

# THE MINIMAL E-DEGREE PROBLEM IN FRAGMENTS OF PEANO ARITHMETIC

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ABSTRACT. We study the minimal enumeration degree (e-degree) problem in models of fragments of Peano arithmetic ( $PA$ ) and prove the following results: In any model  $\mathcal{M}$  of  $\Sigma_2$  induction, there is a minimal enumeration degree if and only if  $\mathcal{M}$  is a nonstandard model. Furthermore, any cut in such a model has minimal e-degree. By contrast, this phenomenon fails in the absence of  $\Sigma_2$  induction. In fact, whether every  $\Sigma_2$  cut has minimal e-degree is independent of the  $\Sigma_2$  bounding principle.

## 1. INTRODUCTION

This paper is motivated by the study of Turing degrees in fragments of Peano arithmetic ( $PA$ ), a subject which was developed in the 1980's. We recommend Chong and Yang [4] [5] for the basic notions and results, as well as some open problems in this subject. The study of enumeration degrees in  $PA$  is however relatively new. In this paper we take the first step by investigating the existence of minimal degrees in this context.

There is a historical significance to this choice. The minimal degree problem has inspired much work in ordinal recursion theory. Its resistance to a complete solution underscores the general difficulty of executing priority arguments in weaker systems, whether over an ordinal satisfying a restricted replacement axiom, or over fragments of arithmetic with a limited induction scheme.

Spector [15] showed the existence of minimal Turing degrees. This was later improved by Sacks [12], who constructed a minimal degree

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below  $\mathbf{0}'$ . The situation in enumeration degrees is quite different: Gutteridge [10] in his thesis showed that there is no minimal e-degree (see also Cooper [6]). The major difference between Turing reducibility and enumeration reducibility, which leads to different conclusions regarding minimal degrees, is that unlike Turing functionals, enumeration operators only deal with positive information. There is also an asymmetry between  $\Sigma_n$  and  $\Pi_n$  sets in the setting of enumeration degrees — a feature which also appears in models of fragments of  $PA$  satisfying the  $\Sigma_n$  bounding principle ( $B\Sigma_n$ ) as against the  $\Pi_n$  bounding principle ( $B\Pi_n$ ). However, when one investigates proof-theoretic strengths of theorems in both of these degree structures, there do exist similarities, as the results of this paper will illustrate.

Chong and Mourad [1] studied the minimal Turing degree problem in fragments of  $PA$ . They showed that in any nonstandard saturated model, the degree of a cut is minimal. This applies in particular to those which satisfy  $B\Sigma_n$  but not  $\Sigma_n$  induction ( $I\Sigma_n$ ). It is still an open question whether a minimal degree exists in any  $B\Sigma_2$  (nonsaturated) model.

In this paper, we show that a necessary and sufficient condition for a model  $\mathcal{M}$  of  $I\Sigma_2$  to have a minimal e-degree is that  $\mathcal{M}$  is nonstandard. Hence ‘e-minimality’ may be viewed as essentially a set-theoretic notion rather than a recursion-theoretic notion. Indeed our argument proves more: In a nonstandard model of  $I\Sigma_2$ , any cut is of minimal e-degree. By contrast, this situation fails under the weaker assumption of  $B\Sigma_2$ . The existence or nonexistence of codes of certain  $\Sigma_2$  sets turns out to be the differentiating factor.

After a brief introduction of the basic concepts, we show in Section 3 that Gutteridge’s theorem holds for regular degrees (i.e., degrees which contain regular sets) in any model of  $B\Sigma_2$ . The corollary is that minimal e-degrees are nonregular degrees (Corollary 3.1). In Section 4 we strengthen this observation by proving (Corollary 4.5) that any cut in a nonstandard model of  $I\Sigma_2$  has minimal e-degree. Finally in Section 5 we exhibit (Theorem 5.1 and Theorem 5.2 respectively) two models of  $B\Sigma_2$ : one in which a  $\Sigma_2$  cut has minimal e-degree and one in which no  $\Sigma_2$  cut has minimal e-degree. The paper concludes with some open problems.

## 2. PRELIMINARIES

In this section, we describe briefly the basic facts relating to fragments of  $PA$  and computability theory referred to in the sequel. Further materials may be found in Chong and Yang [4]. For basic computability theoretic notation see Cooper [8] or Soare [14].

A model  $\mathcal{M}$  in this paper satisfies  $PA^- + I\Sigma_0$ .  $\mathcal{M}$  is a  $B\Sigma_n$  model if it satisfies  $B\Sigma_n$  but not  $I\Sigma_n$ . We are primarily interested in the case when  $n \geq 2$ . A feature of all  $B\Sigma_n$  models is the existence of a  $\Sigma_n$  cut  $I$ , which is a nontrivial subset of  $\mathcal{M}$  closed downward and under the successor function.

**Lemma 2.1.** *Let  $\mathcal{M}$  be a  $B\Sigma_n$  model. Then there is a  $\Sigma_n$  cut  $I$  and a  $\Sigma_n$  map  $f : I \rightarrow M$  whose range is unbounded in  $\mathcal{M}$ .*

In any model  $\mathcal{M}$ , a set is  $\mathcal{M}$ -finite if and only if it has a code in  $\mathcal{M}$ . We will not distinguish an  $\mathcal{M}$ -finite set from its code (a number in  $\mathcal{M}$ ) if the context is clear. We denote by  $[0, a]$  the set  $\{x \in \mathcal{M} : x \leq a\}$ . A set  $A \subseteq \mathcal{M}$  is *regular* if for every  $m$  in  $\mathcal{M}$ ,  $A \cap [0, m]$  is  $\mathcal{M}$ -finite.

**Lemma 2.2.** *Assume that  $\mathcal{M}$  is a model of  $PA^- + I\Sigma_n$ , ( $n \geq 1$ ).*

- (1) *If  $A$  is  $\Sigma_n$  in  $\mathcal{M}$ , then  $A$  is regular;*
- (2) *If  $f$  is a partial  $\Sigma_n$  function whose domain is bounded, then the range of  $f$  is also bounded.*

The following definitions and lemma capture the essence of coding in  $B\Sigma_n$  models. Details can be found in [2].

**Definition 2.1.** *Let  $A$  be a subset of  $\mathcal{M}$ . A set  $X \subseteq A$  is coded on  $A$  if there is an  $\mathcal{M}$ -finite set  $K$  such that  $K \cap A = X$ .*

For example, if  $X$  is coded on a cut  $I$  then  $X$  is an initial segment of an  $\mathcal{M}$ -finite set.

**Definition 2.2.** *Let  $A$  be a subset of  $\mathcal{M}$ . We say that a set  $X$  is  $\Delta_n$  on  $A$  if both  $A \cap X$  and  $A \cap \bar{X}$  are  $\Sigma_n$ .*

**Lemma 2.3** (Chong and Mourad). *Let  $\mathcal{M}$  be a model of  $PA^- + B\Sigma_n$  and  $A$  be an arbitrary subset of  $\mathcal{M}$ . Then every bounded set which is  $\Delta_n$  on  $A$  is coded on  $A$ .*

Consequently, if  $\mathcal{M}$  is a model of  $PA^- + B\Sigma_n$ , then any  $\Delta_n$  subset of  $\mathcal{M}$  is regular.

We now turn to the discussion of enumeration reducibility and enumeration degrees in fragments of arithmetic. For more information on classical e-degree structures, we refer the reader to Cooper [7].

An *enumeration operator* (or *e-operator*) is a computably enumerable set  $\Psi$  such that

$$\Psi(B) = \{x : (\exists F)(\langle x, F \rangle \in \Psi \wedge F \subseteq B)\}$$

where  $F$  ranges over  $\mathcal{M}$ -finite sets. For any set  $A$  and  $B$  we say that  $A$  is *weakly* (or *pointwise*) *enumeration reducible to  $B$*  (or  $A$  is *weakly e-reducible to  $B$* , written  $A \leq_{e,w} B$ ) if  $A = \Psi(B)$  for some e-operator

$\Psi$ . We say that  $A$  is *strongly e-reducible* to  $B$  (written  $A \leq_{e,s} B$ ) if the set  $\{P : P \text{ is } \mathcal{M}\text{-finite and } P \subseteq A\}$  is weakly e-reducible to  $B$ .

