INCOMPUTABILITY IN NATURE

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Abstract To what extent is incomputability relevant to the material Universe? We look at ways in which this question might be answered, and the extent to which the theory of computability, which grew out of the work of Gödel, Church, Kleene and Turing, can contribute to a clear resolution of the current confusion. It is hoped that the presentation will be accessible to the non-specialist reader.

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1. An Historical Parallel

To the average scientist, incomputability in nature must appear as likely as 'action at a distance' must have seemed before the appearance of Newton's *Principia*. One might expect expertise in the theory of incomputability — paralleling that of alchemy in the seventeenth century — to predispose one to an acceptance of such radical new ideas. But specialist recursion theorists as

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a group have shown remarkably little interest in the more general implications of their work.[1] Part of the reason for this is tricky technical obstacles to even formulating appropriate notions and questions concerning computability in the real world.

However, at a purely intuitive level, one has a seemingly unproblematic model of a deterministic, even mechanical, Universe. Its scientific foundations were Newtonian mechanics, and its philosophical basis most clearly traced back to Laplace's predictive demon.[2] The model lives on in contemporary science surviving, at least in the recesses of the scientists mind, the proliferating challenges to old certainties. In particular, it has achieved an uneasy coexistence with notions of randomness (a very special kind of incomputability) forced on us by greater knowledge of subatomic phenomena.

And so, despite the development of the mathematical theory of computability initiated by Gödel, Turing, Church, Post and Kleene in the 1930s, one is still left with uncertainty as to how it applies to the real world. One can get an idea of the scope for quite contrasting, but arguably valid, approaches to quantifying incomputability in natureincomputability, in Nature from the second author's discussion of Church's thesis and its extensions (in the first volume of his book on "Classical Recursion Theory"). So what is missing from the picture we have? How could a belief (or otherwise) in incomputability in nature be substantiated? And how can one enrich, improve, and make more explicit a mathematical model for what is happening?

One approach — the search for overt incomputability — is to take mathematical equations known to accurately describe some natural phenomenon, and to extract solutions exhibiting incomputability in some generally convincing form. Or, on the other hand, to argue that the persistent failure of this approach increasingly confirms there to be no such incomputability to identify.

What weakens this latter position is the obvious observation that efforts to consolidate the Laplacian model, on a foundation of new science and an appropriate model of the underlying mechanisms, seem even more hopeless. While one can point to quite plausible obstacles to overtness. There may, for instance, be *mathematical* constraints on what incomputabilities can be described, which make it hard to get to grips with the computability, or otherwise, of those natural phenomena for which we do have correlative mathematics.

[1] Some have seen the developing terminology of the subject itself — the replacing of Turing's 'computable' by Kleene's 'recursive' — as being intimately related to a turning away from its roots in real problems. It may be no coincidence that the recent reinstatement of Turing's terminology has come at the same time as a growing concern, both within and without the specialism, about whether classical recursion theory *has* any significant general implications.

[2] Of course, Laplace himself did not set out to provide any such model, and it is only twentieth century science which allows us to view the conceptual framework he helped establish in such terms.

For instance, we have in mind the failure to discover 'natural' examples of incomputable sets of integers from amongst the rich variety of incomputable 'almost computable' (i.e. *computably enumerable*) sets. This current lack of mathematically natural examples of incomputable sets has been used as an argument against being interested in such things. But naturalness in the rarified atmosphere of the university pure mathematics department is a very different thing to that of everyday usage. That very mathematical unnaturalness of most sets of numbers may be what presents an obstacle to overt incomputability of objects 'existing in or caused by nature'.[3] It is well-known that one needs no more than high school mathematics to describe incomputable sets (all computably enumerable sets are *diophantine*, got by looking for solutions to appropriate equations from basic school arithmetic); but that those *known* to be incomputable come with descriptions which can be tied in closely to diagonalising techniques in computability theory (definitely not part of the normal school syllabus!).

One can relax the overt approach by looking for incomputability which emerges from mathematics which *looks somewhat like* the mathematics one sees applied scientifically. But despite various proposals[4] in this direction, sceptics tend not to be impressed. They find plenty of scope for arguing that the mathematics involved is *not* typical, or for throwing doubt on the role of incomputability in it.

Finally one must admit that to those not involved, such detailed discussions seem fairly academic. Why should those without a direct career interest care whether actual incomputability (suitably formalised) occurs in Nature? Even if it did occur, for all practical purposes, how would it be distinguishable from theoretically computable but very *complex* phenomena? Whether chaotic phenomena – such as turbulence – involve complexity or incomputability is interesting, but does it really *matter*?

Fortunately, there is another approach — let's call it the *mathematical* approach — which renews the link with Newton. This is a direction rooted in the old debate about whether computability theory has any useful consequences for mathematics other than those whose statements depend on recursion theoretic terminology. Until recently the evidence was even less promising. Recently Soare has sought to present some interesting mathematics originating with researchers from outside computability theory, which both depends on the theory of incomputable sets, and at least looks like mathematics with real-world ex-

[3]From the definition of 'natural' given in *The New Oxford Dictionary of English*, 1998 edition.

[4]Probably the best-known and most convincing of these is that of Pour-El and for non computable solutions to the wave equation. In fact, some claim this does rather more than "look .. somewhat like the mathematics one sees applied scientifically".

planatory power. This gives an, admittedly very weak, parallel with Newton and action at a distance.

It is worth noting that action at a distance was not Newton's direct concern, but was an incidental, if remarkable, ingredient in a comprehensive mathematical theory with what was rapidly acknowledged to be invaluable explanatory power. Not even the inverse square law originated in isolation with Newton, but was familiar to Robert Hooke and others. What Newton had done was to go far beyond Hooke's observation, and to develop a body of mathematics which, with the sort of difficulty one is familiar with in correlating mathematics with nature, was capable of explaining observations which had no other known explanation. He proved a striking parallel between mathematical theory and observed physical phenomena, to the point where *predictions* could be made and confirmed. And acceptance of action at a distance became, at that time, an unavoidable by-product of the theory.

