Clockwork or Turing U/universe? – Remarks on Causal Determinism and Computability

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ABSTRACT. The relevance of the Turing universe as a model for complex physical situations (that is, those showing both computable and incomputable aspects) is discussed. Some well-known arguments concerning the nature of scientific reality are related to this theoretical context.

The close relationship between computability, mechanism and causal determinacy[1] is basic to post-Newtonian science, and underpins the familiar notion of a Laplacian 'clockwork' Universe, which provided a clear mathematical model of physical reality for over a century and a half — the discovery in the 1930s of the possibility of an explicit mathematical description of the model providing a key ingredient in its eventual demise. Acceptance of such a model has never been total of course, even amongst scientists. But in recent times the need for a mathematical alternative to that of Laplace, subsuming and extending according to the changing theoretical and empirical environment, has become increasingly overdue. The purpose of this note is to argue that the genesis of such a precise and intuitively natural model lies in Alan Turing's response, in a more limited mathematical context, to the newly discovered incomputabilities of the decade 1927–1936.

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[1] Or, with David Hume [1739], [1748] in mind, *apparent* causal determinacy.

1. Laplacian determinism

A perceived algorithmic content for reality is peculiar to our time and culture. Its origins are commonly traced back to the ancient Greeks, as is the development of the notion of *proof*, providing a useful infrastructure for mathematical and scientific truth, having been rediscovered (largely via Arab texts) in the late Middle Ages and developed during and after the Renaissance.

The extent to which scientific activity, at least since the time of Newton, has been directed towards the inductive identification of the algorithmic content of nature is illustrated by the emphasis on *prediction* in the formulation of satisfactory theoretical explanations (see for example Casti [1990]). According to Einstein [1950], p. 54:

> When we say that we understand a group of natural phenomena, we mean that we have found a constructive theory which embraces them.

The key element in the revolution in conceptual framework which Newton was responsible for was the new mathematics. This secured the theoretical basis for the mechanistic Universe implicit in Laplace's [1819] description of his predictive[2] 'demon':

> Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situations of the beings who compose it — an intelligence sufficiently vast to submit these data to analysis — it would embrace in the same formula the movements of the greatest bodies and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes.

Newton's embracing of the idea of gravity at a distance, independent of any transmitting medium, involved a replacement of Descartes' extreme mechanistic framework by one in which algorithmic content became an accepted replacement for proximate interaction. This revolutionary change in how causality was viewed has been related (see Betty Jo Dobbs [1991] or Richard Westfall [1984]) to Newton's alchemical interests, his awareness of the possible significance of a more global causality presaging current dissatisfaction with existing models of determinism. Such a broad view of mechanism became increasingly necessary, as during the nineteenth and early twentieth century new fields, electrical and magnetic, with no clear

[2] In focusing on the ontological content of determinism rather than the epistemological, the predictive element of Laplace's formulation, which is open to differing and confusing interpretations, will be ignored in favour of the widespread association of Laplacian determinism with mechanism.

explanation in terms of the interactions of physical bodies, assumed the scientific centre-stage.

Over time, a closer inspection of the Universe's mechanical credentials revealed a number of problems. While twentieth century questioning of Newtonian (in the sense of mechanistic) rationalism was paralleled (cf. Pope, Blake, Berkeley and Coleridge) by that of the eighteenth and early nineteenth century.

David Hume was the first to refer to fundamental difficulties in actually pinning down the basic workings of a putative mechanical Universe, in that the intuitive sense of there being a *connection* between a cause and effect appeared to lack formal content. The working scientist, from before Newton even, has been able to ignore the underlying, and very real, conceptual deficiency (which we return to in section 5), which makes it difficult to even formulate what we mean by determinism.

Moreover, the apparent fragmentation of science into heterogeneous bodies of knowledge, each with its own individual methodological and technical frameworks (see, for example, Dupré [1993]), belies the expected reductionism characteristic of a machine. This further relates to the ostensible gaps in known computability of material phenomena, and (see sections 2 and 5) to the more formal attempts to extend the picture of the role of mechanism as it relates to complex material systems.

Finally, changes to the dominant scientific theories have weakened the intuitive basis for mechanism. It is possible to argue (Earman [1986]) that even:

> Newtonian space-time, whose structure is rich enough to support the possibility of Laplacian determinism, nevertheless proves to be a none too friendly environment.

And of current theories, only special relativity comes with convincing deterministic credentials. Associating mechanism with standard formulations of computability (as in the following section) promises precise criteria by which to decide the significance of such theories for Laplacian determinism. In doing this one must follow Kreisel [1974] in distinguishing between *phenomena* and *theories* — a theory is mechanistic if " every sequence of natural numbers or every real number which is well defined (observable) *according to theory* is recursive or, more generally, recursive in the data (which, according to the theory, determine the observations considered)". Shipman [1998] notes that even very successful theories such as QED (the theory of quantum electro-dynamics — see Lawrie [1990], pp. 201–212), described by Feynman [1985] as "the jewel of physics ... our proudest possession", do "not have a fully satisfactory mathematical and computational foundation". Geroch and Hartle [1986] point to the possible incomputability of physically measurable numbers predicted by Hartle's version of quantum gravity, noting that the implementation of the algorithm suggested by the theory needs one to detect effectively homeomorphism affecting pairs of simplicial 4-manifolds (not possible in general — see Haken [1973]). But in the absence of relevant techniques for *excluding* computable models of these theories, their uncertain status tells us little about the underlying reality.

As already mentioned, an association between computability (in the practical sense) and mechanism goes back to the time of Newton and before, but the *identification* between material structures and those governing information is a comparatively recent development. Different insights arise according to the particular perspective — for Turing [1936], [1950] it was how real phenomena can be described in terms of mathematical models of computability, while for Shannon [1948] (representing the other main theme) it was the recognition of informational structure based on physical laws. Accordingly, one can describe the Laplacian model of the informational structure of the Universe in terms of a (schematically) well-understood shifting around of information content according to the second law of thermodynamics (which one can roughly paraphrase as saying that any nontrivial restructuring of information content corresponding to a change in an isolated physical system cannot be perfectly reversed, due to a loss in *available* information). That is, there is no *creation* of enhanced information content, although there can be (algorithmically translatable) transmutation. There may be local concentrations of high information content (e.g., in biological organisms), but a concomitant entropic loss of a proportion of the antecedent information content to disorderly, non-retrievable manifestations. Incomputability, if it does arise, can only originate with initial conditions and can never be explained in terms of what happens in the observable Universe. Or so the argument goes.