If the oracle set  $A$  in  $\Phi(A)$  is a cut, then we may assume that the  $\mathcal{M}$ -finite set  $F$  in the definition of  $\Phi$  is a singleton  $\{a\}$  which is its maximum element. We write  $\langle x, a \rangle$  instead of  $\langle x, \{a\} \rangle$  in this case.

In fragments of  $PA$ , it is often necessary to distinguish between the notions of “weak” and “strong” reducibilities, as shown by Groszek and Slaman [9] for Turing degrees. The following propositions hold for e-reducibility and their proofs are straightforward.

**Proposition 2.1.** *Let  $\mathcal{M}$  be a model satisfying  $PA^- + B\Sigma_1$ . Then strong e-reducibility is a transitive relation on subsets of  $\mathcal{M}$ .*

**Proposition 2.2.** *Let  $\mathcal{M}$  be a model satisfying  $PA^- + B\Sigma_n$ . Then strong and weak e-reducibilities coincide for  $\Sigma_n$  sets. Hence both reducibilities are transitive.*

Thus the notion of an e-degree is well-defined in relation to strong reducibility. It is not clear if weak e-reducibility is transitive.

### 3. NONMINIMALITY OF REGULAR SETS

Gutteridge’s theorem states that there is no minimal e-degree for the standard model. In this section we show that this result applies to regular sets in all models of  $B\Sigma_2$ . The proof makes use of the idea presented in Cooper [6], coupled with the blocking method to carry out an  $\mathcal{M}$ -finite injury priority argument (of the unbounded type) over  $B\Sigma_2$  models. It consists of two parts: Firstly we show that there is no  $\Delta_2$  minimal e-degree in any model  $\mathcal{M}$  of  $B\Sigma_2$ ; secondly we show that any candidate for a regular minimal e-degree (i.e., one that contains a regular set) must be  $\Delta_2$ . The latter is actually provable under  $\Sigma_1$  induction. It follows as a corollary that in  $\mathcal{M}$ , if there is a minimal e-degree, then it is neither regular nor  $\Delta_2$ .

Recall that an *s-operator* (*singleton operator*) is an e-operator of the form

$$\Theta = \{\langle \langle i, j \rangle, \emptyset \rangle : \text{for some } i, j\} \cup \{\langle \langle i, j \rangle, \{j\} \rangle : \text{for some } i, j\}.$$

We say that  $\langle i, j \rangle$  is *crossed* in the former case, and is *ticked* in the latter case.

Clearly, if  $\Theta$  is an s-operator, then for any set  $B$ ,

$$\Theta(B) = \{\langle i, j \rangle : \langle i, j \rangle \text{ is crossed}\} \cup \{\langle i, j \rangle : \langle i, j \rangle \text{ is ticked and } j \in B\}.$$

Notice that we can tick  $\langle i, j \rangle$  first then cross it later. We will say that a number is *properly ticked* if it is ticked but not crossed. Intuitively crossing has  $\Theta(B)$  ‘moving towards’ a  $\Sigma_1$  set, whereas proper ticking

causes it to approach  $B$ . In particular, if there is an  $i$  such that the whole  $i$ -th column is ticked, then  $B \leq_e \Theta(B)$  since

$$j \in B \Leftrightarrow \langle i, j \rangle \in \Theta(B).$$

We are now ready to prove the first part of Gutteridge's Theorem.

**Theorem 3.1.** *Let  $\mathcal{M} \models B\Sigma_2$ . In  $\mathcal{M}$ , if  $B$  is  $\Delta_2$  and not  $\Sigma_1$ , then there is an  $s$ -operator  $\Theta$  such that*

$$\emptyset <_e \Theta(B) <_e B.$$

*Consequently, there is no  $\Delta_2$  minimal  $e$ -degree.*

We modify the original proof as presented in Cooper [6], and use the blocking method in order to carry out the argument for  $B\Sigma_2$  models.

$\Theta$  is built in stages. Fix a standard enumeration of c.e. sets  $\{W_e : e \in \mathcal{M}\}$  and  $e$ -operators  $\{\Phi_e : e \in \mathcal{M}\}$ . The requirements are:

- $P_e$ :  $W_e \neq \Theta(B)$ .
- $N_e$ :  $B \neq \Phi_e(\Theta(B))$ .

Let  $L(e, s)$  (and  $l(e, s)$  resp.) be the length of agreement functions between  $W_e$  and  $\Theta(B)$  (and  $B$  and  $\Phi_e(\Theta(B))$  resp.) at stage  $s$ . Let

$$u(e, x, s) = \begin{cases} \min\{z : B^s(x) = \Phi_e^s(\Theta^s(B^s \upharpoonright z); x)\} & \text{if such a } z \text{ exists;} \\ 0, & \text{otherwise.} \end{cases}$$

The strategy for  $P_e$  is to code  $B$  into  $\Theta(B)$  by ticking the unmarked  $\langle e, j \rangle$  for all  $j < L(e, s)$ .

The strategy for  $N_e$  is to make  $l(e, s)$  less dependent on  $B^s$  by crossing those  $\langle i, j \rangle$ 's which are "used" in the calculation  $\Phi_e(\Theta(B))$  with  $i > e$ . This crossing action fixes all  $\Phi_e(\Theta(B))(x)$ -computations up to  $x \leq l(e, s)$ , whenever it is defined. Thus if all numbers in some column  $i > e$  are crossed, then  $\Phi_e(\Theta(B))$  is c.e., since it only depends on the first  $e$  columns, which has only finitely many marked elements (as we shall show).

To deal with the limitation of induction available, we use the blocking method which arranges the set of requirements into  $J$  many blocks, where  $J$  is a  $\Sigma_2$  cut to be determined in the course of the construction. Let  $I$  be a  $\Sigma_2$  cut in  $\mathcal{M}$  and let  $f : I \rightarrow \mathcal{M}$  be the associated unbounded function with computable approximation  $f(k, s) : [0, a] \times \mathcal{M} \rightarrow \mathcal{M}$  such that  $\lim_s f(i, s) = f(i)$  for all  $i \in I$ , where  $a$  is an upper bound of  $I$ . We first form  $a$ -many blocks of requirements at stage  $s$ :

$$B_{k,s} = \{R_e : b_{k,s} \leq e < b_{k+1,s}\}$$

dynamically, where  $R_e$ 's are requirements all of which are of the type  $P_e$  or all  $N_e$ . Initially we have  $b_{k,0} = f(k, 0)$  and keep  $b_{k,s} \geq f(k, s)$  for

every stage  $s$ ; moreover, if at stage  $t$  an  $R_e$  in block  $B_{k,t}$  acts, we define  $b_{k+1,t} = \max\{t, f(k+1, t)\}$ .

**Construction:** Stage 0. Set  $\Theta = \emptyset$ , and for all  $k < a$ ,  $b_{k,0} = f(k, 0)$ .

Stage  $s+1$ . Suppose that we have blocks (for  $R = P$  or  $N$ )  $B_{k,s}^R = \{R_e : b_{k,s} \leq e < b_{k+1,s}\}$ . Let the length of agreement functions for block  $k$  be defined by  $\widehat{L}(k, s) = \max\{L(e, s) : P_e \in B_{k,s}^P\}$  and  $\widehat{l}(k, s) = \max\{l(e, s) : N_e \in B_{k,s}^N\}$  respectively. Also define  $\widehat{r}(k, s)$  by replacing  $l(e, s)$  in the definition of  $r(e, s)$  by  $\widehat{l}(k, s)$ .