Of course there is no corresponding 'big picture' obviously emerging from what Soare describes. But it does seem important as an indicator of how certain aspects of the complicated mathematics needed to describe the material universe may well be implicit in the richness of the theory of computability. This would not be surprising given that algorithms seem to underlie the generation of structure in the classical universe. But before speculating on how material phenomena can be located within a theoretical framework based on this underlying algorithmic content, we should first ask: Is there anything very important left for computability theory to explain? Instead of looking for a role for computability theory, let's very briefly examine the nature of gaps in the scientific 'big picture', and the ways in which the content of proposed remedies points to an added role for mathematical theory.

2. Matter and Mathematical Definability

There is no shortage of writings, both popular and specialist, looking at the many fundamental questions still facing science, and most of us have a fairly good idea of what remains to be explained. Let us step back, and get an overview of some of the more intractable questions puzzling workers in the sciences and humanities, and ask: What is the missing mathematics (if any), and how can it help?

A key element in very many controversies appears to be the interaction between the local and the global, and what appear to be breakdowns in the reductive structures commonly relied on in science and epistemology. The deterministic structures we rely on appear to be punctuated by what one can best describe as phase transitions between different levels of familiar relationships. Apart from the obvious puzzle of quantum wave reductions and associated non-locality, there are debates concerning the reductive nature of evolution, and its relevance to social and psychological phenomena; about the origins and exact nature of consciousness, and of concepts and creative ideas; concerning the origins of life on earth, and of the laws of nature themselves; and even concerning the nature of truth in mathematics. At the same time, new science is often based on situations where the traditional reductions are no longer adequate (chaos theory being particularly relevant here). As one observes a rushing stream, one is aware that the dynamics of the individual units of flow are well understood. But the relationship between this and the continually evolving forms manifest in the streams surface is not just too *complex* to analyse — it seems to depend on globally emerging relationships not derivable from the local analysis. The form of the changing surface of the stream appears to constrain the movements of the molecules of water, while at the same time being traceable back to those same movements. The mathematical counterpart is the relationship between familiar operations and relations on structures, and globally arising new properties based on those locally encountered ones. For example (one relevant to new theories of the Universe), consistency is a property of theories based on the atomic proof theoretic features of the theory — namely, the assumptions and permitted deductions. Consistency in turn can be perceived as a constraint on those same atomic elements, and on what the theory describes.

The drive for a completely satisfactory interpretation of quantum theory has led to theories, such as that of decoherence, which routinely have recourse to such global notions, notions which were until relatively recently of interest only to those with a special interest in logic. Now even Gödel's Incompleteness Theorem is cited (usually inappropriately!) in relation to quantum and epistemological uncertainty. Unfortunately, the essential service logic has given to mathematics and other areas, such as computing, does not earn it remission from its traditional marginalisation! New notions coming out of current research are absorbed as slowly as ever. In this context, the little understood — even by mathematicians — notion of mathematical *definability* is the key mathematical concept, with the potential to clarify a broad range of fundamental problems. Many questions in quantum theory, proof theory, and epistemology, can be best understood as a breakdown of definability in an appropriate underlying mathematical model. At the same time, other mysteries, such as how the classical universe escapes the underlying quantum ambiguity, and how natural laws arise, can be traced back to the right notion of definability in the right mathematical structure.

Without going into mathematical detail, it is easy to give an intuitive idea of what definability is, and how it relates to another useful notion, that of *invariance*.definability,and invariance This not necessarily because the notions are very simple ones (they are not), but because they do correspond to phenomena in the real world which we already, at some level, are very familiar with. We have already had the example of the stream, with its micro- and macro-dynamics. The emergence (a vogue word) of form from chaos, of global relations within turbulent environments, is a particularly vivid metaphor for the assertion of definability, or invariance. Let us take a simple mathematical example from arithmetic.

Given the usual operation $+$ of addition on the set $\mathbb{Z}$ of integers, it is easy to see that the set Ev of even integers is *le* from $+$ within $\mathbb{Z}$ via the formula

$$x \in Ev \iff (\exists y)(y + y = x).$$

So all we mean by a relation being *definable* from some other relations and/or functions on a given domain is that it can be *described* in terms of those relations and/or functions in some agreed standard language. Of course, there are languages of varying power we can decide on, in which say one can allow say different levels of quantification. In the above example, we have used very basic *first order* language, with finitary quantification over individual elements of the intended domain — we say that Ev is *first order definable* from $+$ over $\mathbb{Z}$. What has happened is that we started off with just an arithmetical operation on $\mathbb{Z}$, but have found it distinguishes certain subsets of $\mathbb{Z}$ from all its other subsets. Intuitively, we first focused on a dynamic flow within the structure given locally by applications of the form $n+m$ to arbitrary integers m, n. But then, standing back from the structure, we observed something global — $\mathbb{Z}$ seemed to fall into two distinct parts, with flow relative to even integers constrained entirely within Ev, and flow from outside Ev being directed into Ev — with Ev being a maximal such subset of $\mathbb{Z}$. From within the structure, $+$ is observable and can be algorithmically captured. Further than that, we are dealing with 'laws' which cannot be related to the local without some higher analysis. This feature of the integers is not of course a deep one, but it does act as a basic metaphor for other ways in which more or less unexpected global characteristics of structures emerge quite deterministically from local infrastructure.

The notion of *invariance* gives a useful, if slightly more abstract, way of looking at such phenomena. Being able to uniquely *describe* a feature of a structure is a measure of its uniqueness. But some feature of a structure may be quite unique, without one being able to describe that uniqueness in everyday language. Mathematically, we use the notion of *automorphism* to capture the idea of a *reorganisation* of a structure which does not change any of its properties. A feature of that structure is *invariant* if it is left fixed by any automorphism of the structure. Obviously if one can uniquely describe such a feature, it must be invariant, but not necessarily conversely. On the other hand, any relation invariant in a structure can be defined in that structure, if one allows a suitable strong (but no longer very natural maybe) language, so definability can be thought of as providing a hierarchy of invariant relations.

The main remaining question is: How can one turn a metaphor into a model? In what sort of mathematical structure can we find definable relations,definability and corresponding failures of definability, with real world explanatory power?

