Extending and Interpreting Post's Programme

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Abstract

Computability theory concerns information with a causal – typically algorithmic – structure. As such, it provides a schematic analysis of many naturally occurring situations. Emil Post was the first to focus on the close relationship between information, coded as real numbers, and its algorithmic infrastructure. Having characterised the close connection between the quantifier type of a real and the Turing jump operation, he looked for more subtle ways in which information entails a particular causal context. Specifically, he wanted to find simple relations on reals which produced richness of local computability-theoretic structure. To this extent, he was not just interested in causal structure as an abstraction, but in the way in which this structure emerges in natural contexts. Posts programme was the genesis of a more far reaching research project.

In this article we will firstly review the history of Posts programme, and look at two interesting developments of Posts approach. The first of these developments concerns the extension of the core programme, initially restricted to the Turing structure of the computably enumerable sets of natural numbers, to the Ershov hierarchy of sets. The second looks at how new types of information coming from the recent growth of research into randomness, and the revealing of unexpected new computability-theoretic infrastructure. We will conclude by viewing Posts programme from a more general perspective. We will look at how algorithmic structure does not just emerge mathematically from information, but how that emergent structure can model the emergence of very basic aspects of the real world.

Key words: Computability, Post's Programme, Ershov hierarchy, randomness, Turing invariance.

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1 Introduction

Turing's [57] 1939 model of the algorithmic content of those structures which are presented in terms of real numbers can be seen in implicit form in Newton's *Principia* [42], published some 272 years earlier. Newton's work established a more intimate relationship between mathematics and science, and one which held the attention of Turing, in various guises, throughout his short life (see Hodges [30]). Just as the history of arithmetically-based algorithms, underlying many human activities, eventually gave rise to models of computation such as the Turing machine, so the oracle Turing machine schematically addresses the scientific focus on the extraction of predictions governing the form of computable relations over the reals. Whereas the inputting of data presents only time problems for the first model, the second model is designed to deal with possibly incomputable inputs, or at least inputs for which we do not have available an algorithmic presentation. One might reasonably assume that data originating from observation of the real world carries with it some level of computability, but we are yet to agree a mathematical model of physical computation which dispenses with the relativism of the oracle Turing machine. In fact, even as the derivation of recognisable incomputability in mathematics arises from quantification over algorithmic objects, so definability may play an essential role in fragmenting and structuring the computational content of the real world. The Turing model of computability over the natural numbers appears to many people to be a poor indicator of what to expect in science.

Typically, specialist computability theorists are loath to extrapolate to real-world significance for their work. It is a matter of opinion whether this due to honesty (a belief there is little such significance), or arises out of caution and narrowness of interests, or from a combination of these and other factors. Whatever the reason, the outcome is that since the time of Turing, the theory of computability has taken on a Laputa-like [2] aspect in the eyes of many people, an arcane world disconnected from naturally arising information. An important exception here is Piergiorgio Odifreddi's two-volume *Classical Recursion Theory* [45,46], where twenty pages of his first volume is dedicated to Church's Thesis and its extensions (see Section I.8), partly based on communications with Georg Kreisel, and to the relation of computability theory to physics and other areas.

Below, we look at Post's legacy of relating computability-theoretic concepts to intuitively immediate information content, and examine how that can be further extended to an informative relationship with the mathematics of contemporary science.

[2] Swift even has a Laputan professor introduce Gulliver to *The Engine*, an (appropriately useless) early anticipation of today's computing machines, and more.

Some of the ground covered may be familiar, though the focus will be on meaning and interpretation, rather than problem solving and the honing of tools. Within a contemporary overview of part of Post's legacy,[3] we will be guided by the spirit of Post, as seen most clearly in his posthumous paper *Absolutely Unsolvable Problems and Relatively Undecidable Propositions – Account of an Anticipation* (refused publication in his lifetime) and other unpublished material to be found in [17]. For example, we find Post writing in the Appendix on p.378 of [17]:

> " ... perhaps the greatest service the present account could render would stem from its stressing of its final conclusion that mathematical thinking is, and must be, essentially creative. ... it [has] seemed to us to be inevitable that these developments will result in a reversal of the entire axiomatic trend of the late 19th and early 20th centuries, with a return to meaning and truth."

This is followed by a footnote containing:

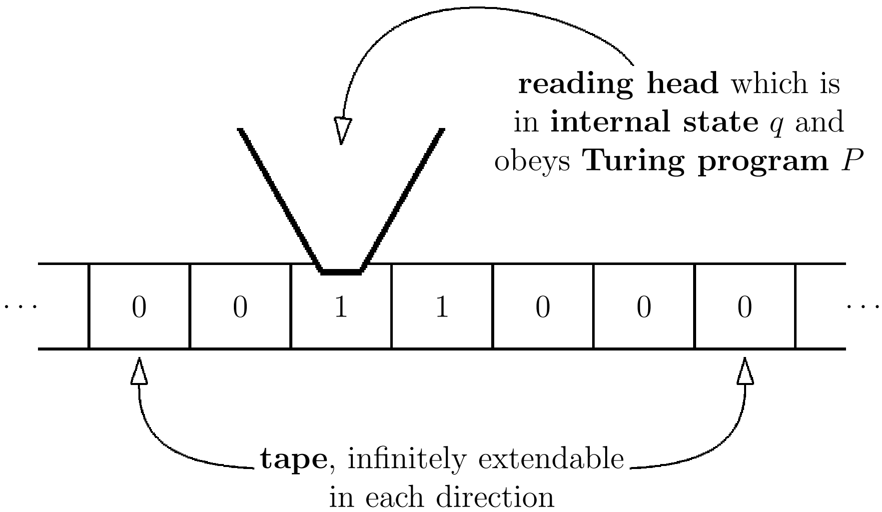
> "Yet, as this account emphasises, the creativeness of human mathematics has a counterpart inescapable limitations thereof - witness the absolutely unsolvable (combinatory) problems. Indeed, with the bubble of symbolic logic as universal logical machine finally burst, a new future dawns for it as the indispensable means for revealing and developing those limitations. For, in the spirit of the Appendix, Symbolic Logic may be said to be Mathematics self-conscious."

2 The Turing Landscape

The oracle Turing machine, which made its first appearance in Turing [57], should be familiar enough. The details are not important, but can be found in most reasonable introductions to computability (see for instance [9]).

One just needs to add to the usual picture of a Turing machine (see below) the capacity for questioning an oracle set about the membership status of individual natural numbers.

[3] See Roman Murawski's article [40] for a more comprehensive discussion of Post's contribution to logic.



The basic form of the questioning permitted is modelled on that of everyday scientific practice. This is seen most clearly in today's digital data gathering, whereby one is limited to receiving data which can be expressed, and transmitted to others, as information essentially finite in form. But with the model comes the capacity to collate data in such a way as enable us to deal with arbitrarily close approximations to infinitary inputs and hence outputs, giving us an exact counterpart to the computing scientist working with real-world observations. If the different number inputs to the oracle machine result in 0-1 outputs from the corresponding Turing computations, one can collate the outputs to get a binary real computed from the oracle real, the latter now viewed as an input. This gives a partial computable functional Φ, say, from reals to reals.

As usual, one cannot computably know when the machine for Φ computes on a given natural number input, so Φ may not always give a fully defined real output. So Φ may be partial. One can computably list all oracle machines, and so index the infinite list of all such Φ, but one cannot computably sift out the partial Φ's from the list.

Anyway, put $\mathbb{R}$ together with this list, and we get the Turing Universe. Depending on one's viewpoint, this is either a rather reduced scientific universe (if you are a poet, a philosopher, or a string-theorist), or (if one is vainly looking for the richness of algorithmic content contained on our list in the physical context, being familiar with the richness of emergent structure in the Turing universe) a much expanded one. But we will defer difficult comparisons between the information content of the Turing universe and that of the physical universe until later. For the moment we will follow Emil Post in his search for the informational underpinnings of computational structure in a safer mathematical context.

Post's first step was to gather together binary reals which are computationally indistinguishable from each other, in the sense that they are mutually Turing computable from each other. Mathematically, this delivered a more standard mathematical structure to investigate — the familiar upper semi-lattice of the *degrees of unsolvability*, or *Turing degrees*. There is no simple scientific counterpart of the mathematical model, or any straightforward justification for what Post did with the Turing universe for perfectly good mathematical reasons — if one wants to get a material avatar of the Turing landscape one needs both a closer and a more comprehensive view of the physical context.

3 Zooming in on the Information Content of the Landscape

Just as computer technology frees us from everyday constraints on how we access the world landscape, allowing us to zoom in from schematic overview of the oceans and continents to a recognisable view of our own street, so the Turing landscape presents us with familiar landmarks, and — as Post [48] anticipated in 1944 — a local richness of interest, full of information content waiting to be revealed. Post's intuition was, in essence, that it may be possible to extract information underlying distinctive features of the Turing landscape, and, going beyond the superficiality of the Google Earth analogy, this information may turn out to actually play a determining role in the formation of these features. This connects with our contemporary intuition, coming out of our growing familiarity with complexity and its emergent formations, that there is an elusive but very solid connection between causal structure and particular types of information.