We say that the requirement block  $B_{k,s}^P$  *requires attention* if  $\widehat{L}(k, s)$  increases. Also,  $B_{k,s}^N$  *requires attention* if

$$\exists \langle i, j \rangle \leq \widehat{r}(k, s) [\langle i, j \rangle \in \Theta^s(B^s) \text{ and } i > e].$$

The actions are: Find the least requirement block requiring attention. If it is  $B_k^P$ , then tick all  $\langle k, j \rangle$  which are less than  $\widehat{L}(k, s)$ . If it is  $B_k^N$ , then cross those  $\langle i, j \rangle \leq \widehat{r}(e, s)$  such that  $\langle i, j \rangle \in \Theta^s(B^s)$  and  $i > k$ .

Let

$$b_{k,s+1} = \begin{cases} s+1, & \text{if block } k \text{ requires attention} \\ & \text{during stage } s+1; \\ \max\{b_{k,s}, f(k, s+1)\}, & \text{otherwise.} \end{cases}$$

### End of Construction.

Let  $J = \{j : \lim_s b_{j,s} \text{ exists}\}$ . Clearly,  $b_{j,s} \geq f(j, s)$ ,  $J$  is a  $\Sigma_2$  subset of  $I$  and  $J$  is downward closed. For each  $j \in J$ , let  $b_j = \lim_s b_{j,s}$  and  $B_j^R = \{R_e : b_{j-1} < j \leq b_j\}$ , which we shall call the  $j$ -th *permanent block* of requirement  $R$ . (It is understood that when  $j = 0$ ,  $b_{-1,s}$  is  $-1$  for all  $s$ ). We now show that  $J$  is a cut and verify that the construction works along  $J$ .

**Lemma 3.1.** *For all  $j \in J$ ,*

- (1) *Every  $P_e \in B_j^P$  is satisfied. Furthermore, there is a stage  $s$ , after which  $P_e$  has no action for all  $P_e \in B_j^P$ ;*
- (2) *Every  $N_e \in B_j^N$  is satisfied. Furthermore, there is a stage  $s$ , after which  $N_e$  has no action for all  $N_e \in B_j^N$ ;*
- (3)  *$j+1 \in J$ , hence  $J$  is a  $\Sigma_2$  cut;*
- (4) *For any  $e \in M$  there is  $j \in J$  such that  $P_e$  and  $N_e$  are in the permanent block  $B_j^P$  and  $B_j^N$  respectively.*

*Proof.* Fix  $j \in J$ . By the definition of  $J$ , there is a stage  $s_0$  such that for all  $t \geq s_0$ , for all  $j' \leq j$ , we have  $b_{j',t} = b_{j',s_0} = b_{j'}$ . Consequently, all permanent blocks  $B_{j'}^R$  are formed by stage  $s_0$ . In the discussion below, all stages are greater than or equal to  $s_0$ .

We first establish statement (1). Suppose, for the sake of contradiction, that some  $P_e \in B_j^P$  is not satisfied. Fix such an  $e$ . Then  $\Theta(B) = W_e$ .

Notice that  $B\Sigma_2$  ensures that any  $\Delta_2$  set  $D$  settles down uniformly on initial segments, i.e., if  $D = \lim_s D^s$  then

$$\forall a \exists t \forall s > t \forall x < a [D^s(x) = D^t(x)].$$

Since  $B$  is  $\Delta_2$  and  $\Theta$  is an s-operator,  $\Theta(B)$  is  $\Delta_2$ . Thus,  $\Theta^s(B^s)$  settles down uniformly on initial segments. Hence  $\lim_s L(e, s) = \infty$ . Thus all elements in column  $j$  are marked. Since no requirement in block  $B_{j-1}^N$  acts after stage  $s_0$ , only  $\mathcal{M}$ -finitely many are crossed. Thus, with the exception of an  $\mathcal{M}$ -finite set, all elements in column  $j$  are properly ticked. Hence  $B \leq_e \Theta(B) = W_e$ , contradicting the fact that  $B$  is not c.e. We conclude that  $P_e$  is satisfied.

Thus for all  $P_e \in B_j^P$ , there is a stage  $s_e$  after which  $L(e, s)$  never changes. By  $B\Sigma_2$ , there is a stage  $t$ , after which  $\widehat{L}(j, s)$  never changes. Therefore, no action is taken for the sake of  $P_e \in B_j^P$  after stage  $t$ .

We now prove (2). Suppose, for the sake of contradiction, some  $N_e \in B_j^N$  is not satisfied, i.e.,  $B = \Phi_e(\Theta(B))$ . First we show that  $\lim_s l(e, s) = \infty$ . Fix  $a$ . By  $B\Sigma_2$ , there is a stage  $s_1$  such that for all  $t > s_1$ ,  $B^{s_1} \upharpoonright (a+1) = B^t \upharpoonright (a+1)$  and  $B^t \upharpoonright (a+1) \subseteq \Phi_e^t(\Theta^t(B^t))$ . Then for all  $x \leq a$  and  $x \notin B$ , we have  $\forall t > s_1 \Phi_e^t(\Theta^t(B^t))(x) = 0$ . This is true since otherwise the computation for the least  $x$  is preserved forever by the construction. Thus  $l(e, s) \geq a$  forever. Clearly  $\lim_s \widehat{l}(j, s) = \infty$ .

Thus for each  $\langle x, y \rangle \in \Theta(B)$ , with  $x > j$ , there is a step  $s$  for which  $\langle x, y \rangle \in \Theta^s(B^s)$  and  $\langle x, y \rangle \leq \widehat{l}(j, s)$ . So each such  $\langle x, y \rangle$  is crossed by construction. By (1), there are only  $\mathcal{M}$ -finitely many marked elements in columns  $\mathcal{M}^{[\leq j]}$ . So we conclude that  $\Phi_e(\Theta(B)) = \Phi_e(F \cup C)$ , where  $F$  is an  $\mathcal{M}$ -finite subset of  $\mathcal{M}^{[\leq j]}$  and  $C$  is a set containing only the crossed part in  $\Theta$ . Consequently,  $\Phi_e(\Theta(B))$  is  $\Sigma_1$ . If  $B = \Phi_e(\Theta(B))$ , then  $B$  is c.e., which gives us the desired contradiction.

Furthermore, since  $B \neq \Phi_e(\Theta(B))$ , there is a point of disagreement  $a$ . Let  $s_2$  be a stage such that for all  $t > s_2$ ,  $B^t \upharpoonright (a+1) = B^{s_2} \upharpoonright (a+1)$  and

$$\forall t > s_0 B \upharpoonright (a+1) \subseteq \Phi_e^t(\Theta^t(B^t)).$$

At any stage  $t > s_2$ , for all  $x \leq a$  and  $x \notin B$ , if  $\Phi_e^t(\Theta^t(B^t))(x) = 1$ , then the computation for the least  $x$  is preserved. It can only happen

once (in fact, only for  $a$ ). After that  $N_e$  never acts. By  $B\Sigma_2$  again, there is a stage after which no  $N_e$  from the block  $B_j^N$  will act.

Now (3) follows easily from (1) and (2): Let  $t_1, t_2$  be the stages after which no  $P_e \in B_j^P$  and  $N_e \in B_j^N$  ever acts, and let  $t_0$  be the stage that  $f(j+1, t_0) = f(t)$ . Obviously, for any  $t > \max\{t_0, t_1, t_2\}$ ,  $b_{j+1, t}$  never moves. Hence  $J$  is a cut.

Finally, we prove (4). Given any  $e$ , let  $s_e$  be the least stage such that there is an  $i < a$  with  $e < f(i, s_e) \leq b_{i, s_e}$ . Then the (partial) map  $g : [0, i] \rightarrow \mathcal{M}$  defined by  $g(j) = s$  which is the stage when  $e$  leaves the  $j$ -th block, i.e.,  $b_{j, s-1} \leq e$  and  $b_{j, s} > e$ , is  $\Sigma_1$ . By  $I\Sigma_1$ , the range of  $g$  is bounded, so that there is a stage after which  $e$  never changes its block.  $\square$

We now build the second s-operator which will imply that any candidate for a regular minimal e-degree is  $\Delta_2$ .