3. Finitism in a Universe with Algorithmic Content

One often hears the view put that considerations of computability must have limited relevance to a clearly finite Universe. In outline, the argument is that one can analyse the Universe in terms of its quantum structure. This entails a discrete — and according to all the evidence — finite model. Since the model is large, computability is relevant to the scientific project. But incomputability has about as much significance for a complete description of the Universe as it does for any other finite relational structure, such as a graph — that is, none. In fact (see the discussion of Church's thesis in volume I of *Classical Recursion Theory*) no discrete model — finite or otherwise — presents a likely host for incomputable phenomena.[5]

Quantum indeterminacy presents little problem for such an outlook. One either expects an improved scientific description of the Universe in more classical terms, or, more commonly, one takes quantum *randomness* as a *given*, and superimposes more traditional certainties on top of that.

The latter perspective is also common to world views that make no assumptions about discreteness. It has the advantage (for the Laplacian in quantum clothing) of incorporating incomputability in the particular form of randomness, without any need for any *theory* of incomputability. The origins of incomputability in mathematics may be theoretical, but not in the real world, the view is.

Let us try to develop a more coherent alternative world view.

We first look at how science is done, and what relevance this has to what scientists are trying to describe. As Richard Feynman once pointed out[6] "It is really true, somehow, that the physical world is representable in a discretized way, and ... we are going to have to change the laws of physics." But it does not follow that the mathematics needed is correspondingly discrete. There do exist certain arcane proposals in logic for reconstructing fragments of mathematics, but nothing of practical use to the working scientist can be claimed. Much of applied mathematics seems irrevocably dependent on limiting processes and descriptions in terms of real numbers. Moreover, work on nonlinear phenomena have made us very aware of the fact that one cannot accurately describe complex environments within a fixed level of approximation — one cannot

[5]On the other hand, if one limits oneself to the usual computability models, the notion of randomness of *finite* strings seems to provide a first step toward a much needed theory of incomputability of finite objects.
[6]see [Fe82]

avoid working within the full system of real numbers. In the broader context, there is widespread awareness of the inadequacy of finitary language — everyday sentences — for capturing truth of a more abstract nature than scientific facts. Philosophy often seems an unending process of translation of intuition into words, while the reading of a philosophical text usually entails a creative, rather than formal, recreation of intuitive content. The epistemological role of sentences in everyday language seems not unlike that of the rational numbers in science.

We have to ask: Is it plausible that a Universe that compels us to use the mathematics of the reals to describe it, actually derives its properties from a mathematics qualitatively no different to that of a very big graph? One surely finds such a discontinuity between reality and description even less satisfactory than that between the mathematics of incomputability and an unexplained quantum uncertainty. Better to put to one side for the moment the possibility of a Universe based entirely on the mathematics of the discrete. It seems more likely that the observed discreteness is something *imposed* by natural laws on an underlying indiscrete mathematical model. And that these natural laws emerge from the model itself much as does the surface form of our rushing stream envisaged previously.

There are also more mathematical reasons for looking for an indiscrete model.

There may be little science can say about why things exist, but it is with mathematics that the scientific project begins. The way in which form develops in that which comes to exist is based on mathematical structure. Mathematics seems to prefigure. But — like joining a club, or enlisting into a culture — the opportunities for development provided come with accompanying rules, pressures and responsibilities. And these may not be predictable — the ordinary citizen of nineteen-thirties Germany had little idea that they were part of a social formation that would build so monstrously on their individual aspirations and ideals. From its origins (whatever that means!) the Universe relies on a mathematical blueprint to narrow down its various potential incarnations. It 'signs up' to an underlying mathematical model, not by choice, but just by the act of coming into existence. Can we say anything about the mathematics prefiguring the Universe?

What transforms inchoate development and gives form to creation, is the way in which change become process. Whatever takes place in the nascent Universe appears to be comprised of basic kinds of atomic acts of development, with repetition forming the basis of further development. Processes are fundamental to matter and the laws which govern its formations and development. They do not seem to be finitarily determined, for example by piecemeal dependency on the members of their real world domain, but can be envisaged relative to a wide range of extrapolations from what we actually observe. These processes are what we broadly describe as 'the algorithmic content' of the Universe.

Most of these fundamental processes identified by science are characterised by enough predictability of immediate effect to be covered by the connotations of such terminology. Science since the time of Newton, at least, has been largely based on the identification and mathematical description of algorithmic content in the Universe. We will look at phenomena — primarily subatomic phenomena — which appear to defy such description, in section 6 below. It turns out that these exceptions can be understood as arising at the borderline between invariance and noninvariance in an appropriate model. In this context they actually consolidate our view of the algorithmic nature of the fundamental processes underlying material development.

Anyway, the main observation here is that a finite structure with the sort of algorithmic content we find in the Universe is very different to, say, a graph, in that it entails 'uncompleted infinities'. A fragment of the natural numbers over which one allows the usual operations of arithmetic, is best understood in the context of the theory of the whole structure. In the same way, the algorithmic content of the Universe makes it subject to the principles governing the mathematical whole of which it is a part. Finiteness of the collection of all atomic particles (even assuming one can make sense of such a conception) cannot remove the general algorithmic content, the accompanying limiting processes, or obviate the higher mathematical relations these entail.

The admission of algorithmic content is what makes a nonsense of a strictly finitist view of the universe, and, as we will see in section 6, offers yet another example of how particular explanations emerge from larger mathematical contexts than have immediate counterparts in nature — such as with imaginary numbers, or at a mathematical level, results in number theory coming from nondiscrete mathematics.

Moreover, the mathematical model emerging not only transcends finitism, but also discretism. One can envisage how real numbers in the form of uncompleted infinities feed into physical reality, and determine a mathematical model which is not discrete. For instance, people routinely attempt to mentally simulate events in the interests of reconstructing history and predicting future circumstances, and modify their activity accordingly. In that this simulation can involve sequences with no specific bound, as when one say tries to act towards a world without war and famine, one can say that there is an uncompleted infinity which has a direct real world impact. One hesitates to admit cerebral phenomena which qualitatively transcend what goes on in other parts of the real universe. In fact one can argue that such mental simulations derive from the brain's capacity to physically simulate other complex material phenomena. And that the Universe is quite capable of accelerating uncompleted infinities into the here and now via its own nonlinear, often turbulent activity, involving a globally extending network of interactions relevant to a particular local description. The limiting processes underlying local descriptions in terms of global phenomena, intrinsic to turbulent environments, can be better understood via the standard hierarchies of real numbers found in computability theory, and we will return to this in the next section.

