broadly interpreted principle of sufficient reason, which underlies most of our thinking so far. We have to fit our intuitive faith in causality to some assumption about what is given to us without reason. When we say 'without reason' here, we are thinking about immanence, and talking about phenomena originating not from within the known bounds of our universe. And when we say 'given to us without reason', we must also include here rules for actions which violate the principle of sufficient reason.

The pre-scientific scenario was that we were given the world roughly as we see it now, by some divine intervention. This takes many different forms, for instance that of Genesis making some concessions to the modern conception of form created out of formlessness. What at first seems a very different view, that of the quantum theorist who asks us to accept randomness as a given, actually turns out to have a lot in common with this picture. The difference is that in Genesis the formlessness derives information content, and is turned into our familiar classical reality, by a god beyond human comprehension: whereas in the latter scenario the classical world emerges according to various unverified speculations from a quantum world featuring randomness, and an existing high information content. Of course, what is well-known is that randomness is as much a symptom of high (if hidden) information content as is richness of form. And there is no such thing as absolute randomness. Mathematically, randomness is defined relative to the level of information content of those forms it avoids. Since there is no ultimate information content, there is no absolute randomness, but the more randomness displayed, the greater the accompanying information content. You must be very clever to avoid what people expect of you! This makes randomness as a given a vague and unsatisfying assumption, and certainly one which violates our principle of sufficient reason much as Genesis and other creationist conceptions do.

No, if we are to start with little, we must have regularity, predictability, algorithmic content. And as we have seen, it can be a short step from such simple beginnings to complexity and emergent new regularities. And such beginnings are not just capable of providing a basis for real-world-like complexity — they are in keeping with our basic quest for manifest and sufficient reason in all things. Even knowing how far we must be from understanding, or even observing, any 'First Cause', this does not stop us observing the levels of human experience we do have access to, and trying to find unifying principles behind them.

To summarize again: We saw in Secs. 2 and 3 how one could get a better understanding of our experience of the real-world by just looking at the logical framework governing how it is described. To an extent, such things as emergence and the fragmentation of science are aspects of the basic structure of information content. In this section we moved on to attempt to mathematically *build* a universe with some relationship to our own by applying sufficient reason to the germs of an instantiation of such structure.

Of course, underlying this must be some assumptions about the way in which information content is presented (what is information content?) and its relationship to the perceived reality it seeks to capture. Information is what we extract from our experience of the world around us. For the scientist it is framed in terms of real numbers. And it has become increasingly standard practice to view information and the material universe as essentially interchangeable. This is not to say that energy and matter *are* just information — but we find them to be neatly captured in informational packages, which as far as we are concerned, correspond with their physical presences. In mathematical terms, we are talking about the familiar, but far from simple, notion of a *presentation* of a structure. This identification of information and the material universe is a viewpoint that has delivered many valuable insights in both directions, both in information theory and science. The possibility that a true picture of the Universe depends on something that is not describable as 'information', that cannot form part of a model based on information content, cannot be excluded, of course. And this could well be relevant, just as admitting many-worlds can be used to construct a (to some people) satisfying narrative. Or, for that matter, just as the creationist scenario does. But is it *necessary* to admit such a possibility? Let us see if we can run with a mathematical model of an immanently formative Universe which enables us to avoid this level of metaphysics. And let us continue to wield Occham's Razor with gusto, hoping that no damage is done — and that it will make more obvious what sort of structure is most appropriate to describe a germinal version of our universe, and beyond.

5. Towards a Model Based on Algorithmic Content

We want a model that is truly fundamental, that is not already built upon specifics of our universe which we aim to understand better. We want it to capture the essence of how information content is structured in the real world, without losing its wide applicability.

Now, it is common in mathematics to de-emphasize the distinction between object and process. This is seen, for instance, in the construction of certain programming languages, such as LISP, Haskell and Erlang, derived from Church's lambda calculus. There are parallel fluidities in nature, such as between energy and matter, or between waves and particles. However, as participants in the world, the distinction between object and process has an immediacy and qualitative difference which it is not useful to abstract away from. Our reason for maintaining a distinction is based on a need to retain some sense of type structure, where matter is observed, actions need to be predicted. We objectify matter more easily than actions. Our aim is to get a model closer to our experience.