In recent times, a version of Laplacian determinism has been given more precise form via a detailed analysis, and corresponding restatement, of Church's thesis. See Odifreddi [1989] for a guide through the more arcane (but still important) subtleties of terminology[3], and Odifreddi [1996] for a useful introduction to some very relevant contributions of Georg Kreisel which we will need to return to later.

[3]For us, the term 'computable' will be firmly based on the notion of Turing computability and (see section 2) mathematical models of the physical universe will be non-discrete. Although the discussion will certainly point to the probability of an *analogue* computability which transcends Turing computability — depending on the controlled reproducibility of certain (Turing) incomputable natural phenomena coming out of a posited non-mechanistic determinism — the use of terminology will be familiar and, as far as possible, standard, a choice to be justified by the eventual placing of such an extended notion of computability within the framework of classical computability theory.

2. Occurrent incomputability in Nature

To find a single body of *empirical* evidence which is clearly inconsistent with a narrowly mechanistic Laplacian determinism, one must first look to the quantum level. However, much that appears strange there does not in itself require a theoretical interpretation taking us beyond the classical framework of Turing computability.[4]

The familiar overview of the determinism provided by quantum theory is that the quantum-mechanical state of a system of particles, specified as precisely as it can be by its *wave function*, and subject to Schrödinger's equation, contains insufficient information to give more than a probabilistic computation of the precise locations or momenta of the particles at a later time. The lack of determination of the quantum-mechanical parameters, according to Heisenberg's Uncertainty Principle, is a feature of reality. But the quantum states themselves, in the absence of measurements, can be calculated over time, which means that a consistent interpretation of the reality (or unreality) underlying the quantum superpositions is sufficient for a deterministic picture. (Popper's 'propensities', for example, give a formalisation of the intimate connection between the quantum-mechanical probabilities attached to particular states and the contextual contingencies involved — see Popper [1983], p. 351.) What *does* radically challenge determinism — as is expressed in the famous example of Shrödinger's cat — is what happens at the margins of the quantum and classical domains, for instance where a measurement is made, when a superposition is turned into an actual outcome via a so-called 'collapse of the wave function'. This leads to the so-called *measurement problem*, asking for an explanation of exactly why this collapse, with its associated probabilities, takes the particular form it does. But it is the evidence, both theoretical and empirical, of nonlocal causality which is most damaging to the clockwork model. The associated incomputabilities, undeniable but apparently beyond explanation in terms of classical computability/incomputability theory, are discussed in more detail in the next section.

Incomputability also emerges, but less assuredly, in physical situations involving mathematical non-linearity. Kreisel [1967] distinguishes between classical systems and *cooperative phenomena* (not known to have Turing computable behaviour), and proposes [1970] (p. 143, Note 2) a collision problem related to the 3-body problem as a possible source of incomputability, suggesting that this might result in "an analog computation of a non-recursive function (by repeating collision experiments sufficiently often)"

[4]For instance 'quantum computation' (originating with Benioff [1982] and Feynman [1982], with more specific proposals from David Deutsch [1985] and Feynman [1986]) appears to hold few surprises for the classical recursion theorist.

(see also Kreisel [1974]). The role of non-linearity (for instance in relation to chaotic situations) is suggestive of that of primitive recursion in the derivation of recursive functions capable of enumerating noncomputable sets.

An ingredient to the history of chaos theory has been the characterisation, due to Claude Shannon [1948] of physical phenomena — or more precisely, the orderliness or otherwise, of such phenomena — as *information*. From the slightly adjusted perspective of computability theory, one still obtains a multitude of different examples of the generation of informational complexity via very simple rules, and of the emergence of new regularities. The existence of so-called 'strange attractors' (see for example the two classic papers of Robert Shaw [1984], [1981]) provide a small-scale illustration of the macroscopic emergence (see section 5) of new forms from a causal context involving many diverse histories.

A good source of examples of the synthetical approach to finding precise mathematical models of functional complexity in nature, is provided by particular physicalist approximations to mental processes. For instance, more recent connectionist theories potentially transcend the Turing computability arising from the classical McCulloch and Pitts [1943] artificial neuron formalism, as observed by Smolensky [1988], p. 3:

> There is a reasonable chance that connectionist models will lead to the development of new somewhat-general-purpose self-programming, massively parallel analog computers, and a new theory of analog parallel computation: they may possibly even challenge the strong construal of Church's Thesis as the claim that the class of well-defined computations is exhausted by those of Turing machines.

But (and this is relevant to what follows) the analog computations cannot, it seems, be achieved within the discrete computational environment successfully analysed by Turing [1936].

The role of particular *presentations* of mathematical structures (not just in Minkowskian geometry) is basic to any extrinsic discussion of computability or otherwise in the material universe. Although science depends on the fact that observation of the world is independent of any particular description of it (in terms of notational reals, say), the particular presentation must be appropriate to the formal development of the underlying reality. (For instance, it is formally possible that a (Turing) computable relation on a countable set of reals may not be computably presentable as a relation on the natural numbers.) Ignoring such considerations[5], physical atomism and an overview of scientific practice suggest a reduction to discrete sys-

[5] cf. Feynman's [1982] suggested resolution of the uneasy relationship between reality and its discrete representations: "It is really true, somehow, that the physical world is representable in a discretized way, and ... we are going to have to change the laws of physics."

tems — all of which seem to point to the computability, or mechanism, of nature (see for example Church [1957], Kolmogorov and Uspenskii [1958], Kreisel [1965], Greenspan [1973], [1980], [1982], Gandy [1980], La Budde [1980], Vichniac [1984] and Toffoli [1984]). But a more basic problem is that discrete representations suppress many of the asymptotic features of the Universe as a relational system, not only undermining natural ingredients of standard theoretical analysis, but severing the link with higher levels of logical structure. The latter may not appear to have much practical significance, since in dealing with relatively small systems those relevant aspects which might otherwise emerge from the system's logical structure are determined mechanistically within a larger causal context. In general one is not unduly concerned with the extent of the material environment involved (the underlying logic is intrinsic to even finite immanently developing structures). But in practice, particular discrete representations cannot be selected without bypassing the essentially dynamic nature of the information content represented. One should be clear that one is not talking about the physical realisation of asymptotes, but about algorithmic actualisation which, according to their role in the determination of the global properties of the system, define reals essential to their description. In any case, the observational evidence of incomputability in nature directs ones attention to mathematical models which offer at least a reasonable chance of providing a theoretical explanation. Any discrete model *consistent with current physical theory* will, in as much as it can provide a recursive simulation (see Odifreddi [1989], pp. 109–113), be a necessarily probabilistic one describing *possible* behaviour (expressed as a sequence of states with non-zero probability). As we shall see in the next section, this and the absence of a generally accepted realist interpretation of current physical theory effectively entails a crudely built in incomputability, and an inability to convincingly present the classical and quantum levels of the material universe within a coherent logical framework. For the moment it will be assumed (cf. Pour-El and Richards [1989]) that the most natural approach is via computability in analysis.