But to summarise, if the ingredients of a simple mathematical construction — such as that of the reals from the natural numbers — are materially manifest, then one should not expect the Universe to be excluded from that construction. Specifically, this leads to a convergence between the mathematics, and the scientific priority given to descriptions in terms of reals and relations between them. We now take this a little further.

4. The Inseparability of Algorithmic Content, Complexity and Incomputability

Mathematically, algorithmic content and incomputability are inseparably linked in sufficiently complex environments. An elementary construction (involving *diagonalising* through a list of all machine computable reals) gives an incomputable real. But for our purposes, we need to limit our attention to mathematical constructions for which the basic ingredients have an obvious physical counterpart, and diagonalisation at first sight appears to be limited to what mathematicians do. More relevantly, *all* real numbers are derivable as the limit of a sequence of *computable* — in fact rational — numbers. Even if one limits oneself to computable *sequences* of computable numbers — that is *uniformly* computable sequences of reals — one still gets limiting reals which are not computable.

However, one needs to look more closely at how incomputability arises from the mathematics of algorithms to be sure that all the basic ingredients are apparent in, say, the turbulent environments we have focused on previously. What is it makes it impossible to compute a given incomputable, but computably enumerable set of natural numbers? The members of the set are of course enumerated by some computable function from numbers to numbers — but not in order of magnitude. The relation between the magnitude of input and that of the output is broken. And since we are only interested in the output, we would need a possibly uncompletable search to computably discover if a particular number was in the set or not. In the case of an incomputable real appearing as the limit of a uniformly computable sequence of reals, this appears as an absence of a computable *modulus of convergence* of the sequence. The breaking of the link between input and output happens in the sense that one cannot be clear at any given point in the sequence how close an approximation to the limiting real is being currently provided. What is there in the case of chaotic situations which corresponds?

A well-known example of chaotic behaviour is that of a pile of sand being accumulated grain by grain. There is an unpredictable link between the order in which the grains of sand are deposited on the heap, and the consequent impact on the pile in terms of settlement of existing grains. This is just the sort of unpredictability of association between input and output that we seem to need. In fact, it has been said that the "Butterfly Effect", or more technically the "sensitive dependence on initial conditions", is the essence of chaos. So a sufficiently sensitive mathematical model for turbulent environments not only cannot be discrete, but will involve limiting processes which have the sort of ingredients needed to generate incomputable elements. Mathematically, the need for an infinitary sequence of approximations does not depend on what is conventionally regarded as a chaotic — or turbulent — environment. Even apparently simple phenomena — such as a dripping tap — exhibit chaotic behaviour. The nonlinearity of the mathematics can be approached via a superposition of infinitely many linear factors, closely related to how one would naturally approximate the event in question via individual computable interactions. The association of incomputability with simple chaotic situations is not new. For instance, Georg Kreisel sketched in [Kr70] a collision problem related to the 3-body problem as a possible source of incomputability.

At this point the reader might object that we are doing exactly what we earlier suggested was a pointless exercise, namely trying to provide direct argumentation in favour of incomputability in Nature. No, we are under no illusion that what we have said so far provides any more than a preliminary basis for a belief that the universe is incomputable. What we are trying to do is convince the reader that he or she should seriously consider taking the mathematical model suggested by the above discussion 'on approval', as it were. Any good salesperson (and we hope to use the term without its usual pejorative associations) would admit the prospective purchaser their doubts. The aim would be to provide sufficient technical background and informative argumentation to at least justify the time and effort involved in getting a first-hand experience of what the product can actually deliver. The salesperson may aver that there is no substitute for having the product in your hands, and trying it out in the privacy of your own home. But one must be persuaded that there is actually some point in doing this, and for this reason we have hopefully presented the scenario in which one might want to at least consider what the model — the Turing model — potentially provides in the way of explanatory power.

It is now time to must describe the model in more detail, and discuss certain technical questions concerning it which as yet remain unresolved.

5. A Closer Look at the Turing Model

In 1939 Alan Turing, puzzled by the role of incomputability in mathematics and logic, introduced Turing machines with oracles, as a way of comparing the (in)computability of different reals. Essentially, oracle Turing machines were machines capable of working in the real world of information, in the form of real numbers. The result was a natural model for structures describable in terms of computable relations over real numbers. Our Universe would be an obvious candidate for such a structure, were it not for the fact that certain basic natural processes are not known to give rise to *computable* relations on reals. What we have promised though is that such phenomena *are* modelled via an analysis of invariance and definability in the basic Turing model.

Historically, the investigation of the properties of the Turing model were initiated by Emil Post in the early nineteen forties, and a seminal paper written largely by Stephen Kleene, but co-authored with Post, appeared in 1954. By that time, the interest had become almost entirely mathematical, as reflected in the name 'recursive function theory' given by Kleene to the new subject. The original motivations of Turing had been — necessarily it can be argued — sidelined in favour of an intensive technical development, just part of which involved efforts to describe the Turing universe in more detail. It was not surprising that real world applications did not figure throughout the recursion theoretic period (roughly the sixty years starting with the discovery of incomputable objects by Church and Turing in 1936). On the one hand one had the confusing way in which real world incomputability manifested itself, and on the other one had an as yet technically inadequate subject, quite insufficient to develop and explain any modelling process it might be part of.

By the 1990s few could ignore the signs that not everything was going well with the subject. The long-term survival of a mathematical field with no real-world applications depends on what it delivers, year in, year out, to the wider mathematical community. And in particular, what it offers to new students and professionals, to mathematicians working in other areas, and to the institutions within which they work. To outsiders classical computability had become a deep subject which was maybe *too* deep; in which core research had become hazardous and, even by the standards of fundamental scientific research, lacking predictability of outcome; in which mathematical applications depended on recursion theoretic terminology; and in which the undoubted contribution to theoretical computer science and constructive mathematics did not depend on the sort of things that recursion theorists currently occupied themselves with. Even insiders who instinctively reacted against any consciously political response found problems in pursuing a purely mathematical agenda, and were themselves often caught up in the drift away from core research.