**Theorem 3.2.** *Let  $\mathcal{M}$  be a model of  $B\Sigma_2$ . There is an s-operator  $\Theta$  such that for any regular set  $B$ ,*

- (1) *If  $\Theta(B)$  is c.e., then  $B$  is  $\Delta_2$ ;*
- (2) *If for some e-operator  $\Phi_e$ ,  $\Phi_e(\Theta(B)) = B$ , then  $B$  is  $\Sigma_1$ .*

*Consequently, any regular minimal e-degree is  $\Delta_2$ .*

We follow the proof as in Cooper [6]. We build  $\Theta$  and a  $\Delta_2$  function  $h$  satisfying the following requirements.

- $P_j$ :  $j \in B \Leftrightarrow \langle h(j), j \rangle \in \Theta(B)$ .
- $N_{\langle e, x, F \rangle}$  (where  $\max F < x$ ):  $\langle x, D \rangle \in \Phi_e$  and  $\Theta(F) \subseteq \Theta(D)$  implies that  $\Theta(D)$  only depends on  $\Theta(F)$ .

If we index the requirement this way, then  $\Theta$  can be constructed via a typical Friedberg-Muchnik type finite injury argument, which can be carried out inside  $PA^- + I\Sigma_1$ . We include the proof for the sake of completeness and of pointing out where  $I\Sigma_1$  is used.

The idea is to make  $\Phi_e(\Theta(B))(x)$  only depend on  $B \upharpoonright x$ .

The strategy for  $P_j$  is as follows. We will make the  $j$ -th row  $M$ -finitely marked, with the last element  $\langle h(j), j \rangle$  properly ticked and for all  $k < h(j)$ ,  $\langle k, j \rangle$  is crossed. The intuition is that we code  $B(j)$  into  $\Theta(B)$  in a “ $\Delta_2$  way”. Observe that  $P_j$  is satisfied as long as we do not cross the whole row.

The strategy for  $N_{\langle e, x, F \rangle}$  is as follows. If  $\langle x, D \rangle \in \Phi_e$  such that  $D = F \cup S$  where  $\max F < x$  and  $\min S \geq x$ , then we cross every element in  $S$ . (This will fix  $x$  in  $\Phi_e(D')$  for any  $D'$  “consistent” with  $F$ .)

**Construction:** Stage 0. Set  $\Theta = \emptyset$ .

Stage  $s + 1$ . We say that the requirement  $N_{\langle e, x, F \rangle}$  requires attention if

$$\exists D[\langle x, D \rangle \in \Phi_e, D = \Theta^s(F) \cup S \text{ and } S \subseteq \{\langle i, j \rangle : j \geq e, x\}],$$

and at least one of the elements of  $S$  is not yet crossed.

The actions are: Find the least requirement requiring attention. Cross all numbers in  $S$ ; for each  $j < s$  and  $j \in B_s$ , make sure that there is a unique number in row  $j$  which is properly ticked.

**End of Construction.**

We now verify that the construction works.

**Lemma 3.2.** *Let  $\mathcal{M}$  be a model of  $PA^- + B\Sigma_2$  and  $B$  be a regular subset of  $\mathcal{M}$ .*

- (1) *For all  $j \in \mathcal{M}$ , the  $j$ -th row of  $\Theta$  is  $\mathcal{M}$ -finitely marked.*
- (2) *If  $\Theta(B)$  is  $\Sigma_1$  then  $B$  is  $\Delta_2$ .*
- (3) *If  $\Phi_e(\Theta(B)) = B$  then  $B$  is c.e.*

*Proof.* For (1), notice that row  $j$  can only be marked by  $N_{\langle e, x, F \rangle}$  with  $e, x < j$  and  $F$  an  $\mathcal{M}$ -finite subset of numbers less than  $j$ . Fix  $j$ . There are only  $\mathcal{M}$ -finitely many such  $N_{\langle e, x, F \rangle}$ 's, say  $d_j$  many. Since each  $N$  acts only once in isolation,  $N_{d_j}$  can act at most  $2^{d_j+1}$  times. Let  $g : [0, d_j] \times [0, 2^{d_j+1}] \rightarrow \mathcal{M}$  be defined by  $g(d, m) = s$  if  $N_d$  acts for the  $m$ -th time at stage  $s$ . Then  $g$  is a  $\Sigma_1$  function with a bounded domain. By  $I\Sigma_1$ , the range of  $g$  is bounded. Therefore, there is a bound  $s$  after which there is no action of  $N_d$  for any  $d \leq d_j$ . Hence the  $j$ -th row is  $\mathcal{M}$ -finitely marked.

(2) follows easily from (1) by construction.

We now prove (3). Suppose that  $\Phi_e(\Theta(B)) = B$ . Define a computable approximation  $A^s$  as follows.

$$A^0 = B \upharpoonright (e + 1). \text{ (It is } \mathcal{M}\text{-finite by regularity.)}$$

$$A^{s+1} = A^s \cup \Phi_e^s \Theta^s(A^s).$$

$$\text{We claim that } B = \bigcup_s A^s.$$

*Proof of Claim.*

“ $\supseteq$ ”: Fix an  $x$ . By regularity  $B \upharpoonright (x + 1)$  is  $\mathcal{M}$ -finite. Using  $B \upharpoonright (x + 1)$  as a parameter, we can apply  $I\Sigma_1$  to prove

$$\forall s A^s \upharpoonright (x + 1) \subseteq B \upharpoonright (x + 1).$$

“ $\subseteq$ ”: Suppose that there is an  $x \in B \setminus \bigcup_s A^s$ . By regularity,  $B \upharpoonright (x + 1)$  is  $\mathcal{M}$ -finite. Hence the set

$$\{y \leq x : y \in B \upharpoonright (x + 1) \wedge y \notin \bigcup_s A^s\}$$

is  $\Pi_1$ , thus has a least element  $x_0 > e$ . Note that  $B \upharpoonright x_0 \subseteq A^t$  for some  $t$ . Since  $x_0 \in B$ ,  $x_0 \in \Phi_e(\Theta(B))$ . Therefore, there is an  $\mathcal{M}$ -finite set  $D$  of  $B$ , such that  $x_0 \in \Phi_e(\Theta(D))$ . By construction  $N_{\langle e, x_0, F \rangle}$  will receive attention (possibly for some other  $D'$ ) at some stage  $v$ , where  $F = D \upharpoonright x_0$  which is also  $B \upharpoonright x_0$ . Hence  $x_0 \in \Phi_e^v \Theta^v(A^v) = \Phi_e^v \Theta^v(B \upharpoonright x_0) \subseteq \Phi_e^v \Theta^v(A_t)$ , hence  $x_0 \in A^{v+1}$ .

This ends our proof of the Claim and the Theorem.  $\square$

The following corollary may be seen as a generalization of Guttridge's Theorem for nonstandard models.

**Corollary 3.1.** *If a model of  $B\Sigma_2$  has a minimal  $e$ -degree, then the degree is not regular.*

#### 4. MINIMALITY OF CUTS UNDER $\Sigma_2$ INDUCTION

We focus our attention primarily on models of  $PA^- + I\Sigma_2$  in this section. However, the analysis of the  $e$ -degree of cuts proceeds from the ground up, starting with  $\Sigma_2$  cuts, which only exist in  $B\Sigma_2$  models. As will be seen in the course of the proofs, the arguments apply to all models which satisfy  $B\Sigma_2$ , except at a crucial point where  $\Sigma_2$  induction is required (and for good reason as will become evident in the next section).

Suppose that in  $\mathcal{M}$  there is a  $\Sigma_2$  cut  $I$  with an unbounded  $\Sigma_2$  map  $f : I \rightarrow \mathcal{M}$ . Let  $a$  be an upper bound of  $I$ . Let  $\bar{I}$  be the set  $\{y \leq a : y \notin I\}$ . Clearly  $\bar{I}$  is computably isomorphic to a  $\Pi_2$  cut. By abusing the terminology "cut", we regard  $\bar{I}$  also as a  $\Pi_2$  cut.