However, one cannot precisely describe the exact nature of a global presentation of essential *information content* (and one must even allow the possibility of objects of higher type than reals being necessary) without a deeper understanding of how information is recorded in an immanently developing material environment. What are the mathematical and physical characteristics of the most basic unit of information? How does atomism, and its specific forms, relate to presentations of such information? For instance, what is the role of the various approaches to quantum gravity, and attempts to directly quantise general relativity (see Smolin [1991], [1993], Rovelli and Smolin [1990], Ashtekar, Lewandowski, Marolf, Mourão and Thiemann [1995]) or string theory (a seminal reference being Green, Shwarz and Witten [1987])? (Of course, given countability of a structure one may present the state of a particle as a real in terms of suitably presented relationships to others.) How does global context impact as locally available information content? How can an absolute structure of space-time be derived from the local? What is the relationship with the current working descriptions used in physics? Such questions may be hard to answer, but fortunately one can say a lot without a completely precise notion of local information content for the material universe. In section 5 below alternatives of optimal plausibility will be identified, sufficiently constrained for the development of a useful theoretical picture.

Confronted with incomputability of empirical origin, one is driven to look for natural parallels between the ways in which incomputability arises in the mathematical context and the relevant physical scenarios. One can of course accept high information content as a given in nature, but success of the scientific project has always been related to the search for mathematical structures through which observed complexity can be reduced to more fundamental features of the natural environment.

The way in which incomputability arises theoretically, by merely taking an overview of a sufficiently advanced mechanical process, is at first sight extremely simple, providing an obvious basis for any extrinsic approach to the problem. By analogy with Penrose's Mandelbrot question (see below), one can view the well-known failure to find any orderly pattern in the decimal expansion of π (cf. Gandy [1988], p. 66) as a manifestation of incomputability of the (computably enumerable) set of finite configurations contained therein. Davis, Matijasevič, Putnam and Robinson's proof (see Matijasevič [1970]) of the Diophantine nature of *all* computably enumerable sets shows how simple are the *mathematica*l levers underlying incomputability. The main obstacle to a straightforward transfer into nature is the difficulty in identifying the physical link corresponding to the shift from the local mechanics to the more global manifestation of incomputability, and a resulting jump in information content with observable consequences of a similar level of logical complexity as that of the originating environment. Despite his apparent non-materialism in relation to the mind (see Hao Wang [1974]), one can detect in Gödel [1972], p. 306, an attempt to grapple with such problems, in particular in his observation (in relation to Turing's [1936] argument that "a machine can reproduce all steps that a human computer can perform"):

> ... that *mind, in its use, is not static, but constantly developing*, i.e., that we understand abstract terms more and more precisely as we go on using them, and that more and more abstract terms enter the sphere of our understanding. There may exist systematic methods of actualizing this development, which could form part of the

> procedure. Therefore, although at each stage the number and precision of the abstract terms at our disposal may be *finite*, both (and, therefore, also Turing's number of *distinguishable states of mind*) may *converge toward infinity* in the course of the application of the procedure.[6]

In the closing pages of Chaitin [1987], one finds parallel speculation concerning contingent incomputability of biological origin. Letting Ω be the halting probability for a suitably chosen universal computer U (p. 164 of the revised edition, 1992):

> We have seen that Ω is about as random, patternless, unpredictable and incomprehensible as possible; the pattern of its bit sequence defies understanding. However with computations in the limit, which is equivalent to having an oracle for the halting problem, Ω seems quite understandable: it becomes a computable sequence. Biological evolution is the nearest thing to an infinite computation in the limit that we will ever see: it is a computation with molecular components that has proceeded for 10^9 years in parallel over the entire surface of the earth. That amount of computing could easily produce a good approximation to Ω, except that that is not the goal of biological evolution. The goal of evolution is survival, for example, keeping viruses such as those that cause AIDS from subverting one's molecular mechanisms for their own purposes.
>
> This suggests to me a very crude evolutionary model based on the game of matching pennies, in which players use computable strategies for predicting their opponent's next play from the previous ones. I don't think it would be too difficult to formulate this more precisely and to show that prediction strategies will tend to increase in program-size complexity with time.
>
> *Perhaps biological structures are simple and easy to understand only if one has an oracle for the halting problem.* (italics added)

In order to overcome this obstacle to convergence of incomputability in nature and in theory, one needs to be more specific about what one means by a 'computable' set of reals or of a 'computable' function on the reals. For this, one must refer to the early work of Lacombe [1955a], [1955b] and Grzegorczyk [1955], [1957] framing the computability of a real-valued function f in terms of the recursiveness of the corresponding functional (see Pour-El and Richards [1989], Chapter 0, for further discussion). One notes that, modulo certain specifics connected with the role of the metric in analysis, the relevant definitions can be formulated naturally in terms of Turing

[6]Further, Kreisel [1972] refers to problems connected with the way in which new mental states are mechanically included in the computer program.

relative computability of reals. This framework, giving, for instance, a natural definition of 'computably enumerable' set of reals, is the familiar one in which reals and sets of reals are accessed piece-wise by a computing machine. However computability of information content in nature, unlike that of an observed mechanical process, is not a primary property and involves problematic features of the role of the reals as a presentation, as pointed out by Penrose [1989] in relation to the question of the computability of certain simply generated mathematical objects, such as the Mandelbrot and Julia sets (see Mandelbrot [1982]).

Penrose (p. 124) points to the apparent unpredictability of structure in computer generated approximations to the Mandelbrot set (the latter readily obtainable via the set's computably enumerable complement) as indications of an underlying incomputability:

> Now we witnessed ... a certain extraordinarily complicated looking set, namely the Mandelbrot set. Although the rules which provide its definition are surprisingly simple, the set itself exhibits an endless variety of highly elaborate structures.