The genesis of Post's Programme as a public property goes back to his February 26, 1944 address to a New York meeting of the American Mathematical Society, and the subsequent May 1944 publication of his brilliant Bulletin of the A.M.S. paper *Recursively enumerable sets of positive integers and their decision probems*. Of course this, Post's best paper, became best known for what at first sight appeared to be a question of technical interest to a small number of researchers in the new area of recursion theory. In the terminology of the time:

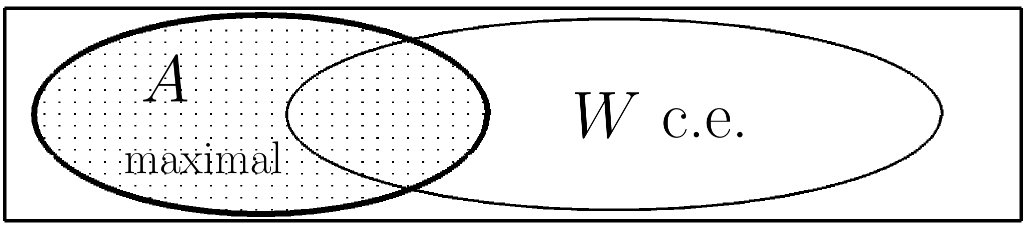
Question: *Is there a non-recursive r.e. set of strictly lower degree of unsolvability than K with respect to arbitrary recursive reducibility?*

Or, as we would ask for the property nowadays: *Is there an incomputable computably enumerable set of strictly lower degree than $\mathbf{0}'$?*

In a sense, Post's Problem lasted no more than twelve years. But the positive solution of Muchnik [41], and independently by Friedberg [24], in 1956, technically groundbreaking as it was, was not what Post had been looking for before his death in 1954. Post's own approach was quite different, and it was this which kept the question alive, and which grew it into a research programme which would interest not just recursion theoretic problem solvers, but would be widely recognised for its intractability and fundamental importance. Post's idea was to seek out classes of sets naturally definable — if possible, in the lattice $\mathcal{E}$ of computably enumerable sets — which contained only incomputable sets of degree $< \mathbf{0}'$. And this had some success. For many-one reducibility, simple sets have this property. For truth-table reducibility, hypersimplicity works. This latter example shows Post to be not too strict about his natural information content being known to be definable in $\mathcal{E}$. Anyway, arising from this project we have:

Post's Programme (narrow version): *Find a naturally definable property of c.e. sets which delivers a non-computable c.e. set of strictly lower Turing degree than* $\mathbf{0}'$.

And this gave rise to a range of immunity properties which one can use to say, roughly speaking, that "the complement of $A \in \mathcal{E}$ avoids the c.e. sets". The simple sets, which gave a successful outcome to Post's problem for the many-one degrees, were those with *immune* complements, where A is immune if its complement is infinite and contains no infinite c.e. subset. There are different equivalent definitions for hypersimplicity (see [9], pp. 228–232, or Odifreddi [45], pp. 272–277), but the original definition of Post says, roughly speaking, that a c.e. A is hypersimple if, given any standard computable array of disjoint finite sets, its complement manages to avoid intersecting with at least one of these finite sets. The strongest of Post's immunity properties, for which there was some hope of it settling the above problem, was that of a set being *cohesive*. The corresponding property of a c.e. set with cohesive complement was that of *maximality*, referring to maximality in the lattice $\mathcal{E}$. The infinite set $\overline{A}$ is cohesive if it is not split by any c.e. set into two infinite parts — that is, the situation below is avoided:



As one can see, it will be hard to use just $\mathcal{E}$ to devise even stronger immunity properties. To describe what became of this candidate for an information-theoretic solution to Post's problem, we need to remind ourselves of more features of the Turing terrain. Remember — the Turing jump A' is what one gets from considering the halting problem for a Turing machine with oracle A. And A is high if $A' = \emptyset''$, low if $A' = \emptyset'$. Degrees are high or low if they contain high or low sets. The surprising and quite beautiful outcome is then:

Martin's Theorem (1966): *Not only are there maximal sets in* $\mathbf{0}'$ *(Yates, 1965) - but the Turing degrees of maximal sets are exactly those which are high. So the high degrees are lattice invariant.*

In general, a class of c.e. degrees is *lattice invariant* if there is an $\mathcal{E}$-definable property of c.e. sets inhabiting exactly the degrees of that class.

With this theorem, Post's problem came of age:

Post's Programme (broader and deeper version): *Discover new relationships between natural information and computability-theoretic structure.*

One of the many developments coming later from this extended research programme was anticipated in Post's 1944 paper. As Martin Davis writes in his Introduction to [17]:

> "Post's remark in this paper that Hilbert's tenth problem 'begs for an unsolvability proof' had a major influence on my own work."

Davis [15,16] proposed a strategy for finding such an "unsolvability proof", which entailed the brave conjecture that all c.e. sets were diophantine. And as we know, a number of major steps variously due to Davis, Putnam and Julia Robinson culminated in Matiyasevich's 1970 solution of the problem, and confirmation of Davis' conjecture — see p. 99 of Matiyasevich's excellent book [39] for more historical detail. This work constituted a major contribution to Post's Programme, in that it established that *all* computably enumerable sets had a previously unknown naturalness of information content. Essentially, each such set arose from familiar high-school arithmetic via one added existential quantifier. A natural definition of a particular c.e. set which is neither computable nor computably equivalent to Turing's halting set may still elude us, but wherever c.e. sets occur, there lies very familiar, and in a sense very naturally arising, information.

There were other results, both positive and negative in import. Martin, Soare and others developed automorphism techniques for breaking the link with $\mathcal{E}$. An early discovery in this direction, emphasising the restrictiveness of $\mathcal{E}$-definability, was:

D. A. Martin: *Hypersimplicity is not definable in* $\mathcal{E}$ *— since one can choose an automorphism* Φ *of* $\mathcal{E}$ *so that* $\Phi(\textit{h-simple}) \not\subseteq \textit{h-simple}$.

Another remarkable negative result, with important consequences for Post's Programme (narrow version), was:

Cholak [6]**, Harrington-Soare** [28]**:** *Given any incomputable c.e. A, one can build an automorphism Φ with $\Phi(A) \in$* **high**.

So, for instance, the low degrees are not $\mathcal{E}$-invariant. Nor are any of the other downward-closed classes coming out of the usual definition of the high-low hierarchy, within which $\mathbf{a} \in \mathbf{low}_n \iff \mathbf{a}^{(n)} = \mathbf{0}^{(n)}$ and $\mathbf{a} \in \mathbf{high}_n \iff \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$. But the dialectical nature of computability theory emerges once again — if one tries to make $\Phi(A) \in \mathbf{0}'$, Harrington and Soare found obstacles which not only pointed to the likelihood of an optimal positive outcome to Post's Programme (narrow version), but actually gave the elusive $\mathcal{E}$-definable property. Their analysis (see [27]) of the obstacles provided the basis for the extraction of a dynamic property, translatable into an $\mathcal{E}$-definable property $\mathcal{Q}$, which does guarantee incomputability and incompleteness of any set A satisfying $\mathcal{Q}$.

When it comes to upward closure, the situation is very different. Post showed that the incomputable c.e. degrees were exactly those containing simple sets (in fact hypersimple sets), while Martin's 1966 theorem was the first sign that the upward closed classes coming out of the high-low hierarchy were all lattice invariant. However, in the Turing landscape little is as straightforward as one might hope, and the challenge is to understand why not. In 1996 another striking outcome of the long-lasting partnership between Harrington and Soare was a proof that non-low is *not* lattice invariant (see [27]). But then normality returned in 1999 with Cholak and Harrington's proof [7] that this was the one exception — one can use coding methods to show that the non-low$_{n+1}$ and high$_n$ degree classes are lattice invariant for all $n \geq 1$.

However, once again we have an ostensible contribution to (the broader and deeper) Post's Programme, which somehow does not quite fit with the spirit of the programme as we understand it. This time it is not $\mathcal{E}$-definability of the information content which is lacking, but the *meaning* and understanding of the jump classes which we derive from the information content — it fails the naturalness criterion in a rather down-to-earth way, in a way that Martin's characterisation of the high c.e. degrees does not.