**Proposition 4.1.**

- (a) *If a set  $A$  is  $e$ -reducible to  $I$  then  $A$  is  $\Sigma_2$ .*
- (b) *If a set  $A$  is  $e$ -reducible to  $\bar{I}$  then  $A$  is  $\Pi_2$ .*

*Proof.* (a) follows from the definition of  $e$ -reducibility and the fact that  $I$  is  $\Sigma_2$ , and (b) follows from  $B\Sigma_2$  by pushing the bounded existential quantifier inside.  $\square$

**Proposition 4.2.**  *$I$  and  $\bar{I}$  form a minimal pair.*

*Proof.* Suppose  $A = \Phi(I)$  and  $A = \Psi(\bar{I})$  where  $\Phi$  and  $\Psi$  are  $e$ -operators. Then  $x \in A$  if and only if

$$\exists i, n [i < n \wedge \langle x, i \rangle \in \Phi \wedge \langle x, n \rangle \in \Psi].$$

Therefore  $A$  is  $\Sigma_1$ .  $\square$

**Proposition 4.3.** (a) *All nonregular bounded  $\Sigma_2$  sets have the same  $e$ -degree.*

(b) *All nonregular bounded  $\Pi_2$  sets have the same e-degree.*

*Proof.* The proof is similar to the Turing degree case, (see Chong and Yang [4]). For the sake of completeness, we give a proof below.

Let  $Y$  be a nonregular  $\Sigma_2$  set bounded by  $b$ . Assume that  $Y$  is defined by  $\exists v\psi(v, y)$ , where  $\psi(v, y)$  is a  $\Pi_1$  formula. For each  $y \in Y$ , let  $v_y$  be the least  $v$  such that  $\psi(v, y)$ . As  $Y$  is not regular, the set  $\{v_y : y \in Y\}$  is unbounded in  $\mathcal{M}$ .

Given any  $\Sigma_2$  set  $X = \{x : \exists u\varphi(u, x)\}$  bounded by  $a$ , we show that  $X \leq_{e,s} Y$  and  $\bar{X} \leq_{e,s} \bar{Y}$ . Consider the following set  $D$  of pairs in  $a \times Y$ :

$$\langle w, y \rangle \in D \Leftrightarrow \exists u < v_y \varphi(u, w).$$

It is easy to see that  $D$  is  $\Delta_2$  on  $a \times Y$ . By the Coding Lemma (Lemma 2.3),  $D$  is coded on  $a \times Y$ . Let  $D^*$  be a code. For any  $\mathcal{M}$ -finite set  $E$ ,  $E \subseteq \bar{X}$  if and only if the second components of  $(E \times b) \cap D^*$  are in  $\bar{Y}$ , which establishes (b). Next, by  $B\Sigma_2$ ,  $E \subseteq X$  if and only if there is a  $u$  which bounds all witnesses  $u_x$  for  $x \in E$ . Thus  $E \subseteq X$  if and only if there exists  $y \in Y$  such that  $(E \times \{y\}) \subseteq D^*$ , which establishes (a).  $\square$

**Corollary 4.1.** (a) *All  $\Sigma_2$  cuts have the same e-degree.*

(b) *All  $\Pi_2$  cuts have the same e-degree.*

(c) *If  $A$  is nonregular and  $A \leq_e I$  then  $I \leq_e A$ .*

(d) *If  $A$  is nonregular and  $A \leq_e \bar{I}$  then  $\bar{I} \leq_e A$ .*

As the structure of cuts are simple, they are natural candidates for minimal e-degrees. The following results point to that direction. A total enumeration degree is an e-degree containing a set  $A$  and its complement  $\bar{A}$  in the degree. Recall the definition of quasi-minimality:

**Definition 4.1.** *We say that an enumeration degree  $\mathbf{a}$  is quasi-minimal if there is no total enumeration degree  $\mathbf{b}$  such that  $\mathbf{0} <_e \mathbf{b} <_e \mathbf{a}$ .*

**Lemma 4.1.** *Let  $\mathcal{M}$  be a model of  $PA^- + B\Sigma_2$ . Let  $I$  be a  $\Sigma_2$  cut in  $\mathcal{M}$ . If a  $\Delta_2$  set  $A$  is weakly e-reducible to  $I$ , then  $A$  is  $\Sigma_1$ .*

*Proof.* Let  $\Phi$  be an e-operator such that

$$x \in A \Leftrightarrow (\exists p)[\langle x, p \rangle \in \Phi \wedge p \in I].$$

Notice that if  $x \notin A$  and  $\langle x, j \rangle \in \Phi$  then  $j \in \bar{I}$ . (Informally we call such a  $j$  an error.) Consider the following two cases.

*Case 1.* There exist (downward) unboundedly many errors in  $\bar{I}$ . Formally

$$(\forall n \in \bar{I})(\exists j < n)(\exists x)[x \notin A \wedge \langle x, j \rangle \in \Phi].$$

It follows that

$$n \in \bar{I} \Leftrightarrow (\exists j < n)(\exists x)[x \notin A \wedge \langle x, j \rangle \in \Phi].$$

As  $A$  is  $\Delta_2$ , this would imply that  $\bar{I}$  is  $\Sigma_2$ . Hence Case 1 is vacuous.

*Case 2.* The negation of Case 1 holds. In other words there is a bound on errors. Formally

$$(\exists n \in \bar{I})(\forall j < n)(\forall x)[\langle x, j \rangle \in \Phi \rightarrow x \in A].$$

Using  $n$  as a parameter, we have

$$x \in A \Leftrightarrow (\exists p < n)[\langle x, p \rangle \in \Phi],$$

which says that  $A$  is  $\Sigma_1$ .  $\square$

**Corollary 4.2.** *The degree of the  $\Sigma_2$  cut  $I$  is properly  $\Sigma_2$  and it is quasi-minimal.*

*Proof.* If  $X \oplus \bar{X} \leq_{e,w} I$ , then  $X$  is  $\Delta_2$ .  $\square$

By a similar argument, we get the symmetric results for  $\Pi_2$  cuts.

**Lemma 4.2.** *If a  $\Delta_2$  set  $A$  is weakly  $e$ -reducible to  $\bar{I}$ , then  $A$  is  $\Sigma_1$ .*

*Proof.* Let  $\Phi$  be an  $e$ -operator such that  $A = \Phi(\bar{I})$ . As  $A$  is  $\Delta_2$ ,  $A$  is regular. Thus  $A \upharpoonright f(i)$  is  $\mathcal{M}$ -finite for each  $i \in I$ , where  $f$  is the unbounded function associated with  $I$ .

*Case 1:*  $A$  requires full  $\bar{I}$  to compute. Formally

$$(\forall n \in \bar{I})(\exists x)[x \in A \wedge (\forall n' > n)[\langle x, n' \rangle \notin \Phi].$$

It follows that

$$n \in \bar{I} \Leftrightarrow (\exists x)[x \in A \wedge (\forall n' > n)[\langle x, n' \rangle \notin \Phi].$$

As  $A$  is  $\Delta_2$ , we conclude that  $\bar{I}$  is  $\Sigma_2$ , which implies that Case 1 is empty.