For us, objects present observable, sometimes very complex, information (this is what makes them objects for us), whereas processes are received as relations between entities. Objects we are used to having *high information content*. They are the subjects of events (such as being observed). We try to *predict* processes, to reduce them to simpler components, and are more disconcerted by a lack of algorithmic content. One may say, there is no strict dividing line here, and the differences are ones of degree, but this does not detract from the usefulness of the distiction. Scientifically, the dichotomy is an essential one, and can be made precise. What the working scientist tries to achieve is firstly a presentation of some physical configuration as a real number, and then an algorithm for computing its new value displaced (such as in time or space) by some other appropriate real. In essence, the scientist aims to make predictions in the form of algorithmic relations or functions between reals. If successful, he or she arrives at a relationship between reals which can be computably *approximated* — in that, a close rational approximation to the input to the algorithm yields a correspondingly close approximation to the output. Mathematically, we want our algorithmically described process to be *continuous*. Of course, many processes in nature cannot be presented in such a continuous way. But everything we know (with some notable exceptions at the quantum level, which we will return to later) tells us that these involve non-linear phenomena describable in terms of more basic processes which *are* continuous. What lies behind such phenomena is the infinitary interactivity discussed earlier, and what comes out of our model is a presentation of emergence in terms information-theoretic structure.

All we are trying to do here is to justify a model which gives different roles to information content and relations over it. And which deals with those relations in the first instance in terms of their algorithmic content. One could say more regarding the feasibility and naturalness of doing this. But for now, it suffices to notice that for particular applications the discussion becomes vacuous, and in very general contexts the usefulness of the model motivates a necessary search for greater clarity. In any case, it is useful to have an analysis of the algorithmic content of computationally complex environments, for which one needs to objectify that which is not algorithmic. Then the usefulness of the analysis grows with our conviction of the naturalness of its basis.

We owe to Alan Turing [**47**], who thought a lot about interactive computing, the precise mathematical ideas on which is based the standard model of computationally complex environments. Having described in 1936 [**46**] a mathematical model of mechanical computability, which is still the basis of much of computability and complexity theory, his 1939 paper sought to relate computability and certain describable relations over the natural numbers. In doing so, he allowed his computing machines to interact with an 'oracle', providing an exterior source of information, which may or may not be computable information. These oracle machines, in computing finite pieces of information from finite pieces of information, can be used to compute from reals to reals via suitable corresponding rational approximations. The associated functions are algorithmic, and continuous over the reals. They exactly correspond to how our working scientist aims to capture the algorithmic content of the universe.

Mathematically, Turing's oracle machines give the real numbers an algorithmic infrastructure, which comprises the *Turing universe*. Emil Post [**39**] gathered together the computably equivalent reals of this structure, and called the resulting ordering the *degrees of unsolvability* —

later called the *Turing degrees* — and this has become the mathematical context for the study of the Turing universe.

It is not surprising that attention has turned to Turing's universe of computably related reals as providing a model for scientific descriptions of a computationally complex real universe (see [**4**], [**7**], [**8**], etc.) What is surprising is that it has taken so long to happen — see [**6**] for some comments on why this should have been so. This new interest in the Turing universe is based on a growing appreciation of how algorithmic content brings with it implicit infinities, and, as we have already mentioned, a science — increasingly coming to terms with chaotic and non-local phenomena — necessarily framed in terms of reals rather than within some discrete or even finite mathematical model. However, most of the research activity concerned with the computational significance of evolutionary and emergent form, and emergence in more specific contexts, has inevitably been ad hoc in nature. The potential for drawing out unities and universalities here is as yet almost untapped. Turing's work on emergence of form in nature, and his seminal papers on the topic — for example [**49**] — rather fit this pattern. After 1939, Turing the inventor of oracle computing machines seems to have had no direct impact on his later work on interactive computing and morphogenesis.

Of course, the relevance of such a model in a particular situation depends on the relative importance of specific properties of the algorithmic content present and those common to a wide range of algorithmic structures. It may be that in certain closely constrained situations, the general analysis does little more than provide a conceptual context for results arrived at by non-computability theoretic means.