There is another very natural characterisation of a jump class as being lattice invariant:

Lachlan 1968 + Shoenfield 1976: *The non-low$_2$ c.e. degrees are exactly those which contain c.e. sets with no maximal supersets.*

Lachlan [36] had done much of the hard work here in showing that every (co-infinite) low_2 c.e. set has a maximal superset, and constructing a range of c.e. sets with no maximal supersets, but it took that extra clarity which Shoenfield and a few extra years could provide to finish off [53] this very pleasing result. Of course, there are limits presumably to how far one can go in finding natural characterisations of jump classes. The following is a long-standing and very interesting question:

Open question: *Find a natural $\mathcal{E}$-definable property characterising the high_2 degrees.*

4 Extending the Programme to the Ershov Hierarchy

So far, the focus has been on the relationship between computational structure and arithmetical statements involving a single existential quantifier. Post's Theorem computationally relates iterations $\emptyset'$, $\emptyset''$, $\emptyset^{(3)}$, ... of the Turing jump and the sets c.e. in those iterations, to the arithmetical hierarchy which classifies arithmetical statements according to their quantifier forms (see [9], Section 10.5, or Odifreddi [45], Section IV.1). In particular, the Turing degrees below $\mathbf{0}'$ are exactly those consisting of Δ_2 sets of numbers. Intuitively, the Δ_2 sets are those which one can approximate in a finitary but non-monotonic way (an intuition captured by Schoenfield's limit lemma in [52]). So instead of being able to computably change one's mind about the membership in A of a given number n, as one does in enumerating a c.e. set, one can change one's mind as many times as one likes, so long as this indecision eventually terminates. As well as being a very natural idea, the Δ_2 sets fit in with our intuition that monotonic reasoning is insufficient. As Turing put it in a talk to the London Mathematical Society, on February 20, 1947 (quoted by Andrew Hodges in *Alan Turing - the enigma*, p.361):

> " . . . if a machine is expected to be infallible, it cannot also be intelligent. There are several theorems which say almost exactly that."

There are more recent theorems than those Turing would have had in mind which also suggest that c.e. sets fail to capture quite interesting and important types of information. For instance, by *Arslanov's Completeness Criterion* (as anticipated by Martin and Lachlan, see [45], p.255), every complete extension of Peano arithmetic which is of c.e. degree is actually of degree $\mathbf{0}'$. Whereas we know from the *Low Basis Theorem* that there are such extensions which are even low, and so are far from being in $\mathbf{0}'$.

Of course, there is a big difference between being allowed to computably make just one mistake or change of mind, and being given an indeterminate finite number of such opportunities. The Ershov [22] difference hierarchy (the finite levels of which originated independently with Gold [25] and Putnam [50]) was an attempt to introduce some infrastructure into the Δ_2 sets, giving a comprehensive classification based on bounding numbers of mistakes. Alternatively, one can view the inductive construction of the levels of the Ershov hierarchy via iteration of boolean operations on the c.e. sets.

At the bottom level of this hierarchy one gets the c.e. sets. At the next we arrive at an important first generalisation of the c.e. sets $\mathcal{E}$, this being the differences $\mathcal{E}_2$ of c.e. sets — where we say A is 2-*c.e.* or *d.c.e.* if and only if $A = B - C$ for some c.e. sets B, C. Dynamically, expressed in terms of mistaken approximation, A is d.c.e. if A has a computable sequence of finite approximations A^s such that for each x

$$|\{s \mid A^{s+1}(x) \neq A^s(x)\}| \leq 2.$$

In general, replacing ≤ 2 with $\leq n$ here we get the notion of n-c.e.

Does this new information content have any interesting computability-theoretic characteristics? Interestingly, Arslanov's completeness criterion extends to the n-c.e. sets [4] (see [32]). One immediately gets some very basic and natural questions, most importantly –

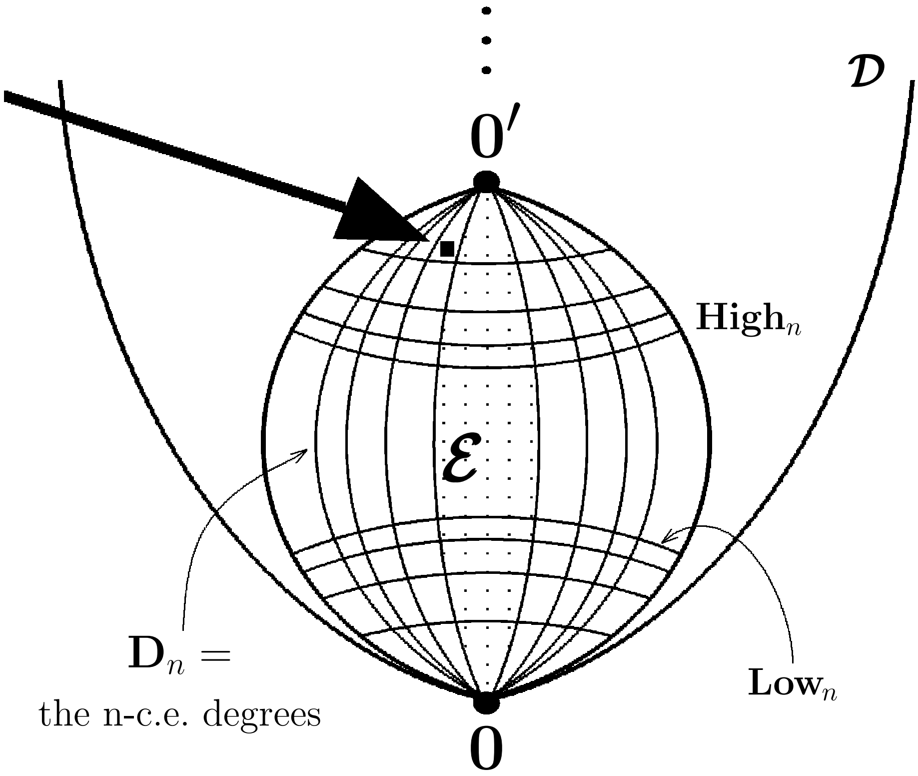
Question: *How does Posts programme extend to other finite levels of the Ershov hierarchy?*

In particular:

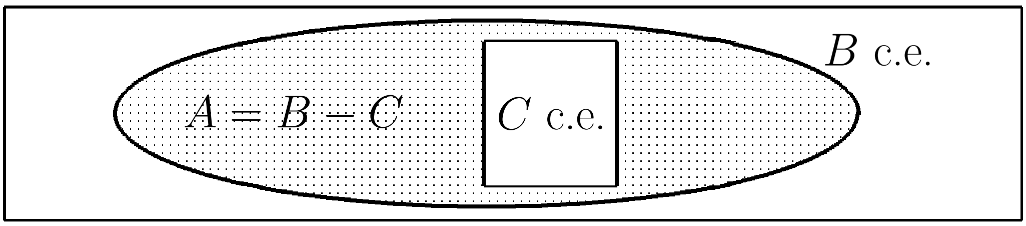
- Is there an easy analogue of Martins Theorem?
- Specifically, are the high d.c.e. degrees exactly those which contain a cohesive d.c.e. set?
- And are the high n-c.e. degrees exactly those which contain a cohesive n-c.e. or cohesive co– n-c.e. set?

One might expect positive answers to these specific questions, as always, the key to a general result being the inductive step from c.e. to d.c.e. For this we are very much zooming in on the information contained locally in the Turing landscape:

[4] And in relation to other reducibilities, it is worth mentioning that Arslanov's completeness criterion characterizes the Turing complete and wtt-complete but not the tt-complete c.e. sets.



Early on, Alastair Lachlan observed a very basic fact relating the c.e. degree structure to that of the d.c.e. sets. Looking at the following diagram, where we are given the d.c.e. $A = B - C$, with B, C c.e.:



one immediately sees that A is c.e. in C. So either A is c.e., or C is incomputable. There is no reason why C should be computable from A, of course. But if we replace C with $E = \{\langle x, s\rangle \mid x \in B^s - A\}$, we have that E is still a c.e. set. And A is still c.e. in E — *and* $E \leq_T A$, in fact $\leq_m A$. So:

- **Useful fact:** Every incomputable d.c.e. A has an incomputable c.e. $E \leq_T A$.

Following in Post's footsteps, we make a promising start in transferring his programme for $\mathcal{E}$ to $\mathcal{E}_2$ and beyond.

Given a Turing functional Φ computing on an input x with an oracle A, if $\Phi^A(x) \downarrow$ then there is largest y which A is queried about during the computation, called the *use* $\varphi^A(x)$ of the computation. We say that X is *weak truth-table reducible* to Y, i.e. $X \leq_{\mathrm{wtt}} Y$, if and only if X is computable from Y, with $\varphi^Y(x)$ computably bounded for the given X and Y. Then, paralleling Post's result for the c.e. degrees (see [1], [2]):

Theorem (Afshari, Barmpalias and Cooper, 2006): *Every d.c.e. wtt-degree $\neq \mathbf{0}$ contains an immune d.c.e. set.*

To see this: Let $\#_E(x)$ = the number of members of $E \leq x$, and $B = \{\langle x, \#_E(x)\rangle \mid x \in A\}$, a d.c.e. set. Then $B \leq_{\mathrm{wtt}} A$ since $E \leq_m A$, and so $E \leq_{\mathrm{wtt}} A$. Also, $A \leq_{\mathrm{wtt}} B$ since $x \in A$ iff $\langle x, y\rangle \in B$, some $y \leq x+1$. And B is immune. Otherwise, assume $F \subseteq B$ to be infinite. We cannot have F computable, since can compute E from F as follows : Given x, look for a $\langle y, \#_E(y)\rangle \in F$ with $x \leq y$, and then enumerate E to see if x is in E or not. Nor can F be be c.e., otherwise we could repeat the argument above with some infinite computable subset of F.