*Case 2:* The negation of Case 1 holds. Informally this says that  $A$  only requires a finite piece of  $\bar{I}$  to compute. Formally

$$(\exists n \in \bar{I})(\forall x)[x \in A \rightarrow (\exists n' > n)\langle x, n' \rangle \in \Phi].$$

Using  $n$  as a parameter, we obtain

$$x \in A \Leftrightarrow (\exists n' > n)[\langle x, n' \rangle \in \Phi],$$

which shows that  $A$  is  $\Sigma_1$ .  $\square$

**Corollary 4.3.** *Any  $\Pi_2$  cut  $\bar{I}$  is quasi-minimal.*

The general restriction on cuts now follows:

**Corollary 4.4.** *If  $J$  is a cut in a model  $\mathcal{M}$  of  $PA^- + B\Sigma_2$ , then any  $\Delta_2$  set that is weakly  $e$ -reducible to  $J$  is  $\Sigma_1$ .*

*Proof.* By Lemma 4.1 and Lemma 4.2 we may assume that  $J$  is neither  $\Sigma_2$  nor  $\Pi_2$ . Repeating the proof of Lemma 4.1, we conclude that Case 1 does not apply since it would otherwise show that  $J$  is  $\Pi_2$ . Hence Case 2 applies and  $A$  is  $\Sigma_1$ .  $\square$

**Theorem 4.1.** *Let  $\mathcal{M}$  be a nonstandard model of  $PA^- + I\Sigma_2$ . If  $J$  is a cut in  $\mathcal{M}$ , and  $A$  is regular set which is weakly e-reducible to  $J$ , then  $A$  is  $\Sigma_1$ .*

*Proof.* By Corollary 4.4, it is sufficient to show that  $A$  is  $\Delta_2$ . Let  $\Phi$  be an e-operator such that  $\Phi(J) = A$ . Since  $J$  is a cut, we may again assume that the positive conditions are single points in  $\Phi$ . Call a number  $j$  *good* if there is an  $x$  such that  $\langle x, j \rangle \in \Phi$  and no  $j' < j$  satisfies  $\langle x, j' \rangle \in \Phi$ .

Observe that the collection  $G$  of all good  $j$ 's is a  $\Sigma_2$  subset of the interval  $[0, a]$ , where  $a$  is an upper bound of  $J$ . Our first step is to effectively collapse  $G$  into an interval of the form  $[0, b]$  for some  $b$ , such that  $G \cap J$  will collapse to a new cut. We will make essential use of the downward closeness of the new cut.

*Claim 1.* One may assume that there is a  $b$  such that every  $x < b$  is good, and  $A$  is enumerated by a cut in the set  $[0, b]$  via some e-operator.

*Proof of Claim 1.* First of all, if  $G \cap J = G \cap [0, j]$  for some  $j \in J$ , then  $A$  is enumerated via  $\Phi$  by the  $\mathcal{M}$ -finite set  $[0, j]$ , which immediately implies that  $A$  is  $\Sigma_1$ .

Thus assume that the set of  $G \cap J$  is unbounded in  $J$ . Now if the set  $G \cap \bar{J}$  is bounded below above  $J$ , i.e. there is a  $c \notin J$  such that  $G \cap J = G \cap [0, c]$ , then again  $A$  is enumerated by  $\Phi$  via the  $\mathcal{M}$ -finite set  $[0, c]$ , again showing that  $A$  is  $\Sigma_1$ .

By  $I\Sigma_2$ , the set  $G$  is an  $\mathcal{M}$ -finite subset of  $[0, a]$ . Thus there is a computable order-preserving bijection between  $G$  and  $[0, b]$  for some number  $b < a$ . This bijection effectively ‘compresses’  $G$  down to a closed interval, proving the claim. Note that under this compression operation,  $G \cap J$  is mapped to a cut. We will still use  $J$  to denote this cut. This should not cause any confusion as we will be dealing with the cut for the rest of the proof.

Fix the interval  $[0, b]$  as given by Claim 1. If  $\langle x, j \rangle \in \Phi$ , let  $j(x)$  be the least such  $j$ .  $B\Sigma_2$  ensures that such a number exists and is computable in  $\emptyset'$ .

*Claim 2.* For any  $t$ , there is a  $j_t \in J$  such that if  $x < t$  is in  $A$ , then  $j(x) < j_t$ .

*Proof of Claim 2.* Since  $A$  is regular,  $A \cap [0, t]$  is  $\mathcal{M}$ -finite. Now by  $B\Sigma_2$  and the fact that  $j(x)$  is a  $\Sigma_2$  map, the range of  $j(x)$  is  $\mathcal{M}$ -finite, hence a uniform bound  $j_t$  as required exists. This proves Claim 2.

*Claim 3.* Let  $x$  be such that  $\langle x, j \rangle \in \Phi$  for some  $j \in [0, b]$ . Then there is a  $t$  such that

$$\{j(x') | x' < t \wedge (\exists j < b)(\langle x', j \rangle \in \Phi)\}$$

contains either the interval  $[0, j(x)]$  or the interval  $[j(x), b]$ . Moreover, the former occurs if and only if  $x \in A$ .

*Proof of Claim 3.* First of all, assume that  $x \in A$ . By Claim 1, for each  $j \leq j(x)$ , there is an  $x_j$  such that  $j(x_j) = j$ . Hence if  $t > x_j$  for each  $j \leq j(x)$ , it will satisfy the requirement of Claim 3.

On the other hand, suppose  $x \notin A$ . Then  $j(x) \notin J$ . By Claim 1, there is a  $t$  such that for all  $j$  with  $b \geq j \geq j(x)$ , there is an  $x_j$  satisfying  $j(x_j) = j$  and  $x_j < t$ . Moreover, by the regularity of  $A$ ,  $A \cap [0, t]$  is  $\mathcal{M}$ -finite, whose image under  $j$  is bounded above in  $J$ . Thus, if  $x \in A$  then there is some  $j' \in J$  which is not in the set

$$\{j(x') | x' < t \wedge (\exists j < b)(\langle x', j \rangle \in \Phi)\}.$$

This proves Claim 3.

We now prove that  $A$  is weakly Turing reducible to  $\emptyset'$ , hence  $\Delta_2$ . Given  $x$ , first check if there is a  $j \in [0, b]$  such that  $\langle x, j \rangle \in \Phi$ . If no such  $j$  exists,  $x$  is not in  $A$ . Otherwise, apply Claim 3 to find a  $t$  such that the set

$$\{j(x') | x' < t \wedge (\exists j < b)(\langle x', j \rangle \in \Phi)\}$$

either contains all  $[0, j(x)]$  or all  $[j(x), b]$ . By Claim 2, we see that in the first case  $x$  belongs to  $A$ , while in the second case  $x$  lies in  $\overline{A}$ .  $\square$

**Theorem 4.2.** *Let  $\mathcal{M}$  be a model of  $PA^- + B\Sigma_2$ . If  $J$  is a cut in  $\mathcal{M}$  and  $A$  is a nonregular set that is weakly  $e$ -reducible to  $J$ , then  $J$  is  $e$ -reducible to  $A$ .*

*Proof.* It is sufficient to assume that  $A$  is bounded, say by  $a$ . Let  $\Phi$  be an  $e$ -operator such that  $A = \Phi(J)$ . Choose an upper bound  $b$  of  $J$ . We are thus considering an  $\mathcal{M}$ -finite set

$$X = \{\langle x, j \rangle \in \Phi | x < a \wedge j < b\}.$$

For each  $x$  in the first co-ordinate of  $X$ , let  $j(x)$  be the least corresponding  $j$  in the second co-ordinate. The map from  $x$  to  $j(x)$  is computable in  $\emptyset'$ . By  $B\Sigma_2$ , it is actually an  $\mathcal{M}$ -finite map. Note that if  $x \in A$ , then  $j(x) \in J$ . We consider two cases.

*Case 1.* There is a  $j \in J$  such that  $j(x) < j$  for all  $x \in A$ . Then  $x \in A$  if and only if  $j(x) < j$ . But then  $A$  would be  $\mathcal{M}$ -finite, which is a contradiction.

*Case 2.* There is no  $j$  as in Case 1. Define  $\Theta$  such that

$$\langle j, x \rangle \in \Theta \iff j \leq j(x).$$

Then it is straightforward to verify that  $J$  is e-reducible to  $A$  via  $\Theta$ .  $\square$

The following corollary proves the main result of this section.