There are other considerations. For instance, do we want a model which tells us about the local escalation of information content into the incomputable, or one which tells us something global about already computationally complex environments? There is a strand of thought — the hypercomputational, as Jack Copeland and others term it — concerned with contriving incomputability via explicitly physical versions of the Turing universe. What is common to both the computability-theoretic and hypercomputational strands is that both the emergence of incomputability, and the emergence of new relations in a universe which admits incomputability, are based on a better understanding of how the local and the global interact. Whatever the context, the key mathematical parallel here is that of definability or invariance, even if within rather different corresponding frameworks. This is not very explicit in building hypercomputational models, which enables Martin Davis [**13**] and others to trivialise what is happening as being the use of oracles to shuffle around existing incomputability.

It is not surprising that the human mind points us in the direction of particular hypercomputational models.

Speculations regarding the potential of new connectionist theories to transcend the classical McCulloch and Pitts [**33**] artificial neuron formalism have been around for some time — for instance, in 1988 Smolensky [**42**, p. 3] observed:

> *"There is a reasonable chance that connectionist models will lead to the development of new somewhat-general-purpose self-programming, massively parallel analog computers, and a new theory of analog parallel computation: they may possibly even challenge the strong construal of Church's Thesis as the claim that the class of well-defined computations is exhausted by those of Turing machines."*

Turing himself anticipated the importance now given to connectionist models of computation — see his discussion of "unorganized machines" in [**48**], and Jack Copeland and Diane Proudfoot's article [**10**] "On Alan Turing's Anticipation of Connectionism." What we are seeing now for the first time is the adoption of Turing's own oracle model of interactive computation in a real world setting, and the enlisting of the explanatory power of the mathematical theory of the Turing universe based on that model.

The fact that the hypercomputational case has brought us little but confusion so far does not mean that it is a bad project. What it may mean is that on the one hand the mathematical tools for analysing hypercomputational proposals are just too crude, that those who have to evaluate the results of their application are either lacking mathematical sophistication in some part, or have less expertise in thinking about the real world than they have in the rigours of recursion theory — and most importantly, that there is still thinking to be done before vague intuitions can be convincingly communicated. The picture we carry forward from our earlier discussions is of processes operating over some raised level of information content, sufficient to give us a stucture in which new relations can be described. This picture is hierarchical, information defined within structures which can only be fully contained within new structures. The Turing model may be appropriate for clarifying some big scientific mysteries, but maybe needs refining for a better understanding of hypercomputation, perhaps incorporating little-world constraints on time and space. And at the level of human affairs, which can never be successfully captured by our scientist working over the reals, maybe arguments from analogy need to be carried forward to a more detailed consideration of definability over structures based on even higher information content.

6. Levels of Reality

But let us now return to what the Turing model can do. Let us try to be more clear about how, from very simple beginnings, we can get from the basic fact of existence to what is for us an even greater puzzle — because we have to take what is happening under the umbrella of sufficient reason — the quite amazing emergence of individual entities. From this point of view, it is not quantum ambiguity which is surprising, but the existence of the well-defined world of our everyday experience.

More generally, we have the problem that even though we have natural laws to help us understand much of what happens in the universe, we have no idea where those laws themselves come from. Their apparent arbitrariness lies at the root of the present day confusion of speculative science, verging on the metaphysical.

For Alan Guth [**23**], the problem is:

> *"If the creation of the universe can be described as a quantum process, we would be left with one deep mystery of existence: What is it that determined the laws of physics?"*

While Roger Penrose [**35**] asks for a *strong determinism*, according to which (pp. 106–107):

> *" . . . all the complication, variety and apparent randomness that we see all about us, as well as the precise physical laws, are all exact and unambiguous consequences of one single coherent mathematical structure."*

Science has come a long way since David Hume first set out to 'enquire how we arrive at the knowledge of cause and effect', and insisted, in *An Enquiry Concerning Human Understanding*, that:

> *"I shall venture to affirm, as a general proposition, which admits of no exception, that the knowledge of this relation is not, in any instance, attained by reasonings a priori, but arises entirely from experience, when we find that any particular objects are constantly conjoined with each other. Let an object be presented to a man of ever so strong natural reason and abilities; if that object be entirely new to him, he will not be able, by the most accurate examination of its sensible qualities, to discover any of its causes or effects. Adam, though his rational faculties be supposed, at the very first, entirely perfect, could not have inferred from the fluidity and transparency of water that it would suffocate him, or from the light and warmth of fire that it would consume him. No object ever discovers, by the qualities which appear to the senses, either from the causes which produced it, or the effects which will arise from it; nor can our reason, unassisted by experience, ever draw any inference concerning real existence and matter of fact."*

Hume may still be right, but the match between mathematics and experience has become more all-embracing, with string theory perhaps the most ambitious of the attempts to unify the two. The Turing model may be as yet very far from clarifying the specific details of relativity or quantum theory, but it does promise a release from the arbitrariness to which all less basic theories — superstring theory, M-theory, inflation, decoherence, the pilot wave, gauge theory, etc. — are subject, and is based almost entirely upon experience.