In fact, for the Turing degrees one can do better:

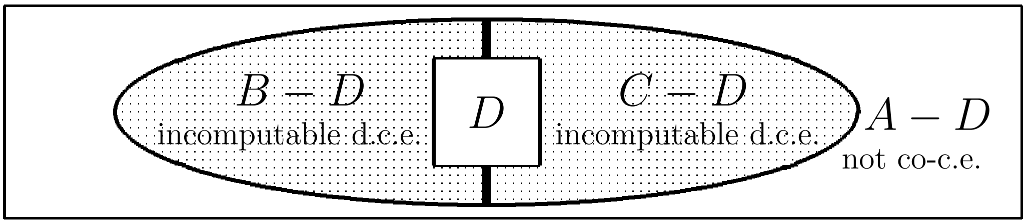
Theorem (ABC, 2006): *Every d.c.e. Turing degree $\neq \mathbf{0}$ contains a hyperimmune d.c.e. set.*

And these results can be extended to every finite level of the n-c.e. hierarchy. But — for odd n we will get *co–* n-c.e. immune or hyperimmune sets of the required degree, for reasons that should be clear — for instance, a non-trivially 3-c.e. set $A = (B - C) \cup D$ cannot be immune, since $B \cap C \cap D \subseteq A$.

Unfortunately, that is as far as one can stretch the straightforward parallels between Post's programme for the two different contexts. As Friedberg's splitting theorem tells us, any uncomplemented member A of $\mathcal{E}$ can be split into two uncomplemented members B, C of $\mathcal{E}$. The Owings splitting theorem lifts this result one level up the Ershov hierarchy. Let $\mathcal{E}(D)$ denote the set of c.e. supersets of a set D. Then:

Owings Splitting Theorem (1967): *If D is c.e., then any uncomplemented member of $\mathcal{E}(D)$ can be split into two uncomplemented members B, C of $\mathcal{E}(D)$.*

From this, it is easy to see from the picture below that the only cohesive d.c.e. sets are co-c.e. If $A - D$ is not co-c.e., then Owings splitting theorem gives the indicated c.e. splitting by B.



Of course, we know by the Cooper-Lempp-Watson density theorem [13] that the Ershov hierarchy collapses nowhere in the high degrees, so one is left looking for other natural information with which to push forward Post's programme extended.

Weaker properties of sets characterising the high Δ_2 degrees include hyperhyperimmune, r-cohesive and dense immune — see [9] or [55] for definitions — but:

- **ABCS, 2007:** *None of these work for n-c.e. with* $n > 1$.

For hyperhyperimmunity, infinitary iteration of the Owings splitting theorem gives a similar result to that for cohesiveness [5] :

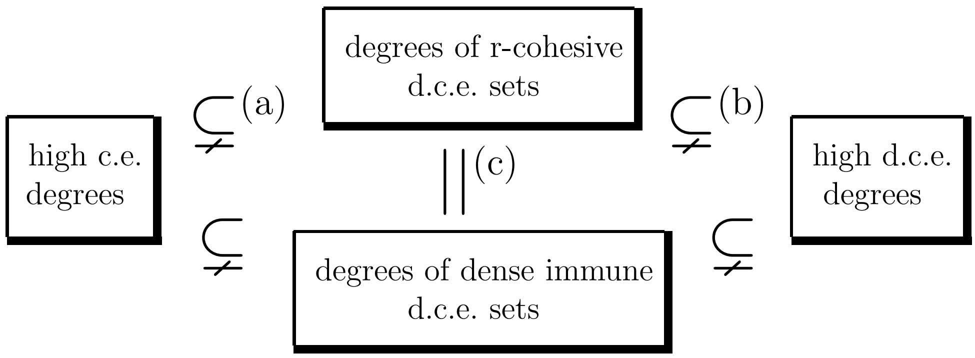
Theorem (ABCS, 2007): *If A Is n-c.e. and hh-immune then A is co-c.e.*

For the other notions the situation is more interesting, and brings out a much more worrying connection between the information content of an n-c.e. set and its computably related c.e. sets — *n-c.e. sets derive basic information content from their c.e. Turing predecessors.* In particular, this gives a new importance to the Lachlan c.e. predecessor of an n-c.e. set. Let A be n-c.e. Then:

Theorem (ABCS, 2007):
(1) A or $\overline{A}$ *can be r-cohesive* $\iff \exists$ *an r-cohesive c.e.* $E \leq_T A$*, and*
(2) A or $\overline{A}$ *can be dense immune* $\iff \exists$ *a dense immune c.e.* $E \leq_T A$.

The following diagram summarises what is known. For simplicity it refers to the 2-c.e. level, but a similar situation pertains to the higher levels of the n-c.e. hierarchy.

[5] We are grateful to one of the referees for pointing out that a more direct proof of this result can be obtained using Lachlan's Theorem 2 of [37].



(a) follows, since high c.e. degrees contain r-cohesive/ dense immune co-c.e. sets. And — by the Cooper-Lempp-Watson density theorem — there are properly d.c.e. degrees above any incomplete c.e. degree.

(b) holds since Since all r-cohesive/ dense immune Δ_2 sets are high. And — Ishmukhametov-Wu [31] (see also [38]) —- there exist high d.c.e. degrees all of whose c.e. predecessors are low.

Finally, (c) holds since the equality holds over the degrees of co-c.e. sets, and this is all that is needed to transmit both properties from c.e. predecessors to d.c.e. degrees bounding them.

So there remain a number of questions:

- What sort of natural information content *does* extend to other levels of the Ershov and high/low hierarchies?
- Can one extend automorphism techniques from the computably enumerable context to break the link between computability theoretic structure and natural information content for the Ershov hierarchy?
- Or will Harrington-Nies type codings work here?
- What about the non-low$_2$ n-c.e. degrees?

Relating to this last question, there are analogous results to those found for the high degrees, see [2] again.

5 New Kinds of Information Content

As we shall see, this section bridges the gap between the traditional mathematical content of Post's programme, and its extension to information originating from the scientific context. Randomness has both powerful mathematically intuitive content, and important physical manifestations; and pinning down the connection between the two presents challenges likely to prove fruitful in both contexts. The intention here is to do no more than point to the importance and relevance of randomness-related information content and its computability-theoretic ramifications. For full treatment of the background and technical content, see [44] and [18].

The mathematical intuition relating to randomness takes various forms. One says that a string $\sigma \in 2^{<\omega}$ is *random* if it has no shorter description than itself. Given a Turing machine U, define the *Kolmogorov complexity* relative to U to be $C_U(\sigma) = |\tau|$ for the shortest τ with $U(\tau) \downarrow= \sigma$. Moving from finite to infinite objects, we want to say a real A is *random* if the Kolmogorov complexity of each beginning $A \restriction n$ is no less than $n - c$, for some fixed constant c, where we remove the dependence on a particular U by making U here universal. But unfortunately — as Martin-Löf found — there are no such reals. We ask too much.

To try and do better, let us say the TM M is *prefix-free* if M is defined on no pair $\sigma \subsetneq \rho$. On can easily define a sensible notion of universal prefix-free U. Then the *prefix-free complexity* $K(\sigma) = K_U(\sigma) = |\tau|$ for the shortest τ with $U(\tau) \downarrow= \sigma$.

Definition (Levin, Schnorr, Chaitin): *A is K-random if $K(A \restriction n) > n - c$ for all n, some constant c.*

This is now a good mathematical expression of the intuition. There are other very persuasive formulations of what it means to be random. For a comprehensive guide to some of the issues involved, and a fuller outline of the historical background, see [20]. The intuition that a random real is one whose form is hard to pin down is expressed algorithmically by the above definition. One can also express it measure-theoretically, via the notion of Martin-Löf randomness — see [9]. What is reassuring is the robustness of the various formulations. A number of the different notions of randomness turn out to be equivalent, although relativised notions based on them can produce subtly different outcomes. For instance, we find:

- **Schnorr:** *A is K-random $\iff$ A is Martin-Löf random.*

One of the most important relativisations compares the K-complexity of different reals, and uses this to come up with an important class of reals which are close in this context to the computable reals:

Definition: *(1) $A \leq_K B \iff K(A \restriction n) \leq K(B \restriction n) + c$ for all n, some c.*
(2) A is K-trivial $\iff$ $A \leq_K \mathbb{N}$.

- **Solovay (1975):** *There exist incomputable K-trivials.*

Even at the level of relativised notions, one can encounter striking robustness.