And goes on to observe (essentially) that the incomputability of the identity on the reals leads one, via particular presentations, to a counterintuitive notion of a noncomputable geometric object in two dimensions, even. In this context, Blum and Smale [1993] argue convincingly for a suitably relaxed version of the classical definition of computable set of reals (their *decidable* sets) according to which the Mandelbrot set and most Julia sets still turn out to be incomputable, and in a more basic sense even than that suggested by Penrose's observations on structure. The persuasiveness of Blum and Smale (see also Blum, Cucker, Shub and Smale [1998]) arises from optimal positioning within the classical framework of Turing computability, and the close correspondence with intuition — either in relation to nature (which does not care about different presentations), or to everyday practice (where any limiting process of approximation can be satisfactorily terminated at some finite level).

So the Blum-Smale notion of decidable set of reals provides a formal framework for presenting the evolution of incomputability in nature, in particular, suggesting a scenario in which nature has no problem in producing the basis for sets of reals which can be presented as the halting set of some (reasonably formulated, cf. Blum and Smale) machine. The resulting incomputability will then provide a context at particular moments of time for further physical interaction. If this environment is fairly diffuse or limited in extent, the interactions can be captured classically (that is, mechanically). But if there is sufficient contiguity of causal influences — that is, the contingencies converge to produce systemic rather than classically local effects (see Bohm [1957] for a particularly graphic analysis of such processes)

— then the resulting state of an individual constituent particle can only be computed by taking into account overall the mathematical and logical structure encompassing the incomputability of information content present. This can be envisaged as the iteration of simple operations producing an incomputability of *texture* of time-space, this texture becoming apparent not just as time passes, but physically (in approximation) in finite time (cf. Pour-El and Richards [1983]). The process by which local absorption of this augmented incomputability of information content derived from such a context takes place depends on the presence of adequate form and contiguity to by-pass any entropic factors at work. As we shall see later, the process can be described in terms of the Turing jump, although to fully capture it theoretically one would expect that the notion of E-recursion (based on Kleene's [1959], [1963] formulation of recursiveness of objects of finite type), and particularly the notion of the 1-section in relation to the E-recursively enumerable degrees (see Sacks [1990], Part D), is needed. The Mandelbrot example, presenting digitally observable evidence of undecidability, points to a formal process of algorithmic re-presentation of the associated incomputability, with the global manifest locally. In nature, one encounters this at the onset of turbulence, or of complexity in weather systems, or, for that matter, the human brain (a climate in miniature),[7] with incomputability apparently feeding on itself as the threshold is passed at which nonlocal causality comes to dominate. While the empirical accessibility of incomputability is essential, irreducible higher type features, both set theoretical and recursion theoretic, may well be relevant to a full characterisation of the information content. But thinking in terms of nature collating the results of computably enumerated events while simultaneously computing relative to the total context presented as a Δ_2^0 characteristic function, one can envisage a reduction of this picture to the classical framework. One notes that while the key to the incomputability derivable classically from a computable function is a severing of the link between image and argument, the physical root is the independence of effect from scale of causal origin, characteristic of chaotic situations.

However, having relied on current knowledge of quantum events for circumstantial evidence of incomputability in nature, it is clear that in this case no such reduction is appropriate. The problem is that the incomputability indicated, although unavoidable, does not come with an explanation in terms of local causality, as will be discussed in more detail in the next section. In fact at first sight it is not hierarchically developing incomputability but that of basic laws of nature which is suggested, the underlying reason

[7] Of course, as Penrose [1994], p. 153, observes: "Once it is conceded that *some* physical action might be non-computational, the possibility is laid open for non-computational actions also in the physical brain, ...".

for this only becoming apparent in section 5 below. For the moment one notes that in mathematics one does not need to take incomputability as a given, and that it seems unlikely that the simple mathematical structures that give rise to incomputable phenomena are not reproducible in nature. And while understanding of creation itself may be out of reach (despite some cosmologists' legitimisations of creational scenarios in terms of what is perceived as being mysterious at the quantum level), it being impossible to qualify or quantify the information content of the Universe's 'boundary conditions', there is no real evidence that the absolute origins of the Universe are not governed by very basic mathematical structures. One cannot but agree with Alan Guth [1997], pp. 251–252, who (reminding one of the evolutionary cosmology of Charles Sanders Peirce) imagines in relation to the improbability of the big bang as a "singular act of creation", a biologist who "discovered a bacterium that belonged to no known species":

> Although she believes firmly that life on earth originated from non-living materials, the possibility that this particular cell is the result of such an improbable occurrence would be too preposterous to even consider.

But this is a topic we shall return to in section 5.

On the other hand, if one chooses to approach incomputability in nature via the everyday mathematics used to describe physical phenomena, evidence consistent with the previous schematic picture is provided by Pour-El and Richards [1983], [1989].

What is lacking, of course, is a mathematical model of the Universe which captures enough of its specific infrastructure to closely simulate the *development* of the observed hierarchical structure of information content. Such a framework should be capable of representing in an organic way the mutable balance between *entropy* (described by Hawking [1977] "as a measure of the disorder of a system or, equivalently, as a lack of knowledge of its precise state"), conventionally ruled by the second law of thermodynamics, and the (less familiar) hierarchically organised, nascent incomputabilities of nature. We will argue below that while the Turing universe tells us little about the *dynamic* relationship between computable and incomputable, it provides a particularly illuminating model of the fine structure of *actually existing* information content.

3. Nonlocality

The argument so far is for a 'quasi-mechanical' Universe in which systemic input to the causal relationships take one beyond the computability over time expected of a (Turing) machine-like universe. But the most radical challenge to Laplacian determinism and the associated computability comes at the quantum level where the very nature of causality comes into question, along with the Turing model of what mechanism remains (which, of course, gives the appearance of being considerable). However it is not the observed ambiguities of quantum phenomena — the associated probabilistic analysis is consistent with a deterministic interpretation of the underlying reality — which is so unusual. What is strange is what happens in the collapse of the wave function — concerning the reasons for which, according to Richard Feynman, "we have no idea" (Feynman, Leighton and Sands [1965]). The fact that the collapse appears to be global in origin, involving nonlocal communication of a seemingly causal nature, indicates firstly that some causality involving higher logical structure is involved. This is most clearly illustrated in the EPR thought experiment of Albert Einstein, Boris Podolsky and Nathan Rosen [1935], of which Bell's [1964] inequality provides an empirically testable (and tested, see Aspect, Dalibard and Roger [1982], Aspect, Grangier and Roger [1982]) version. Even though QED provides a mathematical formulation which seems to successfully transcend some of the conceptual and descriptive difficulties inherited from the classical context (e.g., wave/particle ambiguity), it does not remove the essential dichotomy between theoretical descriptions of quantum/classical reality or explain the apparent systemic nature of the transition between the two.