What is specially relevant here is that, far from Hume's comments causing problems for us, they can be used to clarify not just how we 'draw any inference concerning real existence and matter of fact', but, further, how in general entities and relations derive existence from their global context. Reading Hume, we find a graphic description of how we derive predictive patterns from observations of events. We recognize a parallel between how we know things — a process of *definition* by accumulated experience, of establishing an *invariance* emerging from various possibilities — and in the way the Universe can 'know' itself, and immanently establish its own structure and relations. Although what the Turing model primarily tells us about is not an emergence of particular events from events, but of natural laws from the structure of information content.

What does the Turing model suggest regarding the basic structure of matter and the laws governing it? Let us review some of the ground covered in more detail in [**7**].

What we know of the Turing universe is consistent with the possibility that the information content or level of interactivity of a given entity may be insufficient to guarantee it a unique relationship to the global structure. This is what one might expect to apply at an early stage in the development of the universe, or at levels where there is not a sufficiently density of interactions to give information a global role. A number of classic experiments on subatomic particles confirm such a prediction. On the other hand, mathematically entangling such low level information content, perhaps with content at levels of the Turing universe at which rigidity sets in, will inevitably produce new content corresponding to a Turing invariant real. The prediction is that there is a level of material existence which does not display such ambiguity as seen at the quantum level, and whose interactions with the quantum level have the effect of removing such ambiguity — confirmed by our everyday experience of a classical level of reality, and by the familiar 'collapse of the wave function' associated with observation of quantum phenomena. Since there is no obvious mathematical reason why quantum ambiguity should remain locally constrained, there may be an apparent non-locality attached to the collapse. Such a non-locality was first suggested by the well-known Einstein-Podolsky-Rosen thought experiment, and, again, has been confirmed by observation. The way in which definability asserts itself in the Turing universe is not known to be computable, which would explain the difficulties in predicting exactly how such a collapse might materialize in practice, and the apparent randomness involved.

One might hope that in the course of time the theory of Turing definability might explain aspects of subatomic structure. A conjecture is that when one observes atomic structure, one is looking at *relations* defined on some lower level of matter lacking any sort of observable form, out of which arise peaks of definability observed by us as subatomic particles. This may even lead to a theoretical explanation of 'dark matter'. Until such matter is organized into relations, of which particles are the instantiations, we have no structure capable of being interacted with. It would be as alien to the world of particle physics as that world is to our classical level of human existence.

As we have already mentioned, the Turing model may have implications for how the laws of nature immanently arise. And also how they collapse near the big bang 'singularity', and the occurrence or otherwise of such a singularity. What we have in the Turing universe are not just invariant individuals, but a rich infrastructure of more general Turing definable relations. These relations grow out of the structure, and constrain it, in much the same sort of organic way observable in familiar emergent contexts. These relations operate at a universal level. The prediction is that a Universe *with sufficiently developed information content* to replicate the defining content of the Turing universe will manifest corresponding material relations. The existence of such relations one would expect to be susceptible to observation, these observations in turn suggesting regularities capable of mathematical description. And this is what the history of science confirms. The conjecture is that there is a corresponding parallel between natural laws and relations which are definable in an appropriate fragment of the Turing universe.

The early Universe one would not expect to replicate such a fragment. The homogenization and randomization of information content consequent on the extreme interconnectivity of matter would militate against higher order structure. The manifest fragment of the Turing universe, based on random reals, might still contain high information content, but content dispersed and made largely inaccessible to the sort of Turing definitions predicted by the theory. Projected singularities, such as within black holes or associated with boundary states of the Universe, depend on a constancy of the known laws of physics. But immanently originating laws must be of global extraction. This means that their detailed manifestations may vary with global change, and disappear even.

Notice the difference here between what we are saying, and what the upholders of the various versions of Everett's many worlds scenario are. On the one hand, we have an application of the principle of sufficient reason to the world as we know it, which gives a plausible explanation of quantum ambiguity, the dichotomy between quantum and classical reality, and promises some sort of reconciliation between science, the humanities, and our post-modern everyday world. On the other we have something more like metaphysics.