Things started to change in earnest around 1995–96. These changes were rooted in two seemingly unrelated developments, one philosophical and political in content, and the other technical. The first involved a deliberate attempt to reinstate Turing's terminology in keeping with the subject's origins in real world questions — 'computable' in place of 'recursive' etc. — a project outlined in Robert Soare's 1996 paper on 'Computability and recursion'. The other originated in 1995 with the first serious challenge to the mathematical perspective which had dictated the direction of core research for more than ten years, and involved the discarding of mathematical certainties (in more ways than one) in favour of complexities which startlingly seemed to parallel some of those apparent in the real world. We will say more about the specifics of this below. It is not clear to many what will be the eventual outcome to this transitionary period, and so far the impact of the accompanying confusion on the standing of computability theory has been largely negative.

What lost its revered status around 1995 was the so-called 'biinterpretability conjecture', which it was hoped would lead to an extremal characterisation of the Turing definable relations, to the point where the Turing universe, as a mathematical structure, would be rigidly pinned down by those relations. The idea had been that one could avoid direct involvement with the multiplying complexities of Turing's structure, beyond what was needed to translate it into the more familiar theory of (second order) arithmetic. This relationship would deprive the Turing universe of much of its potential to surprise, and make detailed investigation of Turing definability of little more mathematical interest than the details of arithmetical calculations. And for a number of years the bi interpretability conjecture held a gorgon-like fascination for recursion theorists,[7] while offering a mathematically elegant but no less stony end to the subject.

Let us say a bit more about rigidity for the nonspecialist. When one says a structure is 'rigid' it means intuitively, that travellers within the structure can uniquely pin down their location by looking around them. Or more precisely, cartographically the traveller cannot mistake his or her position — for example it is clear if the map is being held the wrong way round! Every feature of the structure has a uniquely determined global context. It is like a team in which every player has a uniquely fulfillable role. If the players exchange jerseys amongst themselves, the team is no longer what it was — the structure has been changed by this relabelling. A relabelling of a structure under which it is *not* changed is called an *automorphism*. A rigid structure is one which has no nontrivial automorphisms. In computability theory a *reducibility* is an attempted ordering of mathematical objects according to a given level of *perception* of information content. It consists of the collection of those specific *reductions* conforming to the permitted level of perception. Turing reducibility is a particularly important reducibility, comprised of reductions based on everyday algorithmic practice,

[7] At an international conference in Helsinki in 1990, the first author's suggestion, in answer to a question from the audience, that the Turing universe might *not* be rigid, attracted indulgent smiles. The next day, a distinguished special sessions speaker found Turing rigidity 'almost certain' in the light of so many known Turing definable relations.

whereby one computes relative to finite collections of data extracted from the real world. Reducibilities can be limited as to the size of the collections of data used, or as to the way they are presented, or as to the manner in which one computes on them. These are called *strong* reducibilities. There is an historically important strong reducibility called *many-one reducibility* which can be regarded as an abstraction of translations between axiomatic theories, and this gives rise to a structure which has few definable relations, and is very un-rigid. On the other hand there are reducibilities which are allowed a level of *infinitary* interrogation of a given real, or even of objects more complicated than reals (say infinite sets of reals, or sets of transfinite ordinals). Such reducibilities are the domain of *generalised* recursion (or computability) theory, and although so far lacking obvious real-world counterparts, often have very strong intuitive content. An example of a structure arising from a more general reducibility is that of the *hyperdegrees*, where algorithms relative to reals are replaced by very general language-based reductions — and this structure does turn out to be rigid.

The intuition was that Turing reductions were mathematically the weakest adequate to extract enough information from a given real to uniquely fix its relationship to the universe of the reducibility. Of course, this is modulo the intercomputability of certain pairs of reals. To take account of this, it is usual to gather together into *degrees* collections of objects of the domain of the reducibility which are mathematically equivalent in this sense. So a Turing *degree* is a maximal set of reals any two of which can be Turing computed from each other. These reals can be thought of as presentations of the same Turing accessible information content. It was the frustration of this intuition that gave rise to what may seem in hindsight a bizarre response to Turing nonrigidity, that is a disappointment that the reducibility was, perhaps, not the 'right' one mathematically. One would now look for some recognition of the parallel with the encountered inadequacies of scientific observation, and more generally, of the power of the Universe to uniquely determine its own structure.

There are two other observations of a technical nature to be made. The first is that Turing nonrigidity is not yet finally established according to the usual criteria of the academic community. What has happened is that progress in the other direction has all but halted, despite the most determined efforts of leading researchers in the field. And no significant technical challenge, either in the form of counterexamples or technical queries, has been made to the increasingly detailed proposal for a nontrivial automorphism. The latter has been presented now at a number of international meetings, most convincingly during a six-hour invited talk at the AMS-IMS-SIAM Joint Summer Research Conference in Computability Theory and Applications at the University of Colorado, Boulder, in June 1999. The fact that the issue remains unresolved is partly a measure of the newness and initial complexity of the techniques, partly a manifestation of a severely under-resourced research area signally failing to keep pace with its own technical development, and (perhaps most significantly!) due to the mathematical and presentational limitations of those few individuals directly responsible. Also — and this is the main reason for writing about the consequences of work in progress — what is happening has every appearance of a paradigm shift in progress. And not just a minor technical shift in an obscure area of pure mathematics, but one of very general importance. A change as global in nature as the phenomena it concerns, which transcends the incremental processes of normal science, and in which an appreciation of context may feed back into a deeper understanding of narrowly technical issues. The development of string theory is an example of the way in which the sheer explanatory power of a theory can actually contribute to the mathematics.

Secondly, one has to mention that just as important as nonrigidity is the counterbalancing body of results establishing a high level of invariance in the Turing universe. One of these says that there are relatively few — at most *countably many* — Turing automorphisms. To understand some of the most relevant of these results one must have some appreciation of the close relationship between the way one might describe a real in standard mathematical language, and its location within the Turing universe. It turns out that if one has such a description of a real, which one knows to be the simplest possible, then one can roughly locate its position in the Turing universe in a very natural way. Important landmarks in this location are provided by naturally arising reals whose descriptions involve a relatively small number of quantifiers. Most importantly, one has the Turing equivalent group of incomputable reals discovered by Church and Turing in the 1930s, degree theoretically denoted by $\mathbf{0}'$. Occupying a position above $\mathbf{0}'$ similar to that of $\mathbf{0}'$ above $\mathbf{0}$ (the set of all computable reals) is a degree $\mathbf{0}''$. What is remarkable is that above this level a Turing automorphism can do no more than move a given real to one which is Turing equivalent to it. That is, the Turing universe above $\mathbf{0}''$ is rigid *within* the whole structure.