The so-called 'EPR paradox' is most usefully considered as an indicator of the incompleteness, as a description of physical reality, of quantum mechanics (and this seems to have been an important aspect of the thinking of Einstein himself). In outline (and the details of the original experiment, its recasting by Bell [1964], and its subsequent empirical scrutiny, can be found in many places — see, for example, Omnès [1994], chap. 9), Einstein and his two colleagues considered (in the more famous of their examples) the behaviour of two particles whose initial interaction means that their subsequent descriptions are derived from a single Schrödinger wave equation, although subsequently physical communication between the two, according to special relativity, may involve a significant delay. If one imagines the simultaneous measurement of the momentum of particle one and of the position of particle two, one can arrive at a complete description of the system. But according to the uncertainty principle, one cannot simultaneously quantify the position and momentum observables of a particle. The standard Copenhagen interpretation of quantum theory requires that the measurement relative to particle one should instantaneously make the measurement relative to particle two ill-defined (a process commonly described in terms of a collapse of the wave function).

One could try to explain this in various ways. Clearly it constituted a major challenge to any existing causal explanation. Kreisel [1971], p. 177, reminds us (in a mathematical context) that the appearance of a process is not necessarily a good guide to the computability of its results. The immediate question was whether there existed undiscovered, but qualitatively familiar, causal aspects of the material universe in terms of which this apparent inconsistency could be explained. This was what EPR expected:

> While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible.

Despite Einstein's later negative comments concerning the particular proposal of Bohm [1952], this has been taken (see Bell [1976]) as a tacit endorsement of a 'hidden variables' approach.

Some early objections to the hidden variables programme (most well-known being that of von Neumann[1932]) are disposed of in Bell [1966]. However, Bell [1964] proposed a testable version of the EPR experiment (recast, following a suggestion of Bohm, in terms of spin rather than position and momentum). This led to the experiments described in Aspect, Dalibard and Roger [1982] and Aspect, Grangier and Roger [1982], and a general acceptance[8] of the predictions of quantum theory as a description (but not a *complete* description) of what actually happens. This means that although there are quite viable *non-local* theories of hidden variables (presaged by de Broglie [1927] and his non-local 'pilot wave'), the complete description of quantum theory in terms of a hidden reality involving *only* local parameters is not possible.

So one can reasonably deduce that there is no mechanical Universe. One could, presumably, have computability emerging from a grossly non-mechanistic causal context, but there seems no reason for allowing such a possibility in this case. Experience tells us that computability of natural phenomena must derive via natural translation from material mechanism. That is, that if the material universe is to *behave* like a machine, then it essentially achieves this in a causal context via mechanism.

It is important to notice that it is not just to our intuitive sense of a mechanistic Universe which suffers, but also *computability*. One can no longer rely on establishing computability by predicting local phenomena via a local analysis. A computable causality depends not only on being able to computably characterise the nature of the causal factors at work, but on being able to effectively identify the particular causal context relevant to a given phenomenon. And any extended notion of 'locality' which is not physically based (for example based on Bohm's [1980] 'enfolded' reality), may retrieve determinism, but is not likely to deliver computability.

So one is left with the question of what is the appropriate mathematical framework, radical enough to clarify questions raised about the logical

[8]But see Franson [1985].

status of contemporary physics — in particular, its *completeness* (in regard to explanations of nonlocality and the measurement problem), and *consistency* (in particular concerning the contrasting descriptions of the classical and quantum worlds). There are well-known examples of structures in which the global properties appear strikingly undetermined by the local ones (cf. the M. C. Escher lithographs 'Waterfall' and 'Ascending and Descending', based on mathematical examples of Penrose and Penrose [1958]). But what is needed is an appropriate mathematical formulation of local causality, with a corresponding, mathematically nonrigid, global structure, capable of confirming the widespread intuition that nonlocality has important consequences for the nature of of the real world and our knowledge of it.

4. The Turing model

We have seen that at a number of levels the standard mathematical model of determinism in nature has been found lacking. Although at first sight this may not appear to be relevant to the working scientist, beyond a background confusion regarding the role of 'truth' in science and an accompanying pragmatism, it may well be that the resulting conceptual vacuum underlies a number of fundamental scientific and philosophical problems, and seriously undermines the hegemonic role of science in society generally. But the very persistence of the model suggests that any alternative, while achieving improved stability in the presence of high information content, will have to retain at least some of the Laplacian strengths of simplicity and aptness.

As we saw in section 2, there is plenty of evidence that structures derived from the Turing model are capable of enriching their information content via the iteration of familiar algorithmic processes. The rejection of determinism itself — for example, Dupré [1993]:

> There are two very powerful reasons for rejecting the doctrine of determinism. The first is that it seems almost entirely, or perhaps entirely, devoid of empirical support. The second is that our most successful scientific theories describe a probabilistic rather than a deterministic world.

— still entails the admission of incomputability, without denying the essential role of computability as a universal force for understanding and structural cohesion. So we are left with a Universe in which incomputability, even if not formally verified, is both theoretically likely and empirically forced on us, but which we have no means of making any sense of in any precise sense except via those computable relationships we can identify. Following Turing [1939], who attempted to use the notion of oracle computability give an explanatory context to Gödel's incompleteness phenomenon, the Turing model can be thought of as the minimal extension of the Laplacian one consistent with the observed incomputability of nature. Our choice of Turing reducibility (or one of its subreducibilities) is based on the empirical evidence of histories being reducible to the analysis of correlations between descriptions of essentially finitary circumstances, and the aesthetically based preference for models based on ourselves, as understood, as reflections of the functionality of the larger Universe.[9] Particular reductions will be intended to capture computable aspects of allowable historical processes, such as those derived from the actions of the basic forces of nature. Fundamental to the model will be transitivity, and a conventional 'arrow of time' identifying events which are permitted historical consequences of each other. From such basic considerations eventually arise closer and more striking correlations between specific physical phenomena and particular features of the Turing model. None of the definitions of the *strong reducibilities* (those properly contained in Turing reducibility) seem to fully capture the range of basic physical processes, nor do what is known of their theories (see Odifreddi [ta]) tie in with these deeper properties of the physical universe (see below).

For details of standard notation and terminology for the Turing degrees, see for example Soare [1987] or Odifreddi [1989].[10] A fuller description of Turing machines and their properties can be found in Davis [1958].