**Corollary 4.5.** *Let  $\mathcal{M}$  be a model of  $PA^- + I\Sigma_2$ . Then  $\mathcal{M}$  has a minimal e-degree if and only if  $\mathcal{M}$  is nonstandard. Indeed, if  $\mathcal{M}$  is nonstandard, then every cut in  $\mathcal{M}$  is of minimal e-degree.*

*Proof.* If  $\mathcal{M}$  is standard, then Gutteridge's Theorem states that  $\mathcal{M}$  has no minimal e-degree. Suppose  $\mathcal{M}$  is nonstandard. Then Corollary 3.1 says that one needs to look only at nonregular sets. Let  $J$  be a cut in  $\mathcal{M}$ . By Theorem 4.1 and Corollary 4.4, any regular set e-reducible to  $J$  is  $\Sigma_1$ . On the other hand, by Theorem 4.2, any nonregular set weakly e-reducible to  $J$  also enumerates  $J$ . Hence  $J$  is of minimal e-degree.  $\square$

## 5. E-MINIMALITY OF CUTS UNDER $\Sigma_2$ BOUNDING

Let  $\mathcal{M}$  be a  $B\Sigma_2$  model with  $f$  a  $\Sigma_2$  function mapping a  $\Sigma_2$  cut  $I$  cofinally into  $\mathcal{M}$ . The proof of Theorem 4.1 makes essential use of  $\Sigma_2$  induction. The situation in  $\mathcal{M}$  where only  $\Sigma_2$  bounding holds is much more complicated. To begin with, in contrast with Corollary 4.5, it turns out that not every cut in a  $B\Sigma_2$  model is necessarily of minimal e-degree (in fact it is not even clear that one exists). The source of this bifurcation goes back to the proof of Theorem 4.1, where under  $\Sigma_2$  induction the set of *good*  $j \in J$  is  $\Sigma_2$ , hence  $\mathcal{M}$ -finite (i.e., coded). Such a coding need not exist in  $\mathcal{M}$ . But if it does, then  $J$  will be of minimal e-degree in  $\mathcal{M}$ . To make this precise, we introduce the notion of ' $\Sigma_2$  bi-clustering' on a cut. Call a sequence of  $\mathcal{M}$ -finite sets a  $\Sigma_2$  sequence if it is the range of a  $\Sigma_2$  function with domain  $I$ .

**Definition 5.1.** *Let  $J$  be a cut in  $\mathcal{M}$ . A  $\Sigma_2$  sequence  $\mathcal{C}$  of pairwise disjoint nonempty  $\mathcal{M}$ -finite sets  $C_i, i \in I$ , is said to bi-cluster on  $J$  if for all  $c \in J$  and  $d \in \bar{J}$ , the sets  $\{i | C_i \cap J \cap [c, d] \neq \emptyset\}$  and  $\{i | C_i \cap \bar{J} \cap [c, d] \neq \emptyset\}$  are each unbounded in  $I$ .*

Observe that for such a  $\mathcal{C}$ ,  $\bigcup_{i \in I} C_i$  is not  $\mathcal{M}$ -finite, since by  $B\Sigma_2$ , no  $\mathcal{M}$ -finite set  $F$  can be a union of a  $\Sigma_2$  sequence of nonempty  $\mathcal{M}$ -finite sets. Indeed, suppose that the union is an  $\mathcal{M}$ -finite set. For  $x \in \bigcup_i C_i$ , let  $h(x) = i$  if and only if  $x \in C_i$ . The disjointness of the

sequence ensures that this is well-defined. Then  $h$  is  $\Sigma_2$  so that by  $B\Sigma_2$  its image is bounded in  $I$ . This is a contradiction since the  $C_i$ 's are pairwise disjoint.

**Lemma 5.1.** *Let  $J$  be a cut in  $\mathcal{M}$ . If there is no  $\Sigma_2$  sequence of non-empty  $\mathcal{M}$ -finite sets which bi-clusters on  $J$ , then  $J$  is of minimal e-degree.*

*Proof.* Let  $b$  be an upper bound of  $J$ . First of all, Theorem 4.2 holds for  $J$  as it does not concern  $\Sigma_2$  bi-clustering. For Theorem 4.1, note that since the set  $G$  of ‘good’  $j$ 's is  $\Sigma_2$ , say defined by  $\exists u \forall v \varphi(x, u, v)$ , it may be considered as a union of  $I$  many  $\mathcal{M}$ -finite sets  $\{G_i | i \in I\}$ , where

$$G_i = \{x < a | \exists u < f(i) \forall v \varphi(x, u, v) \wedge \\ \exists w \forall u \leq f(i-1) \exists v < w \neg \varphi(x, u, v)\}.$$

Our assumption on the nonexistence of a  $\Sigma_2$  bi-clustering sequence on  $J$  implies that there exist  $c \in J$  and  $d \in \bar{J}$  such that either  $\{i | G_i \cap J \cap [c, d] \neq \emptyset\}$  is bounded in  $I$  or  $\{i | G_i \cap \bar{J} \cap [c, d] \neq \emptyset\}$  is bounded in  $I$ . In the former case let  $i_0 \in I$  be the least such that for all  $i \geq i_0$ ,  $G_i \cap J \cap [c, d]$  is empty. Now if  $Y_J = \bigcup_{i < i_0} G_i \cap J$  is bounded in  $J$ , then the set  $A$  enumerated by  $\Phi$  using  $J$  actually uses only a bounded segment of  $J$  and is therefore  $\Sigma_1$ . If  $Y_J$  is unbounded in  $J$ , then the set  $Y_{\bar{J}} = \bigcup_{i < i_0} G_i \cap \bar{J} \cap [c, d]$  is not bounded below in  $\bar{J}$  (otherwise, say it is bounded below by  $e$ , then  $J$  is the downward closure of the  $\mathcal{M}$ -finite set  $\bigcup_{i < i_0} (G_i \cap [0, e])$ ). Then using the  $\mathcal{M}$ -finite set  $\bigcup_{i < i_0} G_i$  as the set of ‘good’  $j$ 's for  $J$ , one may repeat the proof of Theorem 4.1 (Claims 1 to 3) to show that  $A$  is  $\Delta_2$ , hence  $\Sigma_1$  by Corollary 4.1.

The case where  $\{i | G_i \cap \bar{J} \cap [c, d] \neq \emptyset\}$  is bounded in  $I$  is argued similarly.  $\square$

Thus a necessary condition for a cut  $J$  not to have minimal e-degree is to construct a  $\Sigma_2$  sequence which bi-clusters on it. Taking a cue from this, we now show that there exists a  $B\Sigma_2$  model in which no  $\Sigma_2$  cut has minimal e-degree.

**Theorem 5.1.** *There is a  $B\Sigma_2$  model  $\mathcal{M}$  in which no  $\Sigma_2$  cut is of minimal e-degree.*

*Proof.* Chong and Yang [3] exhibited a  $B\Sigma_2$  model  $\mathcal{M}$  with the set of natural numbers  $\omega = I$  as a  $\Sigma_2$  cut for which there is a  $\Sigma_2$  subset  $X$

that is not coded on  $I$ . Let

$$X^* = \{\langle i, j \rangle \mid \text{the least } \Sigma_2 \text{ witness of } i \in X \text{ showed up} \\ \text{between } f(j-1) \text{ and } f(j)\}.$$

Then  $X^*$  is  $\Delta_2$  on  $I \times I$  and so by Lemma 2.3 there is an  $\mathcal{M}$ -finite set  $\widehat{X}^*$  whose intersection with  $I \times I$  is  $X^*$ . We may assume that  $\widehat{X}^*$  is a function on  $[0, b]$  where  $b$  is an upper bound of  $I$ . Furthermore, for each  $y$  in the range of  $\widehat{X}^*$ , by taking the least  $x$  such that  $\widehat{X}^*(x) = y$ , we may assume that  $\widehat{X}^*$  is one-one. Then the set  $C = \{\widehat{X}^*(i) \mid i \in I\}$  is  $\Sigma_2$ . We view  $C$  as the union of the  $\Sigma_2$  sequence of singleton sets  $C_i = \{\widehat{X}^*(i)\}$  for  $i \in I$ .