Relativising the notion of Martin-Löf randomness (often called 1-randomness), one gets a natural notion of *A-random*. Then:

Definition: *(1)* $A \leq_{LR} B \iff$ *every B-random real is A-random.*
(2) A is low for random $\iff A \leq_{LR} \mathbb{N}$.

- **Nies:** *A is low for random* $\iff$ *A is K-trivial.*

Notice that $\leq_{LR}$ is an extension of Turing reducibility, giving a special significance to the structure of the LR-degrees.

What emerges from all this analysis of new kinds of information content — and this is an area which generates a confusingly large number of interesting results — is a very different solution to Post's Problem, one which is not just injury-free, but requirement free (see [19]). Even more (see [43] and [20]) we find:

Nies (2005): *(1) The K-trivials* $\subsetneq$ **low** *— in fact* $\subsetneq$ *the superlow degrees. And: (2) There is a low*$_2$ *c.e. degree* $\geq$ *all K-trivials — where this degree cannot be low.*

Recently, Kucera and Slaman have succeeded in constructing a non-c.e. low Turing bound for the K-trivials.

The significance of the work described in this section is not so much its relationship to Post's Problem, since for the disinterested observer Kucera's 1986 priority-free solution (see [9], p.333) is probably quite sufficient in that direction. No, what is being revealed is not just CV-building mathematics, but new and unexpected levels of information, computationally structured in new and exciting ways. For many years, for example, the low degrees had appeared essentially indivisible in the Turing context, and now we are seeing a surprising richness of Turing infra-structure emerging even at this level. As (pursuing our Google Earth analogy) we zoom in, we find not just information on the form of the terrain, but information structured into emergent computability-theoretic relations.

Even so, the Laputan aspect of all this activity can only be countered by indications of some sort of relationship to 'big science'. Pure mathematics can survive on its technical interest, but not without links, direct or indirect, to the real world — either via generally appreciated fundamental concepts such as dealt with by say number theory or combinatorics, or directly to an enhanced understanding of the world we live in. The aim of the rest of this article is to translate the musty recursion-theoretic cupboard into a magic wardrobe, and point to the hidden door at the back leading into an intriguing world with remarkable resonations with our own.

Here, we have the makings of a connection with the material world via quantum randomness. Quantum randomness is a familiar experimental and theoretical phenomenon. As Cristian Calude comments in a recent talk: "It passes all reasonable statistical properties of randomness." And Calude is able to prove, under reasonable assumptions, that it is Turing incomputable (see [4], [5] for more detail and background information). However, there remains a basic and intractable problem:

Open question: *How random is quantum randomness?*

What is needed is a better engagement between the mathematicians and the physicists over such questions. For the mathematicians there is a modelling challenge, for the physicists a need for better understanding of and respect for the mathematical underpinnings of their everyday assumptions about randomness. It is not widely appreciated that randomness is not absolute, and is as much a calibratable attribute as is incomputability. It may seem that by allowing quantum randomness as a a built-in feature of the physical universe one ameliorates the arbitrariness of elements of our theory, and the need to explain further. But then one realises that there is still an explanation needed of the *level* of randomness in evidence (if it is randomness), and once again we are confronted with the inadequacy of our explanations. It may seem a clever idea to allow God to play dice. But the neatness is somewhat dissipated by the need to describe *what sort of dice* is being played!

There is not just an academic interest in pinning down the character of quantum randomness. If we do our work well, we may find that we are not dealing with randomness at all, but rather a less specific form of incomputability arising in a mathematically explicable way from the familiar causal structure of the physical universe.

6 Zooming Out of the Turing Landscape

Notice that the question ending the last section sends us in a rather different direction, all within the framework of the extended and deeper Post's programme. Instead of zooming in on abstract information and seeking to characterise its algorithmic structure, we are now presented with very real and inescapable causal structure, accompanied by the information content of its particular instantiations, and the problem is to explain and characterise this connection. The difficulty is that recognition of these causal structures entails us taking a global view of an environment of which we ourselves are a component. When we look at the mysterious emergence of structure in nature, either subatomic laws, or the richness of life forms, or large-scale galactic or super-galactic structures, we are not just looking at information, but at expressions of patterns of a universal nature. And patterns the origins of which science is as yet unable to to explain.

When we inspect the intricacies of the Cat's Eye Nebula, say, as revealed by the Hubble Space Telescope, we feel we should be able to explain the remarkable complexity observed on the basis of our understanding of the local physics. The intuition is that it should be possible to describe global relations in terms of local structure, so capturing the emergence of large-scale structure. The mathematics pertaining to any particular example will be framed in terms of the specific interactive structure on which it is based. But if one wants to reveal general characteristics, and approach deep problems around the emergence of physical laws and constants, which current theory fails to do, one needs something more fundamental.

Schematically, we will argue below, any causal context framed in terms everyday computable mathematics can be modelled in terms of Turing reductions. Then emergence can be formalised as definability over the appropriate substructure of the Turing universe; or more generally, as invariance under automorphisms of the Turing universe. Simple and fundamental as the notion of definability is, and basic as it is to everyday thought and discourse, as a concept it is not well understood outside of logic. This is seen most strikingly in the physicists' apparent lack of awareness of the concept in interpreting the collapse of the wave function. Quantum decoherence and the many-worlds hypothesis comprise a far more outlandish interpretive option than does speculating that measurements, in enriching an environment, merely lead to an assertion of definability. It appears a sign of desperation to protect consistent histories by inventing new universes, when the mathematics of our observable universes already contains a straightforward explanation. We will argue below that many scientific puzzles can be explained in terms of failures of definability in different contexts, and that the key task is to identify useful theoretical models within which to investigate the nature of definability more fully. One of the most relevant of these models has to be that of Turing, based as it is on a careful analysis of the characteristics of algorithmic computation.

This brings us to another well-known research programme, initiated by Hartley Rogers in his 1967 paper [51], in which he drew attention to the fundamental problem of characterising the Turing invariant relations. Again, the intuition is that these are key to pinning down how basic laws and entities emerge as mathematical constraints on causal structure. It is important to notice how the richness of Turing structure discovered so far becomes the raw material for a multitude of non-trivially definable relations, matching in its complexity what we attempt to model.

Unfortunately, the current state of Rogers' programme is not good. For a number of years research in this area was dominated by a proposal originating with the Berkeley mathematician Leo Harrington, which can be (very) roughly stated:

Bi-interpretability Conjecture: *The Turing definable relations are exactly those with information content describable in second-order arithmetic.*

Most importantly, bi-interpretability is not consistent with the existence of non-trivial Turing automorphisms. Despite decades of work by a number of leaders in the field, the exact status of the conjecture is still a matter of controversy.

For those of us who have grown up with Thomas Kuhn's 1962 book [35] on the structure of scientific revolutions, such difficulties and disagreements are not seen as primarily professional failures, or triggers to collective shame (although they may be that too), but rather signs that something scientifically important is at stake. A far more public controversy currently shapes developments around important issues affecting theoretical physics — see, for example the recent books of Lee Smolin [54] and Peter Woit [59].

As Peter Woit [59, p.1] describes, according to purely pragmatic criteria particle physics has produced a standard model which is remarkably successful, and has great predictive power:

> By 1973, physicists had in place what was to become a fantastically successful theory of fundamental particles and their interactions, a theory that was soon to acquire the name of the standard model. Since that time, the overwhelming triumph of the standard model has been matched by a similarly overwhelming failure to find any way to make further progress on fundamental questions.

The reasons why people are dissatisfied echo misgivings going back to Einstein himself [21, p.63]:

> . . . I would like to state a theorem which at present can not be based upon anything more than upon a faith in the simplicity, i.e. intelligibility, of nature . . . nature is so constituted that it is possible logically to lay down such strongly determined laws that within these laws only rationally completely determined constants occur (not constants, therefore, whose numerical value could be changed without destroying the theory) . . .

If one really does have a satisfying description of how the universe is, it should not contain arbitrary elements with no plausible explanation. In particular, a theory containing arbitrary constants, which one adjusts to fit the intended interpretation of the theory, is not complete. And as Woit observes:

> One way of thinking about what is unsatisfactory about the standard model is that it leaves seventeen non-trivial numbers still to be explained, ...

At one time, it had been hoped that string theory would supply a sufficiently fundamental framework to provide a much more coherent and comprehensive description, in which such arbitrary ingredients were properly pinned down. But despite its mathematical attractions, there are growing misgivings about its claimed status as "the only game in town" as a unifying explanatory theory. Here is how one time string theorist Daniel Friedan [23] combatively puts it:

> The longstanding crisis of string theory is its complete failure to explain or predict any large distance physics. ... String theory is incapable of determining the dimension, geometry, particle spectrum and coupling constants of macroscopic spacetime. ... The reliability of string theory cannot be evaluated, much less established. String theory has no credibility as a candidate theory of physics.

Smolin starts his book [54]:

> From the beginning of physics, there have been those who imagined they would be the last generation to face the unknown. Physics has always seemed to its practitioners to be almost complete. This complacency is shattered only during revolutions, when honest people are forced to admit that they don't know the basics.