For instance, corresponding to the ith Turing machine, Φ_i denotes the ith partial computable (p.c.) functional $2^\omega \longrightarrow 2^\omega$. A set (or binary real) A is *Turing computable from*, or *Turing reducible to*, a set B ($A \leq_T B$) if and only if $A = \Phi_i^B$ for some $i \in \omega$, and A, B are *Turing equivalent* ($A \equiv_T B$) if and only if $A \leq_T B$ and $B \leq_T A$. Since $\equiv_T$ turns out to be an equivalence relation, one can (following Post [1948]) define the *degree of unsolvability* or *Turing degree* of A by

$$\deg(A) = \{X \in 2^\omega \mid A \equiv_T X\}.$$

So the Turing degrees are sets of objects, formally described in terms of reals, the essential information content of their members being algorithmically indistinguishable. We write $\leq$ for the partial ordering induced by $\leq_T$ on the set $\mathcal{D}$ of all degrees, $\mathbf{0}$ for the least degree, consisting of all computable sets of numbers (or, equivalently, of computable reals), and $\boldsymbol{\mathcal{D}}$ for the structure $\langle \mathcal{D}, \leq \rangle$.

Let $W_i^A = \operatorname{dom} \Phi_i^A$ denote the ith *computably enumerable in* A (*A-c.e.*) set ($W_i = W_i^\phi$ being the ith c.e. set). The *c.e. degrees* (collectively

[9]However see, for example, Post's [1965] speculations on the possible relevance of extensions of the familiar Turing model to an understanding of thought processes.

[10]But each must be read in the light of Soare [1996].

denoted by $\boldsymbol{\mathcal{E}}$) are those containing c.e. sets. Feferman [1957] showed the c.e. degrees to be exactly those degrees containing (coded) recursively axiomatisable first-order theories, and there are many other examples of classes of natural, mechanically generated, objects from which the totality of c.e. degrees arise.

Kleene and Post [1954] defined the notion of *jump operator* on sets and degrees. The *jump* ($n+1$ *th jump*) of a set A is defined by $A' = A^{(1)} = \{x \mid x \in W_x^A\}$ ($A^{(n+1)} = (A^{(n)})'$). This induces a *jump operator* on degrees defined by $\mathbf{a}' = \deg(A')$, $A \in \mathbf{a}$, with the special properties that $\mathbf{a} < \mathbf{a}'$, and $\mathbf{a}'$ is the largest of the degrees of sets c.e. in $A \in \mathbf{a}$. Post's Theorem [1948] that $X \in \Delta^A_{n+1} \Leftrightarrow X \leq_T A^{(n)}$ attaches special importance to the ascending sequence $\mathbf{a}, \mathbf{a}', \ldots, \mathbf{a}^{(n)}, \ldots$. The most important noncomputable degree $\mathbf{0}'$ contains, for instance the (coded) undecidable axiomatic theories of Gödel [1934], as well as many other natural mathematical objects. We define the standard ω*-jump* of $\mathbf{a}$ by $\mathbf{a}^{(\omega)} = \deg(\oplus_{n\in\omega} A^{(n)})$, $A \in \mathbf{a}$.

Events originating with computable physical processes operating on given initial conditions lead to a notion related to that of A-c.e.: a set B is said to be *computably enumerable in, and above* a set A (or A-CEA) if $A <_T B$ and B is A-c.e. The CEA sets are those X-CEA for *some* X. There are corresponding degree theoretic notions. This is fortunate, in that it is easy to theoretically provide for a (possibly very small) entropic element in any reduction to an event CEA, due to the (relativised) Sacks splitting and density theorems (Sacks [1963], [1964], respectively).

The structure of $\boldsymbol{\mathcal{D}}$ (and of local structures such as $\boldsymbol{\mathcal{E}}$) turns out to be very rich, and to lead to great technical complexity. In fact, Simpson [1977] was able to characterise the first order theory of $\boldsymbol{\mathcal{D}}$ as being recursively isomorphic to the second order theory of arithmetic.

It is clear that most of the evidence supporting the Laplacian model, making it so intuitively persuasive over such a long period, also supports the algorithmic aspects of the Turing one. While those incomputabilities so damaging to the Laplacian picture either provide the basic information content of the Turing model, or as we shall see (in the case of quantum phenomena arising from nonlocality) are accounted for by it.

Although it is convenient in framing the above definitions to distinguish between information and computable process, there are (for our purposes equivalent) formulations of computability theory, for instance based on Church's [1933] notion of λ-computability, which make no theoretical distinction between function and argument. This is not just mathematically attractive (with correspondingly elegant models, cf. Scott [1975a], [1975b]), but also seems in line with the primacy of process in nature, and the observed flexibility of relationship between matter and energy at the most basic (that is, quantum) level.

Anyway, given a set of reals with corresponding information content, empirical considerations have lead us to look to material expressions with mechanical basic relationships, and this has in turn suggested the Turing model. It is then the full range of such computable reductions, containing as it must a blueprint for any future development of a consistent infrastructure, and allowing for any eventual complexity of information content, which determines the overall logical coherence of the system and, specifically, via which the system must achieve a level of definition of its fundamental causal structure.

We saw above how problems with the logical coherence of the material universe surface via quantum considerations, such as from the EPR paradox, and how current theoretical solutions involve discussion of such logical characteristics of the Universe as its *consistency*. Gödel's incompleteness theorems have also been imported from logic to provide useful analogies relevant to some of the more mysterious aspects of the observed universe. However, in dealing with the real world, the axiomatic theories to which consistency and incompleteness relate seem incapable of providing *more* than illuminating analogies. What is needed is a precise explanation of what Leibniz [1714] describes as the "pre-established harmony" of a Universe exhibiting a high degree of systemic unity. One must look to the neglected and little understood notion of *definability* (and the closely related notion of *invariance*), within a mathematical structure which captures the algorithmic and underlying information content of the material universe, for a more precise analysis of how nature achieves its consistency. This will enable one to model a process whereby basic natural laws both determine and are determined according to hierarchical principles, and which thereby guarantees consistency and coherence. One notes that even though a system may be, according to certain reductionistically arrived at criteria, strictly finite, the incomputabilities and logical forms derived from the algorithmic content are intrinsically present and collectively comprise a logical blueprint for structural development.