6. Scientifically Presenting the Universe

We need to say a little more about what we mean by a mathematical model for the Universe. This has been left deliberately vague up until now, and as we shall see, a degree of vagueness is inherent in the nature of scientific knowledge itself. Flexibility is also a feature, and a strength, of scientific practice as regards the details of mathematical modelling via assignments of reals to parameters. What we are talking about is a *scientific presentation* of the Universe via some informative mathematical structure. The mathematics needs to reflect the ingredients of the current scientific picture we have of the Universe. As we have tried to argue above, scientific descriptions are based on the real numbers. This is not because we necessarily believe that the real numbers actually enable us to capture reality, but because they are what we have found by experience to provide a practical scientific framework. The sort of scientific presentation we have in mind is not one currently used in practice to deal with the sort of fragments of the Universe which science needs to focus on in isolation. We stayed close to such local presentations in our earlier discussion of turbulence and the generation of incomputability. A feature of that discussion was the way in which the new incomputable reals derived their specific properties from the algorithmic content of the chaotic environment. This is what is required in a comprehensive mathematical model which describes the Universe in terms of 'atomic' information content, described via reals. That is, the algorithmic content of the Universe, as revealed to us by the scientific project, needs to be reflected by the comprehensive presentation in terms of reals. This will enable one to correlate the mathematical description with the physical processes in an informative way.

In summary, the model must reflect the picture which science *in principle* provides of the way we experience the Universe, with a comprehensive structuring of the information content of that description in accordance with the observed processes at work. Of course, without making any assumptions regarding underlying reality — even to assert that a given object exists, using some agreed language, is to translate our experience into a scientific statement — all that is being sought is something which is in accordance with what we observe. That is, we look for a mathematical structure within which we may informatively interpret the current state of the scientific enterprise. This presentation may be done in different ways, one must assume, but if differing modes of presentation yield results which build a cohesive description of the Universe, then we have an appropriate modelling strategy.

Part of our experience of science is what seems an intrinsic fragmentation, in which the reductive structure of the Universe is not a good guide to what happens at different levels of scientific application. In section 7 we will look more closely at how the proposed model explains this in terms of internal invariance. All that needs to be observed for the moment is that just as different fragments of the scientific enterprise are based on their own basic relations, whose complex reductive relationships to more basic fragments are suppressed, so it may turn out that our current method of presentation ignores other yet to be discovered basic levels. What is important is that the scientific picture is now sufficiently filled out for the Turing model to have something to tell us about the way the world we live in is as we currently experience it. One may have to make certain assumptions about *how much* of the Turing model is needed to describe our Universe, but the mathematical indications are that a large amount of information content is needed for the emergence of anything like the classical universe of well-defined individual objects we daily encounter.

Anyway, a level of flexibility in the way one envisages using the Turing universe as a model is an advantage, allowing us to apply it with differing perspectives. In particular, it means we can envisage a modelling of the Universe as part of an extended epistemological structure, consisting of statements we can make (in some language) about the Universe and its mathematics.

7. What the Turing Model Delivers

The claim is that many seeming anomalies in science, the humanities, and human affairs in general, stem from failures of reductive analyses which have served us too well in the past to be easily bypassed. Further, that what is lacking is not an attack on the validity of such widely applicable analyses, but a model in which relevant global factors can be clarified. And crucially, that a key global notion is that of definability or more generally, mathematical invariance. That there are sufficient indicators, of both a practical and theoretical nature, for us to look for a model for the Universe based on presentations in terms of reals. And that the appropriate abstraction of such a Universe and its processes is that arising from Alan Turing's work in the late 1930s.

Let us assume for the moment that we are prepared to entertain such a perspective. It is then time to deliver — So what *does* computability theory have to offer? We revisit some of the themes touched on in the first author's 1999 paper. We have tried to make the comments independent of outside references, which might be thought of little help to the majority of readers, who will have read many books and articles of course, but not necessarily those preferred by the authors! We will also try to be concise, at the risk of oversimplification. What we hope will be striking is not the extent to which the model surprises, but the way in which it supports and theoretically substantiates existing perceptions and trends in science and beyond.

Our model says nothing about the mystery of material existence. But it does offer a framework in which a breakdown in reductionism is a commonplace, certainly not inconsistent with the picture given of levels we do have some hope of understanding. It can tell us, in a characteristically schematic way, how *things* come to exist. Our basic premise, nothing new philosophically, is that existence takes the most general form allowed by considerations of internal consistency. Where that consistency is governed by the mathematics of the universe within which that existence has a meaning. Of course, within that universe, not even the terms in which we seek to discuss it can be assumed to have any meaningful existence. But proceeding regardless, assuming our intuitive concept of 'creation' to have some content, we assume, as before, our Universe to embody some process of development, in which processes are basic. We assume the appropriateness of presentations of the Universe in terms of information content and algorithmic relationships on that information content. We can then appeal to the mathematics of the Turing model to tell us something about what we might expect concerning 'thingness' in our Universe. This will be a first step in filling in a picture of the Universe characterised by more determinism, more immanence, and less dependence on metaphysical speculation.

What would the model lead us to expect regarding the basic structure of matter and the laws governing it? The mathematical indications are that a low level of information content goes with nonrigidity and a lack of Turing invariant individuals. In other words, the structure of the Turing universe appears to be consistent with the possibility that a given individuals information content may be insufficient to guarantee it a unique relationship to the global structure. The corresponding prediction for a Universe in tune with that model would be that its most basic components may materialise ambiguously[8] — a prediction confirmed by a number of classic experiments on subatomic particles. (The assumption has been since at least the time of Leibniz, that reality takes all forms consistent the underlying laws of existence, which from a contemporary point of view tends to mean consistency with the *mathematics* pertaining.) On the other hand, mathematically entangling such low level information content, perhaps with content at levels of the Turing universe at which rigidity sets in, will inevitably produce new content corresponding to a Turing invariant real. The prediction is that there is a level of material existence which does not display such ambiguity as seen at the quantum level, and whose interactions with the quantum level have the effect of removing such ambiguity — confirmed by our everyday experience of a classical level of reality, and by the familiar 'collapse of the wave function' associated with observation of quantum phenomena. Since there is no obvious mathematical reason why quantum ambiguity should remain locally constrained, there may be an apparent non-locality attached to the collapse. Such a non-locality was first suggested by the well-known Einstein-Podolsky-Rosen thought experiment, and, again, has been confirmed by observation. The way in which definability asserts itself in the Turing universe is not known to be computable, which would explain the difficulties in predicting exactly how such a collapse might materialise in practice, and the apparent randomness involved.