He goes on to list what he calls the "five great [unsolved] problems in theoretical physics". Gathering these together, and slightly editing, they are [54, pp.5-16]:

1. Combine general relativity and quantum theory into a single theory that can claim to be the complete theory of nature.
2. Resolve the problems in the foundations of quantum mechanics.
3. The unification of particles and forces problem: Determine whether or not the various particles and forces can be unified in a theory that explains them all as manifestations of a single, fundamental entity.
4. Explain how the values of the free constants in the standard model of physics are chosen in nature.
5. Explain dark matter and dark energy. Or, if they dont exist, determine how and why gravity is modified on large scales.

That each of these questions can be framed in terms of definability is not so surprising, since that is exactly how, essentially, they are approached by researchers. The question is the extent to which progress is impeded by a lack of consciousness of this fact, and an imperfect grip of what is fundamental. Quoting Einstein again (from a letter to Robert Thornton, dated 7 December 1944, Einstein Archive 61-754), this time on the relevance of a philosophical approach to physics:

> So many people today – and even professional scientists – seem to me like someone has seen thousands of trees but has never seen a forest. A knowledge of the historical and philosophical background gives that kind of independence from prejudices of his generation from which most scientists are suffering. This independence created by philosophical insight is – in my opinion – the mark of distinction between a mere artisan or specialist and a real seeker after truth.

Smolin's comment [54, p.263] is in the same direction, though more specifically directed at the string theorists:

> The style of the string theory community ... is a continuation of the culture of elementary-particle theory. This has always been a more brash, aggressive, and competitive atmosphere, in which theorists vie to respond quickly to new developments ... and are distrustful of philosophical issues. This style supplanted the more reflective, philosophical style that characterized Einstein and the inventors of quantum theory, and it triumphed as the center of science moved to America and the intellectual focus moved from the exploration of fundamental new theories to their application.

So what is it that is fundamental that is being missed? For Smolin [54, p.241], it is *causality*:

> It is not only the case that the spacetime geometry determines what the causal relations are. This can be turned around: Causal relations can determine determine the spacetime geometry ... Its easy to talk about space or spacetime emerging from something more fundamental, but those who have tried to develop the idea have found it difficult to realize in practice. ... We now believe they failed because they ignored the role that causality plays in spacetime. These days, many of us working on quantum gravity believe that *causality itself is fundamental* – and is thus meaningful even at a level where the notion of space has disappeared.

Citing Penrose as an early champion of the role of causality, he also mentions Rafael Sorkin, Fay Dowker, and Fotini Markopoulou, known in this context for their interesting work on causal sets (see [3]), which abstract from causality relevant aspects of its underlying ordering relation. Essentially, causal sets are partial orderings which are locally finite, providing a model of spacetime with built-in discreteness. Despite the apparent simplicity of the mathematical model, it has had striking success in approximating the known characteristics of spacetime. An early prediction, in tune with observation, concerned the value of Einstein's cosmological constant.

Of course, this preoccupation with causality might suggest to a logician a need to also look at its computational content. Smolin's comment that "Causal relations can determine determine the spacetime geometry" touches on one of the biggest disappointments with string theory, which turns out to be a 'background dependant' theory with a vengeance — one has literally thousands of candidate Calabi-Yau spaces for shaping the extra dimensions of superstring theory. In current superstring models, Calabi-Yau manifolds are those qualifying as possible space formations for the six hidden spatial dimensions, their undetected status explained by the assumption of their being smaller than currently observable lengths.

Ideally, a truly fundamental mathematical model should be background independent, bringing with it a spacetime geometry arising from within.

7 Turing Invariance and the Laws of Physics

There are obvious parallels between the Turing universe and the material world. Each of which in isolation, to those working with specific complexities, may seem superficial and unduly schematic. But the lessons of the history of mathematics and its applications is that the simplest of abstractions can yield unexpectedly far-reaching and deep insights into the nature of the real world.

Most basic, science describes the world in terms of real numbers. This is not always immediately apparent, any more that the computer on ones desk is obviously an avatar of a universal Turing machine. Nevertheless, scientific theories consist, in their essentials, of postulated relations upon reals. These reals are abstractions, and do not come necessarily with any recognisable metric. They are used because they are the most advanced presentational device we can practically work with. There is no faith that reality itself consists of information presented in terms of reals. In fact, those of us who believe that mathematics is indivisible, no less in its relevance to the material world, have a due humility about the capacity for our science to capture more than a surface description of reality.

Some scientists would take us in the other direction, and claim that the universe is actually finite, or at least countably discrete. We have argued elsewhere (see for example [14]) that to most of us a universe without algorithmic content is inconceivable. And that once one has swallowed that bitter pill, infinitary objects are not just a mathematical convenience (or inconvenience, depending on ones viewpoint), but become part of the mathematical mold on which the world depends for its shape. As it is, we well know how essential algorithmic content is to our understanding of the world. The universe comes with recipes for doing things. It is these recipes which generate the rich information content we observe, and it is reals which are the most capacious receptacles we can humanly carry our information in, and practically unpack.

Globally, there are still many questions concerning the extent to which one can extend the scientific perspective to a comprehensive presentation of the universe in terms of reals — the latter being just what we need to do in order to model the immanent emergence of constants and natural laws from an entire universe. Of course, there are many examples of presentations entailed by scientific models of particular aspects of the real world. But given the fragmentation of science, is fairly clear that less natural presentations may well have an explanatory role, despite their lack of a role in practical computation.

The natural laws we observe are largely based on algorithmic relations between reals. For instance, Newtonian laws of motion will computably predict, under reasonable assumptions, the state of two particles moving under gravity over different moments in time. And the character of the computation involved can be represented as a Turing functional over the reals representing different time-related two-particle states. One can point to physical transitions which are not obviously algorithmic, but these will usually be composite processes, in which the underlying physical principles are understood, but the mathematics of their workings outstrip available analytical techniques. Over forty years ago, Georg Kreisel [33] distinguished between classical systems and *co-operative phenomena* not known to have Turing computable behaviour, and proposed [34, p.143, Note 2] a collision problem related to the 3-body problem, which might result in "an analog computation of a non-recursive function (by repeating collision experiments sufficiently often)". However, there is a qualitatively different apparent breakdown in computability of natural laws at the quantum level — the *measurement problem* challenges us to explain how certain quantum mechanical probabilities are converted into a well-defined outcome following a measurement. In the absence of a plausible explanation, one is denied a computable prediction. The physical significance of the Turing model depends upon its capacity for explaining what is happening here. If the phenomenon is not composite, it does need to be related in a clear way to a Turing universe designed to model computable causal structure. We will need to talk more about definability and invariance.

For the moment, let us think in terms of what an analysis of the automorphisms of *any* sufficiently comprehensive, sufficiently fundamental, mathematical model of the material universe might deliver.

Let us first look at the relationship between automorphisms and many-worlds. When one says "I tossed a coin and it came down heads, maybe that means there is a parallel universe where I tossed the coin and it came down tails", one is actually predicating a large degree of correspondence between the two parallel universes. The assumption that *you* exist in the two universes puts a huge degree of constraint on the possible differences — but nevertheless, some relatively minor aspect of our universe has been rearranged in the parallel one. There are then different ways of relating this to the mathematical concept of an automorphism. One could say that the two parallel worlds are actually isomorphic, but that the structure was not able to *define* the outcome of the coin toss. So it and its consequences appear differently in the two worlds. Or one could say that what has happened is that the worlds are *not* isomorphic, that actually we were able to change quite a lot, without the parallel universe looking very different, and that it was these fundamental but hidden differences which forces the worlds to be separate and not superimposed, quantum fashion. The second view is more consistent with the view of quantum ambiguity displaying a failure of definability. The suggestion here being that the observed existence of a particle (or cat!) in two different states at the same time merely exhibits an automorphism of our universe under which the classical level is rigid (just as the Turing universe displays rigidity above $\mathbf{0}''$) but under which the sparseness of defining structure at the more basic quantum level enables the automorphism to re-represent our universe, with everything at our level intact, but with the particle in simultaneously different states down at the quantum level. And since our classical world has no need to decohere these different possibilities into parallel universes, we live in a world with the automorphic versions superimposed. But when we make an observation, we establish a link between the undefined state of the particle and the classical level of reality, which destroys the relevance of the automorphism. To believe that we now get parallel universes in which the alternative states are preserved, one now needs to decide how much else one is going to change about our universe to enable the state of the particle destroyed as a possiblity to survive in the parallel universe — and what weird and wonderful things one must accommodate in order to make that feasible. It is hard at this point to discard the benefits brought by a little mathematical sophistication. Quantum ambiguity as a failure of definability is a far more palatable alternative than the invention of new worlds of which we have no evidence or scientific understanding.