Epistemologically, there is a close link between definability in terms of basic, perhaps physical derived, concepts and human (or at least scientific) understanding. Even in a mathematical context, for instance, Gödel [1946], p. 152, remarks that although the formal notion of ordinal definability might not completely capture the informal one of "comprehensibility by our mind", he believed it to provide "an adequate formulation in an absolute sense ... of [a set's] 'being formed according to a law' ". So there is the half-suggestion of 'definable in terms of something basic' as providing at least an approximation to what one can comprehend. And although Gödel [1964], p. 268, talks about the "remoteness from sense experience" of sets, which might still be perceptible via mathematical intuition — so there is no acceptance of an explicit link between sensory input and mathematical intuitions — in saying (Gödel [1995], p. 383) that "the certainty of mathematics is to be secured not by ... the manipulation of physical symbols, but rather by cultivating (deepening) knowledge of the abstract concepts themselves" one can detect an implicit assumption of an analogy between the process of verifying the role of formal definitions and the empirical process, in that they must be based on an epistemologically irreducible level of input. While (Gödel [1964], p. 268) although:

> ... mathematical intuition need not be conceived of as a faculty giving an *immediate* knowledge of the objects concerned ... as in the case of physical experience, we *form* our ideas of [mathematical] objects on the basis of something else which *is* immediately given ... [and which] may represent an aspect of objective reality ...

The notion of definability relevant to causal structures involving computable basic laws is that of *Turing definability*. (For a recursion theoretic overview of mathematical definability, see Slaman [1998].) Formally, a relation on reals is *(absolutely) Turing definable* if its degree theoretic counterpart can be described, using only the standard first-order language, in terms of the ordering relation $\leq$ on $\boldsymbol{\mathcal{D}}$. Rogers [1967a], looking for a language independent notion, defined such a relation to be (Turing) *invariant* if and only if it was left degree theoretically unchanged under all automorphisms of $\boldsymbol{\mathcal{D}}$. Of course, in the real world one often achieves a level of understanding via the establishment of relative definabilities, not just of a general kind, but involving specific objects, which may not by themselves be definable. This leads to the important formal notion of *definability relative to parameters* (see Slaman and Woodin [1986]). There is of course a corresponding notion of *relative invariance* of relations on the Turing universe, although little is known so far concerning the structures arising (see Cooper [1997]).

As one would expect in comparison with a Universe with many features and general characteristics accessible to human understanding, many of the naturally arising relations on $\boldsymbol{\mathcal{D}}$ turn out to be Turing definable. Particular examples include the definability (Cooper [1994]) of the relation of computably enumerable in — and hence of $\boldsymbol{\mathcal{E}}$, of CEA, of the Turing jump — giving that of $\mathbf{0}'$ and of each $\mathbf{0}^{(n)}$ for $n > 0$, of every level of the arithmetical (that is, Kleene-Post) hierarchy, of (Jockusch and Shore [1984]) the ω-jump and $\mathbf{0}^{\omega}$ (the degree of the theory of $\boldsymbol{\mathcal{D}}$), and (see Nies, Shore and Slaman [1996], [ta]) of every *atomic double-jump class* — where one says that degrees $\mathbf{a}, \mathbf{b}$ with identical n^{th} jumps are n^{th}*-jump equivalent*, and the n^{th}-atomic jump classes are the corresponding equivalence classes. One has to look to the local level (that is below $\mathbf{0}'$) to find specific non-invariant (and hence undefinable) relations. One defines the *high-low hierarchy* (usually relative to $\boldsymbol{\mathcal{E}}$), to be comprised of the following *jump classes*:

$$\mathbf{a} \in \mathbf{High}_n \Leftrightarrow \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}, \qquad \mathbf{a} \in \mathbf{Low}_n \Leftrightarrow \mathbf{a}^{(n)} = \mathbf{0}^{(n)},$$

Then (Cooper [1997]) the class of *low* degrees (= $\mathbf{Low}_1$ is not definable in $\boldsymbol{\mathcal{E}}$ (whereas by Nies, Slaman and Shore [1996], [ta], all the other jump classes are so definable).

It is of course a trivial observation that there is a universal ceiling on that level of definability of a real achievable degree theoretically. Accordingly, just as observational data provide the foundation for all scientific knowledge, at the theoretical level the nature of the observed world itself relies on the *available* definability of the system. This corresponds to the fact (previously noted) that it is in the nature of different presentations of a structure to particularise the essential informational content of a constituent part in ways which have no relevance to the underlying reality.

Before 1927, one might reasonably conjecture that this obvious limitation on definability is all there is, and that, correspondingly, the causal structure of the universe uniquely defines all its characteristics. One might take William James' graphic description (from an 1884 lecture to the Harvard Divinity School, [1897], p. 150, 1956 reprinted edn.) of an 'iron block' Universe as a metaphor for this conjecture:

> What does determinism profess? It professes that those parts of the universe already laid down absolutely appoint and decree what the other parts shall be. The future has no ambiguous possibilities hidden in its womb: the part we call the present is compatible with only one totality. Any other future complement than the one fixed from eternity is impossible. The whole is in each part, and welds it with the rest into an absolute unity, an iron block, in which there can be no equivocation or shadow of turning.

Formally it is subsumed in the so-called Bi interpretability Conjecture. The programme of research stimulated by this productive and mathematically attractive proposal eventually showed there to be deeper theoretical reasons for a locally lower ceiling on the level of definability or invariance achievable. Bi interpretability is fully discussed elsewhere (see for example Slaman [1991] or Nies, Shore and Slaman [1996]), but we will sketch in some of the essentials.

The notion of bi interpretability in computability theory seems to have been imported from model theory (see Ahlbrandt and Ziegler [1986]) by Harrington, although much of the subsequent development and application is associated with Slaman and Woodin [1986].

As described in Cooper [1997], bi interpretability extends some of the benefits of isomorphism (between known and less well-known structures) to apparently dissimilar pairs of structures. Choosing a representative X for a member $\mathbf{x}$ of some degree structure $\mathbf{D}$ over the reals can be thought of as defining a mapping from $\mathbf{D}$ to the standard model for second order arithmetic. Conversely (following Jockusch and Simpson [1976]), a *coding* of second order arithmetic involves *specifying* a collection of degrees, and relations on these degrees to represent addition and multiplication. If one can uniformly define the relationship between $\mathbf{x} \in \mathbf{D}$ and the code for the chosen representative $X \in \mathbf{x}$, then we say that $\mathbf{D}$ is *bi interpretable* with second order arithmetic. In local versions of the bi interpretability conjecture (arising from the work of Harrington and Slaman) one can replace the representative X with a number, namely the index for X in some canonical listing, with a consequent substitution of first order for second order arithmetic.