One can only speculate about the origins of subatomic structure, particularly since we do not seem to have a complete description of it as yet. One guess is that when one observes atomic structure, one is looking at *relations* defined on some lower level level of matter lacking any sort of observable form. This could be envisaged as a kind of formless soup of information content out of which arise peaks of definability observed by us as subatomic particles. Such

[8]But in keeping with the limitation on possibilities presented by the existence of only countably many Turing automorphisms.

an implementation of the model might be used to explain why the hypothesised 'dark matter' in the Universe has not been observed. Until it is organised into relations, of which particles are the instantiations, we have no structure capable of being interacted with. It would be as alien to the world of particle physics as that world is to our classical level of human existence.[9]

The mathematics leads to other scientifically appropriate predictions. In particular, there is the question of how the laws of nature immanently arise, how they collapse near the big bang 'singularity', and what the model says about the occurrence or otherwise of such a singularity.

What we have in the Turing universe are not just invariant individuals, but a rich infrastructure of more general Turing definable relations. These relations grow out of the structure, and constrain it, in much the same sort of organic way that the forms observable in our rushing stream appear to. These relations operate at a universal level. The prediction is that a Universe *with sufficiently developed information content* to replicate the defining content of the Turing universe will manifest corresponding material relations. The existence of such relations one would expect to be susceptible to observation, these observations in turn suggesting regularities capable of mathematical description. But then this is a prediction which adumbrates the familiar historical process of the scientific mapping of natural laws. The conjecture is that there is a corresponding parallel between natural laws and relations which are definable in an appropriate fragment of the Turing universe.

The early Universe one would not expect to replicate such a fragment. The homogenisation and randomisation of information content consequent on the extreme interconnectivity of matter would militate against higher order structure. The manifest fragment of the Turing universe, based on random reals, might still contain high information content, but content dispersed and made largely inaccessible to the sort of Turing definitions predicted by the theory. Projected singularities, such as within black holes or associated with boundary states of the Universe, depend on a constancy of the known laws of physics. But immanently originating laws must be of global extraction. This means that their detailed manifestations may vary with global change, and disappear even. Scientific evidence for time-related variations in natural laws is at best inconclusive, as is evidence for the collapse of the laws of physics near the

[9]On a technical note, one should mention that it is not known if the Turing universe is rigid above $\mathbf{0}'$ — only that $\mathbf{0}'$ it is definable, and so invariant. A material manifestation of a Turing defined relation will still be scientifically perceived as an entity describable by a real, despite the underlying mathematics by which it is constrained. And the information content of that defined relation will correspond to an infinitary join adumbrated by local information information content organised according to some corresponding description. In which case one can speculate that the level of elevation of information content implicit in a Turing definition is due to some canonical form of description needed for persistence in time, or arises from the conjectured subsubatomic level of information content.

putative singularities. But there are persuasive theoretical arguments for the latter, associated with the standard model. And it is not unreasonable to expect some continuity between the current Universe and one bereft of the familiar (and not so familiar!) laws of physics.

The reader will already be aware of the many attempts to stretch the standard model to deal with unsatisfactory gaps and anomalies in the theory. The area abounds with theoretical constructs — such as inflation, many worlds, decoherence, the pilot wave, gauge theory, superstring theory, M-theory — all intended to extend the standard model in ways not yet justified by observation. Some of these are proposals of a limited technical nature designed to address particular problems with the standard model in a relatively piecemeal way. Others are conceptually more radical, and can be located within the present schema. This applies particularly to theories of decoherence, which in trying to reconcile quantum ambiguity with classical reality do seem to deal with, in a very basic way, the underlying mechanics of mathematical definability. However, without locating this within the mathematical theory of definability, one is still left with having to resort to the metaphysics of parallel classical realities, with its unsatisfactory lack of economy, and accompanying problems with explaining our own special relationship to the reality we see around us. Still other attempts to stretch the standard model — such as superstrings and M-theory — present the standard model as a very small fragment of a mathematical theory which goes far beyond that arising from empirical science. These represent an approach which, while not actually inconsistent with that described here, appears to share very little in the way of new thinking.

Logically, it is well known that one can extend a given incomplete theory in many different ways. And that, in general, such an incomplete theory will have infinitely many completions. In the real world one can justify any addition to the standard model via suitably elaborated auxiliary hypotheses.[10] So it is not very surprising that a variety of such proposed extensions to the standard model of the Universe have appeared. Or that more than one of these has achieved the status of a fully-fledged paradigm with its own committed group of supporters. Occam's Razor, and the Principle of Parsimony, are what have been provided as a formal counterpart to our natural scepticism. These say, roughly speaking, that one should not increase, beyond what is necessary, the number of extra factors required to explain something. If one takes into account not just the *number* of added factors, but also the likelihood of those factors being substantiated by observation (either in a positive way or by expected negative data not being found), then the interpretation we have attempted based on physical entities as information is a clear winner. How can we put a computable Universe amongst

[10]This is roughly what the Duhem-Quine thesis says.

many, top of our list? Especially since other assumptions must be added to that of many worlds to deal with the still remaining anomalies and incompleteness of the theory. Superstring and M-theory make very different assumptions, which are however no more accessible to experimental verification. And there is even less clarity about what the theory actually is. There are some very detailed implications for the theoretical basis of the standard model, which the Turing model has as yet nothing to say about. In fact, superstring theory seems capable of sitting quite comfortably within the Turing framework. The aims and consequences of the two approaches are rather different, and potentially complementary.