The Turing model, and its connections with emergence, also lead us to expect the familiar fragmentation of science, and human knowledge in general. As we know from computability theory, a Turing definition of a given relation does not necessarily yield a computable relationship with the defining information content. But working within the relations at a given level, there may well be computable relationships emerging, which may become the basis for a new area of scientific investigation. For instance research concerning the cells of a living organism may not be usefully reduced to atomic physics, but deals with a higher level of directly observed regularities. Sociologically, one studies the interactions governing groups of people with only an indirect reference to psychological or biological factors. Entire relations upon cells (humans) defined in some imperfectly understood way by the evolutionary process provide the raw material underlying the new discipline, which seeks to identify a further level of algorithmic content. This algorithmic content may not be directly expressed in terms of numbers. But inasmuch as the area in question does have basic notions, corresponding to the new emergent relations, shared by workers in the field, and descriptions of entities and regularities are formulated in a shared language, the algorithmic content is not dissimilar in kind to that at lower levels.

In [**4**] we mentioned a number of areas in which one can observe qualitatively similar problems, all connected with parallel issues of definability and nonrigidity. One example is that of the origin of life on Earth. Another concerns the exact nature of evolution — as Stuart Kauffman [**28**] observes (p. 644):

> *"Evolution is not just 'chance caught on the wing.' It is not just a tinkering of the ad hoc, of bricolage, of contraption. It is emergent order honored and honed by selection."*

There is the mysterious emergence of large scale structure in the Universe. Also in the 1999 paper is a section on epistemological relativism. There is a basic intuition that an analysis of the epistemology derived from our Universe is potentially just as complex as that of the Universe itself. So it should not be surprising that emergence and the mathematics of definability should be relevant here. And there is the whole question of the nature of human thought processes, touched on earlier.

There are questions about the range of possibilities embodied in such things as quantum ambiguity: Going from the uniqueness of a defined phenomenon to — what? Are there any overall constraints apart from those imposed by the mathematics specific to the emergent structures? There seems to be one unavoidable rule — obvious when it is pointed out — which is that each superimposed alternative must be viable by itself. Which, in addition to the specifics, demands that the information content develops within the rules experience and the computability theory lead us to expect. In particular, there can be at most countably many such alternatives. It is known that there exist at most countably many Turing automorphisms.

What may be most important of all, though, is the way we get a new model, replacing the one Laplace's predictive demon gave us [**31**] around 200 years ago:

> *"Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective*

situations of the beings who compose it — an intelligence sufficiently vast to submit these data to analysis — it would embrace in the same formula the movements of the greatest bodies and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes."

We now see not only the bottomless mystery in Broch's point at infinity where foundations fail, but a hierarchy of lesser mysteries rising through the entire structure. The idea of a controlling god makes no sense in any context, but, taking god to be the embodiment of an unpredictable creative principle in all things, that of a ubiquitous god does. And, whether we are religious or not, man is indeed made in the image of this god, a microcosm of the wider universe, part of and contributor to its emergent creativity. The all-understanding humanity of enlightenment science may be dead, but the vitality of our participation in the world takes on a new life.

7. Algorithmic Content Revisited

The reader may still be left with some basic objections, not just to what we have been saying, but to the very advocacy of computability theory as applied science. These basic doubts are not going to be dissipated by specific examples of the usefulness of the techniques of computability, such as Robert Soare's beautifully presented paper [**43**] on *Computability theory and differential geometry*. One can address particular objectives — in this case a rebuttal of *Simpson's Thesis*, clarified by S. GṠimpson in a *Foundations of Mathematics (FOM) Network* e-mail communication on Aug. 4, 1999:

> "The concise statement of Simpson's Thesis is:
>
> *Priority methods are almost completely absent from applied recursion theory.*"

One just ends up like the boy trying to seal the breach in the dyke with his finger. Paradigm changes depend on a number of ingredients being in place, including the unifying concept behind the individual pieces of evidence.