Another key conceptual element in the drawing together of a global picture of our universe with a basic mathematical model is the correspondence between emergent phenomena and definable relations. This gives us a framework within which to explain the particular forms of the physical constants and natural laws familiar to us from the standard model science currently provides. It goes some way towards substantiating Penrose's [47, pp.106-107] 'strong determinism', according to which "all the complication, variety and apparent randomness that we see all about us, as well as the precise physical laws, are all exact and unambiguous consequences of one single coherent mathematical structure" — and repairs the serious failure of the standard model pointed to by researchers such as Smolin and Woit. It also provides a hierarchical model of the fragmentation of the scientific enterprise. This means that despite the causal connections between say particle physics and the study of living organisms, the corresponding disciplines are based on quite different basic entities and natural laws, and there is no feasible and informative reduction of one to another. The entities in one field may emerge through phase transitions characterised in terms of definable relations in the other, along with their distinct causal structures. In this context, it may be that the answer to Smolin's first 'great unsolved problem in theoretical physics' consists of an explanation of why there is no single theory (of the kind that makes useful predictions) combining general relativity and quantum theory.

The following table provides a summary of some of the main features of the Turing interpretation, drawing out parallels between scientific activity and what the Turing model provides. For further discussion of such issues, see [8], [10], [11], [12] and [14].

Science	**Turing landscape**
Physical entities modelled as information	Structures information
Theories describing relations over the reals, enabling calculations	Functionals over the reals modelled on real computational capabilities
An extensive basic causal structure which is algorithmic	Models computable causal relations over the reals
Descriptions of globally emerging laws and constants elusive	Problems pinning down the nature of Turing invariance and definability
Features quantum ambiguity and nonlocality	Explanation in terms of putative breakdown in Turing definability
Theoretical fragmentation involving phase transitions	Incomputability, and algorithmic relations over emergent objects

8 Post's Real Anticipation

So what would Post himself have thought of the past fifty years' history of the programme that has been ascribed to him? One view is that his interest would have been satisfied somewhere around the point at which Harrington and Soare discovered their $\mathcal{E}$-definable solution to Post's problem. On the other hand, Post did have a keen interest in broader issues, and the quotations which at the end of Section 1 show his belief in the wider role of logic.

What is certain, is that Post's early recognition of the importance of the relation between information content and computability theoretic structure has been hugely influential. And that, together with Hartley Rogers seminal focus on globally emergent relations, has given rise to a Programme which remains full of interest, surprises, and open questions. Some of these questions are very hard to clarify and to give answers to that would satisfy the mathematician's need for technically verifiable proofs. This raises the question of the relative value given to work with a speculative (or philosphical) component, and work whose significance is encapsulated in theorems. Some logicians question the value of interdisciplinarity in computability, and point to the superior record of model theorists, for instance, in actually proving theorems connecting different areas.

What is being lost here is the distinction between primary and secondary attempts at mathematical descriptions of aspects of the real world. When Newton developed the conceptual framework that enabled him to describe precisely the Turing functional that provided the algorithmic machinery to predict the movements of planets and comets, the link between mathematics and application did not depend on a theorem, but on the comparison between the the way the functional computed and observational data. The initial leap from real world to Turing functional was speculative, because that is the way primary connections with the real world originate. At the time, some people were uncomfortable with Newton's 'action at a distance', and it is said that Newton's alchemical interests were what were conducive to him being able to make the necessary conceptual leap. In recent times, when model theorists have proved theorems which capture aspects of the real world, they are going via a body of existing mathematics with established relevance to the real world, and as such this process of linking takes place entirely within mathematics, and can be described as secondary. On the one hand it is not open to question in the way that primary linking to the real world is. But on the other it is not, for all its mathematical interest, likely to change our view of the world in any radical way. Of course, we now know that Newton was wrong in various ways — we can now replace the picture of action at a distance with a reinstated mechanical one based on a better understanding of particle physics, and we know that at larger scales Newtonian dynamics must be replaced by a Einstein's relativity. But for all that, Newton's primary speculations retain their scientific value.

It may seem a little paradoxical to claim that the focus of science since the time of Newton has been on the extraction of algorithmic content, while at the same time championing the speculative component of scientific progress. But this is certainly true of the person who claims that this is wrong, characterising scientists as pure seekers after truth, while justifying the relevance of his or her own area on the basis of its algorithmic content, in the form of proofs of theorems. The aim of science may be the extraction of algorithmic content of the real world, but this is not to say that science is or even should be algorithmic. Certainly people look for truth, while benefitting greatly from the scientific narrowing of the focus. So when Newton speculated on the laws governing the Universe, he on the one hand was clearly extracting algorithmic content, very important for the communicability of his work and its usefulness in making predictions. But as a representative of the human race, he was also making a leap in the dark, in that he was guessing at truth, and in a way that was imperfect — but with sufficient validity to give his work lasting importance.

Closer to home, Frege's work on logic was primary, and speculative, an attempt to formalise our intuitions about how we desribe the world in language and reason about it. There *were* theorems involved (based on what turned out to be an inconsistent theory), but that is not what Frege is remembered for. Again, when Turing wrote his 1936 paper, a large proportion of it was a speculative attempt to link his machine model to what a resonable observer might understand as the possible scope of human computation (within carefully described bounds). The success of this part of his paper is validated firstly by Gödel's immediate conversion to the truth of the Church-Turing thesis, having previously been unconvinced by Church's arguments using his lambda computability model, and secondly by the durability of the Turing model in the theory and lay understanding of computers. One could go on — Gödel's mathematics and speculations are usually kept separate, unlike those of Turing, but the reason why Gödel and Turing ended up the only mathematicians (unless you count Einstein) in Time magazine's list of 100 makers of the 20th century was their perceived primary relevance to the real world. Gödel was probably there because of the speculative significance, good and bad, that people have extracted from his incompleteness theorem. Of course, Hilbert's work may be valued highly within mathematics, as may theorems connecting different areas of mathematics, but that is not what gives them a place in the wider world.

What is important is that there are different kinds of scientific activity which have validity, and which make science the rich and exciting field it is. If Gauguin had been a worse painter, we would have said that the truth he generated working in his bank was far more worthwhile than the speculative truth he sought after in his paintings in Tahiti. As it is, we are grateful for his bravery and vision, and do not worry that he left his respectable, salaried, solidly algorithmic banking activities. Unfortunately, most of us have to wait for history to tell us if we would have been better served paying more attention to 'normal science'.

References

[1] B. Afshari, G. Barmpalias and S. B. Cooper, Immunity properties and the n-c.e. hierarchy, In *Theory and Applications of Models of Computation - Proceedings of the Third Annual Conference on Computation and Logic, TAMC06, Beijing,*

May 2006 (J.-Y. Cai, S. B. Cooper, A. Li, eds.), Springer Lecture Notes in Computer Science No. 3959, 2006, pp. 694–703.

[2] B. Afshari, G. Barmpalias, S. B. Cooper and F. Stephan, Post's Programme for the Ershov Hierarchy, *J. of Logic and Computation*, **17** (2007), 1025–1040.

[3] L. Bombelli, J. Lee, D. Meyer and R. D. Sorkin, Spacetime as a causal set, *Phys. Rev. Lett.*, **59** (1987), 521–524.

[4] C. Calude, Algorithmic randomness, quantum physics, and incompleteness. In Maurice Margenstern (Ed.), *Machines, Computations, and Universality, 4th International Conference, MCU 2004, Saint Petersburg, Russia, September 21–24, 2004, Revised Selected Papers*, Lecture Notes in Computer Science 3354, Springer, 2005, pp. 1–17.

[5] C. S. Calude and K. Svozil, Quantum randomness and value indefiniteness, *Advanced Science Letters*, to appear.

[6] P. Cholak, Automorphisms of the lattice of recursively enumerable sets, *Memoirs of the Amer. Math. Soc.*, Vol. 113, 1995.

[7] P. Cholak and L. Harrington, Definable encodings in the computably enumerable sets, *Bull. of Symbolic Logic*, **6** (2000), 185–196.

[8] S. B. Cooper, Clockwork or Turing U/universe? - Remarks on causal determinism and computability, in *Models and Computability* (S. B. Cooper and J. K. Truss, eds.), London Mathematical Society Lecture Notes Series 259, Cambridge University Press, Cambridge, New York, Melbourne, 1999, pp.63–116.

[9] S. B. Cooper, *Computability Theory,* Chapman & Hall/CRC, Boca Raton, London, New York, Washington, D.C., 2004.

[10] S. B. Cooper, Definability as hypercomputational effect, *Applied Mathematics and Computation*, **178** (2006), 72–82.

[11] S. B. Cooper, How Can Nature Help Us Compute?, in *SOFSEM 2006: Theory and Practice of Computer Science - 32nd Conference on Current Trends in Theory and Practice of Computer Science, Merin, Czech Republic, January 2006* (J. Wiedermann, J. Stuller, G. Tel, J. Pokorny, M. Bielikova, eds.), Springer Lecture Notes in Computer Science No. 3831, 2006, pp.1–13.