We will look very briefly at the Turing interpretation of other areas. Leaving quantum theory, one can still find abundant evidence for both nonrigidity, and what appear to be new laws corresponding to invariant relations emerging at higher levels, and relating to new entities reducible to ones at previous levels. However, this evidence may be less scientific, and more subject to controversy. Firstly, the Turing model does tell us why there is this fragmentation of science, and human knowledge in general, and why we do not have computable reductions of one to another. As we have said previously, in general a Turing definition of a given relation does not necessarily yield a computable relationship with the defining factors. But working within the relations at a given level, there may well be computable relationships emerging, which may become the basis for a new area of scientific investigation. For instance research concerning the cells of a living organism may not be usefully reduced to atomic physics, but deals with a higher level of directly observed regularities. Sociologically, one studies the interactions governing groups of people with only an indirect reference to psychological or biological factors. Entire relations upon cells (humans) defined in some imperfectly understood way by the evolutionary process provide the raw material underlying the new discipline, which seeks to identify a further level of algorithmic content. This algorithmic content may not be directly expressed in terms of numbers. But inasmuch as the area in question does have basic notions, corresponding to the new emergent relations, shared by workers in the field, and descriptions of entities and regularities are formulated in a shared language, the algorithmic content is not dissimilar in kind to that at lower levels.

The first author's 1999 paper mentioned a number of areas in which one can observe qualitatively similar problems, all connected with parallel issues of definability and nonrigidity. We briefly review two of these, and leave the reader to look there for further comments.

A particularly puzzling problem is that of the early origin of life on Earth. Wherever one encounters explanations in terms of 'emergence', one can expect an understanding of the notion of definability, relative to the basic relations operating, to reinforce the necessarily vague intuitions underlying such explanations. And at any level, there are initially no more basic relations than those presented by the fundamental processes pertaining. There are more specific characteristics established on this basis, and there is no intention to deny the importance of the science based on such specificities. In any case, in this instance there is an undeniably close parallel between the intuitions underlying definability and those feeding into more recent explanations of how life appeared on earth in the time available. And some theoretical breakthrough is needed, given the confusion betrayed by such speculative explanations as those enlisting extraterrestrial intervention.

An even more high profile debate concerns the exact nature of evolution, and again one can find on the side of those dissatisfied with mechanistic determinism appeals to notions of emergence and chaos theory. And again one can consolidate these more global perspectives via the mathematics of definability. Determinism is defended, but not reduced to a clearly inadequate mechanism.

And there are many other ways in which the perspective described here can shed new light on thorny problems. For instance, there is the mysterious emergence of large scale structure in the Universe, and on the other hand the remarkable fact that the Universe appears to present a very similar view from whatever observation point one chooses. Once one accepts the basic underlying nature of process and its mathematical counterpart of algorithm, and admits the possibility that the Universe is deeply imbued with incomputability and its mathematics, then many troublesome problems can be placed within a helpful explanatory context.

In the 1999 paper mentioned above, there is a section on epistemological relativism. There is a basic intuition that an analysis of the epistemology derived from our Universe is potentially just as complex as that of the Universe itself. So it should not be surprising that the mathematics of definability should be relevant here. And it is not surprising that without such a conceptual framework as proposed here, different epistemological approaches should give rise to such differing views of the world. And that such controversies as currently characterise discussions between scientists and those engaged in the humanities should thrive in the unfilled vacuum. For reasons of space, we limit ourselves to a few comments extending the perspective.

We do this via what at first sight seems a less promising example. Mathematics, as we know it, is expressed via finitary statements, and has at best a problematical relationship with the material world. The exact character of this relationship is still argued over, as is the extent to which mathematics owes its existence to mathematicians. A puzzling anomaly arises from the dominance of the axiomatic model for mathematical activity, and its incompatibility with experience of creative mathematicians. It is quite true, one observes, that having discovered some new and interesting mathematical theorem, one can usually translate its proof into one in some standard axiomatic theory. The problem is that the discovery of the theorem was not obviously axiomatic, and the formal proof when assembled may do much to obscure the mathematical intuition telling us why the theorem is true. If we look more closely at how we derive the theory itself, we may better understand how it is — in the context of our model — that our discovery of the theorem comes via a route which apparently bypasses the rules of that theory.

One first makes the observation that if the formal theory does not explain the emergence of the axioms, then what it contributes to our understanding of theorems derived from these axioms must be limited. The unconscious assumption is that since an axiom seems simple, so must be the process of its derivation. And that the formal connection between the axiom and the theorem must be a good guide to why the theorem is 'true'. But even the natural numbers, before any statements are made about them, are complex in origin. Historically their emergence is linked to that of language, and its usage in relation to recognition of patterns, or relations on matter. Numbers as such relations are already present in the material world, and on the other hand seem to emerge in ones consciousness independently of any such exterior source. But what one can say is that there exists some process of definition of these basic relations, which is manifest both in nature and within the human mind. And that this does not depend on the exact balance of defining factors in a particular context. It may well be that recognition of such relations on the real world depends on some sort of parallel definition within our consciousness, which one can regard as a simulation within the mind of phenomena within the exterior world. Mental processes, while being a microcosm of the greater universe of which they are a part, do appear to mirror some of its complexity and hence its capacity for global definition. It is possible to regard the mind as a functional Universe in miniature. Anyway, what is important is that the emergence of number does show the familiar irreducibility of phenomenon so characteristic of a phase transition, to which modelling in terms of definability is relevant. The natural conjecture is that the forgotten but complex process underlying the emergence of the axioms is what is being revisited via the discovery of theorems, and that in both cases the basis is definability in terms of the atomic ingredients that constitute our intuitive grasp of what happens in the real world. Horizontal connections between different mathematical statements may play a role, particularly in the translation of personal intuitions into a shared framework of proof, but that is not always the most natural route to new mathematics.

What we have argued above is that it may be possible to reconstitute the mathematical framework governing the scientific approach, in such a way as to revive the sense of universal relevance once associated with 'natural science'. The state of human knowledge lends an inevitably speculative quality to what we have written, but we hope this article will be accepted with the openness and generosity of spirit in which it is offered. How appropriate that the concluding remarks above have the effect of giving theoretical support to those who view contemporary mathematical research as being very much a part of the scientific enterprise.

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