For those with a finitist view of the Universe, almost everything said, right from the beginning, will be at best irrelevant, including any argument about the role of priority methods. It is hard to dislodge an outlook traceable back to the beginnings of science. Here is Archimedes in the introduction to *The Sand Reckoner*:

> *"Many people believe, King Gelon, that the grains of sand are infinite in multitude; and I mean by the sand not only that which exists around Syracuse and the rest of Sicily, but also that which is found in every region, whether inhabited or unhabited. Others think that although their number is not without limit, no number can ever be named which will be greater than the number of grains of sand. But I shall try to prove to you that among the numbers which I have named there are those which exceed the number of grains in a heap of sand the size not only of the earth, but even of the universe."*

But — despite the fact there are probably less than 10^{87} particles in the universe — for most of us, a finite model of the universe will not do, as the sort of things said in Secs. 2 and 3 should hopefully persuade all but the most incorrigible finitists. But will not, of course, in view of the already extensive literature on the topic!

It is worth trying to be a bit more explicit, though. We argued that presentations of aspects of the universe lead us to particular mathematical models. And that if the model fits closely enough, things describable in terms of that model can be expected to be aspects of the original physical situation, maybe not visible to us, but a very real element in us developing an understanding of that system. How can it be that a structure which masquerades as being finite, on closer inspection necessarily needs an infinite structure to explain it? The key ingredient is algorithmic content, and this derives from our basic principle of sufficient reason. Despite Humean caution, we can go beyond a purely fortuitous link between experience and form.

How does one envisage a germinal Universe, involving minimal information content, in which something recognizable as 'events' occur — which, we may suppose, also have the most basic information content imaginable. In such an impoverished (but very strange!) environment, there cannot be 'sufficient reason' for diversity within (or for unique manifestation of) particular modes of event. In other words, we already find it hard to avoid (there is just not enough information content) to make particular kinds of development non-uniform. But our only constraint on the actuality of this nascent structure is its mathematics, a mathematics which has a general applicability to similar structures, and to this structure at similar stages of its development. And the mathematics of such structures with such a uniformity of infrastructure is what we can only characterize as *algorithmic*. Of course, the mathematics may not actually give us well-defined events. But even the ambiguity must be uniformly instantiated.

So the close relationship between the mathematics and its real-world avatar entails a Universe which is not just hard to understand apart from its algorithmic content, but which actually *embodies* algorithmic content. As Hume would have us know, the exact nature of that algorithmic content may be beyond reason (despite the advances in mathematics and science since his time), but his vision of how we do know things presages a mathematical model of how entities develop in more general contexts. By so closely following his analysis, we come up with a mathematics he would find hard reject the relevance of.

Why do the laws of physics appear so uniform throughout the Universe? Why do they appear to be algorithmic in effect? The more interesting question is: How could they be otherwise?

So the finite model is not just impractical, it fails to describe what is happening. Neither, we suspect, can science live with its close relation, the discrete model. Even if Richard Feynman did suggest [**17**], after a scientific lifetime working with mathematics over the reals, the following radical resolution of the uneasy relationship between reality and its discrete representations:

> *"It is really true, somehow, that the physical world is representable in a discretized way, and ... we are going to have to change the laws of physics."*

But even if Feynman were right applied computability theory is not affected, at least until we gather more convincing evidence for Kreisel's Thesis P.

8. What Is to Be Done?

The theory of Turing definability is a notoriously difficult and dangerous area of research. It is the mathematical equivalent climbing Everest's Kangshung face or K2's Magic Line (and you can fall off). So far, we have only achieved a glimpse of the richness of structure hidden there, a fitting counterpart to that of the real world. As we observed in [**6**], the complexity of the Turing model was not always seen in this light. In the late twentieth century, relative computability became area of research known for its mathematical unlovelyness and forbidding pathology. Its relevance was not at all clear during the recursion theoretic years. The difficulty of the area may have surrounded researchers of the 1960s — pre-eminently Gerald Sacks — with a vaguely heroic, even machismatic, aura. But as time went on this had become a double-edged weapon, and by the 1990s almost no one was impressed by the length and incomprehensibility of groundbreaking new proofs. "Touching the Void" — and having accidents — was all very well for mountaineers but, as the new century approached, mathematics was very much about deliverables. At times the very value of research into relative computability was questioned. Images of such dissent stick in the memory: Sacks himself, lecturing at Odifreddi's CIME summer school in Bressanone, Italy in 1979, illustrating his view of 'Ordinary Recursion Theory' with a slide of the Chinese masses in cultural revolution turmoil — his metaphor for an activity obsessive, formless, pointless; or, ten years later, Robin Gandy's contribution to a discussion on the future of logic, at a conference in Varna, Bulgaria — communicating an impression of the structure of the Turing degrees via exaggeratedly desperate scribbles on a blackboard.