[12] S. B. Cooper, Computability and emergence, in *Mathematical Problems from Applied Logic I. Logics for the XXIst Century* (D.M. Gabbay, S.S. Goncharov, M. Zakharyaschev, eds.), Springer International Mathematical Series, Vol. 4, 2006, pp. 193–231.

[13] S. B. Cooper, S. Lempp and P. Watson, Weak density and cupping in the d-r.e. degrees, *Israel J. Math.*, **67** (1989), 137–152.

[14] S. B. Cooper and P. Odifreddi, Incomputability in Nature, in *Computability and Models* (S.B. Cooper and S.S. Goncharov, eds.), Kluwer Academic/Plenum, New York, Boston, Dordrecht, London, Moscow, 2003, pages 137–160.

[15] M. Davis, Arithmetical problems and recursively enumerable predicates (abstract), *J. of Symbolic Logic*, **15** (1) (1950), 77–78.

[16] M. Davis, Arithmetical problems and recursively enumerable predicates, *J. of Symbolic Logic*, **18** (1) (1950), 33–41.

[17] M. Davis (Ed.), *Solvability, Provability, Definability: The Collected Works of Emil L. Post*, Birkhäuser, Boston, Basel, Berlin, 1994.

[18] R. Downey and D. Hirschfeldt, *Algorithmic Randomness and Complexity*, to appear.

[19] R. Downey, D. R. Hirschfeldt, A. Nies and F. Stephan, Trivial reals, in *Proceedings of the 7th and 8th Asian Logic Conferences* (R. Downey, D. Decheng, T. S. Ping, Q. Y. Hui and M. Yasugi, eds.), Singapore University Press/ World Scientific, 2003, pp. 103–131.

[20] R. Downey, D. R. Hirschfeldt, A. Nies, and S. A. Terwijn, Calibrating randomness, *Bulletin of Symbolic Logic*, **12** (3) (2006), 411–491.

[21] A. Einstein, Autobiographical Notes, in *Albert Einstein: Philosopher-Scientist* (P. Schilpp, ed.), Open Court Publishing, 1969.

[22] Y. L. Ershov, A certain hierarchy of sets, i, ii, iii. (Russian), *Algebra i Logika,* **7** (1968), 47–73; **7** (1968), 15–47; **9** (1970), 34–51.

[23] D. Friedan, A Tentative Theory of Large Distance Physics, *J. High Energy Phys.* JHEP10 (2003) 063.

[24] R. M. Friedberg, Two recursively enumerable sets of incomparable degrees of unsolvability, *Proc. Nat. Acad. Sci. (USA)* **43** (1957) 236–238.

[25] E. M. Gold, Limiting recursion, *J. Symbolic Logic*, **30** (1965), 28–48.

[26] L. Harrington and R. I. Soare, Post's Program and incomplete recursively enumerable sets, *Proc. Nat. Acad. Sci. (USA)* **88** (1991) 10242–10246.

[27] L. Harrington and R. I. Soare, Definability, automorphisms, and dynamic properties of computably enumerable sets, *Bull. of Symbolic Logic*, **2** (1996), 199–213.

[28] L. Harrington and R. I. Soare, The Δ^0_3-automorphism method and noninvariant classes of degrees, *J. of the Amer. Math. Soc.*, **9** (1996), 617–666.

[29] L. Harrington and R. I. Soare, Codable sets and orbits of computably enumerable sets, *J. of Symbolic Logic*, **63** (1) (1998), 1–28.

[30] A. Hodges, *Alan Turing: The Enigma,* Vintage, London, Melbourne, Johannesburg, 1992.

[31] S. Ishmukhametov and G. Wu, Isolation and the high/low hierarchy, *Archive for Math. Logic*, **41** (2002), 259–266.

[32] C. G. Jockusch Jr., M. Lerman, R. I. Soare and R. Solovay, Recursively enumerable sets modulo iterated jumps and extensions of Arslanov's Completeness Criterion, *J. Symbolic Logic*, **54** (1989),1288–1323.

[33] G. Kreisel, Mathematical logic: What has it done for the philosophy of mathematics?, in *Bertrand Russell, Philosopher of the Century* (R. Schoenman, ed.), Allen and Unwin, London, 1967, pp. 201–272.

[34] G. Kreisel, Church's Thesis: a kind of reducibility axiom for constructive mathematics, in *Intuitionism and proof theory: Proceedings of the Summer Conference at Buffalo N.Y. 1968* (A. Kino, J. Myhill and R.E. Vesley, eds.), North-Holland, Amsterdam, London, 1970, pp. 121–150.

[35] T. S. Kuhn, *The Structure of Scientific Revolutions*, Third edition 1996, University of Chicago Press, Chicago, London.

[36] A. H. Lachlan, Degrees of recursively enumerable sets which have no maximal superset, *J. Symbolic Logic*, **33** (1968), 431–443.

[37] A. H. Lachlan, On the lattice of recursively enumerable sets, *Trans. of the Amer. Math. Soc.*, **130** (1968), 1–37.

[38] A. Li, G. Wu and Y. Yang, *Bounding computably enumerable degrees in the Ershov hierarchy*, *Annals of Pure and Applied Logic*, **141** (2006), 79–88.

[39] Y. Matiyasevich, *Hilbert's tenth problem*, MIT Press, Boston, 1993.

[40] R. Murawski, E. L. Post and the development of mathematical logic and recursion theory, *Studies in Logic, Grammar and Rhetoric*, **2** (15) (1998), 17–30.

[41] A. A. Muchnik, On the unsolvability of the problem of reducibility in the theory of algorithms, *Dokl. Akad. Nauk SSSR, N.S.* **109** (1956) 194–197 (Russian).

[42] I. Newton, *Philosophiae Naturalis Principia Mathematica*, London, 1687.

[43] A. Nies, Lowness properties and randomness, *Adv. Math.*, **197** (2005), 274–305.

[44] A. Nies, *Computability and Randomness*, to appear.

[45] P. Odifreddi, *Classical Recursion Theory*, North-Holland, Amsterdam, New York, Oxford, Tokyo, 1989.

[46] P. Odifreddi, *Classical Recursion Theory, Volume II*, Elsevier, Amsterdam, Lausanne, New York, Oxford, Shannon, Singapore, Tokyo, 1999.

[47] R. Penrose, Quantum physics and conscious thought, in *Quantum Implications: Essays in honour of David Bohm* (B.J. Hiley and F.D. Peat, eds.), Routledge & Kegan Paul, London, New York, 1987, pp. 105–120.

[48] E. L. Post, Recursively enumerable sets of positive integers and their decision problems, *Bulletin of the American Mathematical Society*, **50** (1944), 284–316; reprinted in E. L. Post, *Solvability, Provability, Definability: The Collected Works of Emil L. Post*, pages 461–494.

[49] E. L. Post, Absolutely Unsolvable Problems and Relatively Undecidable Propositions – Account of an Anticipation; in M. Davis (Ed.), *The Undecidable*, Raven Press, New York 1965, pp. 340–433; reprinted in E. L. Post, *Solvability, Provability, Definability: The Collected Works of Emil L. Post*, pp. 375–441.

[50] H. Putnam, Trial and error predicates and the solution to a problem of Mostowski, *J. Symbolic Logic*, **30** (1965), 49–57.

[51] H. Rogers, Jr., Some problems of definability in recursive function theory, in *Sets, Models and Recursion Theory* (J.N.Crossley, ed.), Proceedings of the Summer School in Mathematical Logic and Tenth Logic Colloquium, Leicester, August–September, 1965, North-Holland, Amsterdam, pp. 183–201.

[52] J. R. Shoenfield, On degrees of unsolvability, *Annals of Mathematics*, **69** (1959), 644–653.

[53] J. R. Shoenfield, Degrees of classes of recursively enumerable sets, *J. Symbolic Logic*, **41** (1976), 695–696.

[54] L. Smolin, *The Trouble With Physics: The Rise of String Theory, the Fall of Science and What Comes Next,* Allen Lane/Houghton Mifflin, London, New York, 2006.

[55] R. I. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, New York, 1987.

[56] A. Turing, On computable numbers, with an application to the Entscheidungsproblem, *Proceedings of the London Mathematical Society*, Vol. 42, 1936–37, pages 230–265; reprinted in A. M. Turing, *Collected Works: Mathematical Logic*, pages 18–53.

[57] A. Turing, Systems of logic based on ordinals, *Proceedings of the London Mathematical Society*, Vol. 45, 1939, pages 161–228; reprinted in A.M. Turing, *Collected Works: Mathematical Logic*, pages 81–148.

[58] A. M. Turing, *Collected Works: Mathematical Logic* (R. O. Gandy and C. E. M. Yates, Editors), Elsevier, Amsterdam, New York, Oxford, Tokyo, 2001.

[59] P. Woit, *Not Even Wrong: The Failure of String Theory and the Continuing Challenge to Unify the Laws of Physics,* Jonathan Cape, London, 2006.