*Claim 1.*  $\mathcal{C} = \{C_i\}_{i \in I}$  is a bi-clustering on  $I$ .

*Proof of Claim 1.* Observe first of all that  $\{\widehat{X}^*(i) \mid i \in X\}$  is cofinal in  $I$ , since otherwise  $X$  would be  $\mathcal{M}$ -finite. Next, if  $i \in I \setminus X$ , then  $\widehat{X}^*(i) \in \bar{I}$ . This is true since  $\widehat{X}^* \cap I \times I = X^*$ . Now if  $\{\widehat{X}^*(i) \mid i \in I \setminus X\}$  is bounded away from  $I$ , i.e., all of its elements are greater than a  $d \in \bar{I}$ , then one may choose a  $b \in \bar{I}$  such that  $Y = \{i \leq b \mid \widehat{X}^*(i) < d\}$  is an  $\mathcal{M}$ -finite set which codes  $X$  on  $I$ , contradicting our choice of  $X$ . This is the place we made the crucial use of the fact that  $X$  is not coded on  $I$ . Hence  $\mathcal{C}$  is a  $\Sigma_2$  sequence which bi-clusters on  $I$ .

Thus let  $b$  be an upper bound of  $I$ . Effectively partition  $\mathcal{M}$  into  $b$ -many unbounded pairwise disjoint computable sets  $B_i$  for  $i \leq b$ . Without loss of generality, we may assume that  $C$  is a subset of  $[0, b]$  which divides it into a  $\Sigma_2$  part and a  $\Pi_2$  part, each of which is not  $\mathcal{M}$ -finite. We define an e-operator  $\Phi$  that has the following property:

(\*)  $i$  is ‘good’ (in the sense defined in Theorem 4.1) if and only if  $i \in C$ . Furthermore, if  $i \notin C$ , then  $\Phi^{[i]} \subset \Phi^{[0]}$ , where  $\Phi^{[i]} = \{x \mid \langle x, i \rangle \in \Phi\}$ .

Thus assume that  $i \in C$  if and only if  $(\exists u)(\forall v)\phi(u, v, i)$  for some  $\Sigma_0$  formula  $\phi$ . At stage  $s$ , if  $\Phi_s^{[i]} \subset \Phi_s^{[0]}$  then enumerate  $\langle x, i \rangle$  into  $\Phi$  for some  $x \in B_i$  which is larger than any number we have seen. Otherwise, there is some  $x \in B_i \cap (\Phi_s^{[i]} \setminus \Phi_s^{[0]})$ . Check if  $\forall u < x \exists v < s \neg \phi(u, v, i)$ . If this holds, enumerate  $\langle x, 0 \rangle$  into  $\Phi$ . If not, go to the next stage.

Notice that by  $B\Sigma_2$ , if  $i \notin C$ , then for any  $x \in B_i \cap \Phi^{[i]}$  there is a stage  $s$  at which we enumerate  $\langle x, 0 \rangle$  into  $\Phi$ .

It is not difficult to verify that  $\Phi$  satisfies condition (\*). Let  $A = \Phi(I)$ . Clearly  $A$  is  $\Sigma_2$ . We will show that  $A$  occupies an intermediate e-degree, i.e. the e-degree of  $A$  is strictly between  $\mathbf{0}_e$  and the e-degree of  $I$ .

*Claim 2.*  $A$  is a regular set.

*Proof of Claim 2.* Fix any  $m \in \mathcal{M}$ . We show that  $A \upharpoonright [0, m]$  is  $\mathcal{M}$ -finite. Notice that  $A \upharpoonright [0, m]$  is the disjoint union of the following two sets:

$$A_0 = \{x < m \mid \langle x, 0 \rangle \in \Phi\},$$

and

$$A_1 = (\{x < m \mid (\exists i \in I) \langle x, i \rangle \in \Phi\} \setminus A_0).$$

Consider

$$\{i \in I \mid \exists x (x \in A_1 \wedge \langle x, i \rangle \in \Phi)\}$$

which is a subset of

$$\{i \in C \mid (\exists u < m) (\forall v) \phi(u, v, i)\} \cap I.$$

Since  $I$  is  $\omega$ , this set is a (real) finite set. Hence  $A_1$  is a finite set. Therefore  $A \upharpoonright m$  is  $\mathcal{M}$ -finite.

*Claim 3.*  $A$  is not  $\Delta_2$ .

*Proof of Claim 3.* Suppose that  $A$  is  $\Delta_2$ . We show that  $\bar{I}$  is  $\Sigma_1$  in  $A$ . Indeed  $i \in \bar{I}$  if and only if there is an  $i' < i$  such that  $\exists x (\langle x, i' \rangle \in \Phi \wedge x \in \bar{A})$ . To see this, note that if  $i \in \bar{I}$ , then as  $\mathcal{C}$  bi-clusters on  $I$ , there is an  $i' \in C \cap \bar{I}$  below  $i$ . For this  $i'$ , only those  $x$ 's less than the least  $\Sigma_2$  witness (say  $z$ ) for  $i'$  are 'passed on' to  $\Phi^{[0]}$ . The number  $z$  is however retained by  $i'$  and so never enters  $A$ . This implies that  $\bar{I}$  is  $\Sigma_2$ . However, as  $I$  is a  $\Sigma_2$  cut we have a contradiction. This proves Claim 3.

Now  $I$  is not e-reducible to  $A$  since this would imply that there is a regular minimal e-degree, which contradicts Corollary 3.1. This proves that  $I$  is not of minimal e-degree. By Corollary 4.1, no  $\Sigma_2$  cut in  $\mathcal{M}$  has minimal e-degree. This completes the proof of the Theorem.  $\square$

Our final task is to show that in contrast to Theorem 5.1, there is a  $B\Sigma_2$  cut (hence all  $\Sigma_2$  cuts) of minimal e-degree.

**Theorem 5.2.** *There is a  $B\Sigma_2$  model all of whose  $\Sigma_2$  cuts have minimal e-degree.*

*Proof.* A model is *saturated* if every set of natural numbers is coded on  $\omega$ . Mytilinaios and Slaman [11] constructed a  $B\Sigma_2$  model  $\mathcal{M}$  which is saturated and in which the set of natural numbers  $\omega$  is a  $\Sigma_2$  cut. Let  $I = \omega$ . We show that  $I$  has minimal e-degree.

Let  $\Phi(I) = A$ . By Theorem 4.2, it is sufficient to assume that  $A$  is regular. Then for each  $i \in I$  there is a least  $j \in I$ , denoted  $j_i$ , such that

$A \upharpoonright f(i)$  is enumerated by  $\Phi$  using  $[0, j]$  as oracle. This fact follows from  $B\Sigma_2$ . Then the function  $g : i \rightarrow j_i$  is coded on  $I$ . Hence there is an  $\mathcal{M}$ -finite set  $X$  such that  $X \cap I \times I = g$ . But then given  $x < f(i)$ , where  $f$  is  $\Sigma_2$  cofinal map from  $I$  into  $\mathcal{M}$ , we have  $x \in A$  if and only if it is enumerated by  $\Phi$  using  $[0, j_i]$ . This shows that  $A$  is  $\Delta_2$ , hence  $\Sigma_1$  by Lemma 4.1.  $\square$

We end this paper with some open problems:

### Open Problems:

1. Is weak e-reducibility a transitive relation for  $\Pi_n$  sets in  $B\Sigma_n$  models?
2. Is weak e-reducibility a transitive relation for  $\Sigma_n$  sets in  $I\Sigma_{n-1}$  models?
3. Is there a minimal e-degree in every model of  $B\Sigma_2$ ? Despite the negative result presented in this section, we conjecture that the answer is positive.
4. Is there a minimal e-degree in every model of  $\Sigma_1$  induction? In view of Slaman's result [13] that  $B\Sigma_2$  is equivalent to  $\Delta_2$  induction over  $PA^-$ , this question becomes quite natural, as  $I\Sigma_1$  is now the only level of induction where nothing is known about existence of minimal e-degrees.

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