Things have changed, and we have what we described in [**6**] as a 'Turing renaissance'. What is currently so exciting is that the sorts of questions which preoccupied Turing, and the very basic extra-disciplinary thinking which he brought to the area, are being revisited and renewed by researchers from quite diverse backgrounds. What we are seeing is an emergent coming together of logicians, computer scientists, theoretical physicists, people from the life sciences, and the humanities and beyond, around an intellectually coherent set of computability-related problems. The recurring and closely linked themes here are the relationship between the local and the global, the nature of the physical world, and within that the human mind, as a computing instrument, and our expanding concept of what may be practically computable.

The specific form in which these themes become manifest are quite varied. For some there is a direct interest in incomputability in Nature, such as that coming out of the n-body problem or quantum phenomena. For others it is through addressing problems computing with reals and with scientific computing. The possibility of computations 'beyond the Turing barrier' leads to the study of analog computers, while theoretical models of hypercomputation figure in heated cross-disciplinary controversies. There is also intensive research going on into a number of practical models of natural computing, which present new paradigms of computing whose exact content is as yet not fully understood. In many scientific areas it is the emergence of form which is deeply puzzling, and, as we have described above, there is a key role here for the sort of mathematical models we have been discussing.

However, what we have described so far has been largely twentieth-century mathematics, even if many of the ingredients only appeared in the final decade of the last century. Many of the key ideas were described in the 1999 paper [**4**], while the joint paper of Cooper and Odifreddi [**7**] was largely concerned with clarification and elaboration, and with reining in the ambition of that earlier paper. This article is to some extent a further step in that direction. We have dwelled on some basic issues concerning the link between definability and emergence, but for the big picture, [**4**] is still indispensible, in that it draws together so many strands in contemporary science.

We will finish with something very new. Not by any means a retreat, but a making more explicit of some of the limitations of the classical theory of the Turing universe as a model for emergence. And a pointing to other areas in which computability theory can adapt to once again address basic scientific issues. We have pointed to important questions regarding computability theoretic structure and emergence, but not all these relate to standard structures. There appears to be little alternative to the Turing model in relation to ontology and other fundamental questions regarding the origins of the material universe and the emergence of natural laws. There seems to be a specially fundamental role for this analysis in throwing new light on basic puzzles concerning the exact role of entropy, and other areas where thinking is unsatisfyingly ad hoc and bound by scientific cliché. And it is certainly true that at one level, substructures of the Turing model can provide an instantiation of many emergent phenomena. But this is not a useful model for prediction of detail, any more than classical computability addresses practical computational questions in a direct way. What we get is a context, a conceptual resource, a formative influence on the scientific culture, the big picture in the way we expect from logic and philosophy, deep and essential insights — and theoretical foundations on which important practical developments can be based. Of course, Alan Turing's 1936 paper [**46**] did all that.

In [**6**], we made some comments on how in the years following Turing's paper, computability theory became dominated by mathematical concerns which, while stimulating necessary technical developments, took the area away from its real-world context. One early decision [**39**] was to view reducibilities in terms of degree structures. Mathematically, this enabled the new area to be developed in the context of familiar structures such as partial orderings, upper semi-lattices and Boolean algebras. Reducibilities which were not transitive were ruthlessly discarded, despite the fact that in real-life computation, transitivity commonly fails. In fact, part of the puzzling non-locality posed by the EPR thought experiment of Einstein, Podolsky and Rosen [**15**] comes from just such a non-transitivity in relation to events which can be connected in real time and space. The problem is that if one were to take seriously non-transitive reducibilities which correspond to what we meet in the real world, we would have to develop not just new computability-theoretic structures, but new mathematical abstractions with no existing theory. Our orderings would be non-transitive, while our metrics would be non-symmetric — something which physicists, significantly, sometimes talk about. But the computability theory related to such structures does not yet exist. This is a major project, but potentially of great importance. One cannot even begin to imagine how definability in such structures might turn out, or what automorphisms might look like. But one can be confident that the classical theory will continue to play an important role, and to technically underpin new developments. And one can expect to get an even closer relationship between the structures of computability and complexity theory, and their real-life avatars.

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