The *Bi interpretability Conjecture* is that $\boldsymbol{\mathcal{D}}$ *is bi interpretable with second order arithmetic*. Bi interpretability and the rigidity of the standard model of second order arithmetic gives:

(**1**) *Rigidity of* $\boldsymbol{\mathcal{D}}$.

And, since definitions in $\boldsymbol{\mathcal{D}}$ can be read off from those in the standard model via the bi interpretation (if it were to exist), one immediately obtains a complete characterisation of the Turing definable relations:

(**2**) *The definable relations on* $\boldsymbol{\mathcal{D}}$ *are exactly those given by the relations definable in second order arithmetic*.

Bi interpretability turns out to be technically more useful, but formally *equivalent* to Turing rigidity (Slaman and Woodin, private communication). The successes of the research programme based on bi interpretability are remarkable (see Nies, Shore and Slaman [1996] or Cooper [1997]). For instance, Slaman and Woodin have shown that recursion theoretic relativisation is sensitive to double jumps, in that only Turing degrees with identical double-jumps can have isomorphic cones above them. Although Martin [1968] showed that an assumption of projective determinacy was sufficient to guarantee an upper cone of bases of *elementarily equivalent* cones of $\boldsymbol{\mathcal{D}}$.

Lerman's [1977] notion of *automorphism base* (a substructure which locally determines the global action of any automorphism), has successfully reduced rigidity of large structures to that of smaller, even local, ones. Examples of automorphism bases include (Jockusch and Posner [1981]) any comeager set $\boldsymbol{A} \subseteq \boldsymbol{\mathcal{D}}$, the set of all noncomputable degrees *minimal* in $\boldsymbol{\mathcal{D}}$, and (Slaman and Woodin) $\boldsymbol{\mathcal{E}}$. When combined with results on definability, one gets particularly striking results.

Since, any automorphism of $\boldsymbol{\mathcal{D}}$ must preserve every level of the arithmetical hierarchy of Turing degrees, and every level of the high/low hierarchies is invariant under all automorphisms of $\boldsymbol{\mathcal{D}}$, there are some quite dramatic implications for $\mathrm{Aut}(\boldsymbol{\mathcal{D}})$. For instance, if $\boldsymbol{\mathcal{E}}$ or $\boldsymbol{\mathcal{D}}(\leq \mathbf{0}')$ is rigid then so is $\boldsymbol{\mathcal{D}}$. Which means that local structures contain a key to the Turing universe and its more material manifestations. Also, since there is in fact a *finite* automorphism base of c.e. degrees, $\mathrm{Aut}(\boldsymbol{\mathcal{D}})$ is countable.

Further, known *local* automorphism bases provide automorphism bases for $\boldsymbol{\mathcal{D}}$: such as (Lerman [1977]/Jockusch and Posner [1981]) every level of the high/low hierarchy (in particular, the low degrees), and (Ambos-Spies [ta]) every lower cone $\boldsymbol{\mathcal{E}}(\leq \mathbf{a})$ $(\mathbf{a} \neq \mathbf{0})$. Finally, if we define a substructure $\boldsymbol{\mathcal{C}}$ to be *rigid in* $\boldsymbol{\mathcal{D}}$ if and only if $\psi \restriction \boldsymbol{\mathcal{C}} =$ the identity for every $\psi \in \mathrm{Aut}(\boldsymbol{\mathcal{D}})$, then we find that (Slaman and Woodin) $\boldsymbol{\mathcal{D}}(\geq \mathbf{0}'')$ is rigid in $\boldsymbol{\mathcal{D}}$.

The situation for the structures derived from the strong reducibilities is not very well known (see Odifreddi [ta]), although for none of them has there been shown to be a relatively rigid upper-cone counterpart to the classical part of our Universe. For those derived from the various truth-table reducibilities (*truth-table*, *bounded truth-table* and *weak truth-table*), there are as yet no fully worked out proofs of nonrigidity. Although the fact that there are known to be $2^{2^{\aleph_0}}$ automorphisms of the many-one degrees (ruling out any non-trivial invariant subsets) is an indicator for nonrigidity in relation to all the reducibilities intermediate between many-one and Turing reducibility.

5. Causal determinism restored

> *God does nothing by himself which he can do by another* – Isaac Newton
> (Jewish National and University Library, Jerusalem, Yahuda Manuscript Collection Var. I, Newton MS 15.5 (2, n.50), fol.67r.)

Given the present state of knowledge of the Turing universe, and the acceptance of it as a useful representation of the relationship between computable and incomputable aspects of the material universe, it is now possible to present a coherent and intuitively satisfying picture of the underlying causal structure of the Universe, together with a persuasive materialist basis for the associated epistemology, which immediately clarifies the numerous scientific and philosophical puzzles previously alluded to. This will eventually require us to be more specific about certain details of the modelling process.

But first, having discarded Laplace's clockwork Universe, it is necessary to re-examine the relationship between computability and determinism.

The nature of determinism

Allowing for the fact that the nature of Minkowski space-time dictates 'time slices' relative to 'initial' and 'subsequent' times which are appropriately chosen surfaces, one can start with a plausible formulation of determinism given by Bertrand Russell [1953], p. 398, which avoids any mention of mechanism or computability:

> A system is said to be 'deterministic' when, given certain data, $e_1, e_2, \ldots, e_n$, at times $t_1, t_2, \ldots, t_n$ respectively, concerning this system, if E_t is the state of the system at any time t, there is a functional relation of the form
>
> $$E_t = f(e_1, t_1, e_2, t_2, \ldots, e_n, t_n, t).$$
>
> The system will be 'deterministic throughout a given period' if t, in the above formula, may be any time within that period .... If the universe, as a whole, is such a system, determinism is true of the universe; if not, not.

However, one can then follow Russell (p. 401) in observing that:

> It follows that, theoretically, the whole state of the material universe at time t must be capable of being exhibited as a function of t. Hence our universe will be deterministic in the sense defined above. But if this be true, no information is conveyed about the universe in stating that it is deterministic. ... This, however, is plainly not what was intended.

So how does one escape the apparent falsity of Laplacian determinism, without ending up with a trivially true notion? The point is that having noted the existence of the functional relation, one does expect that E_t can be arrived at for given t other than by examining the state of the universe in question at time t. We do not conclude (Russell p. 401) that "the material universe *must* be subject to laws" — since we intuitively expect laws to involve some level of *coersion*, not merely recording. This involves some 'nice' description of E_t, which is logically simpler in some sense than the Universe itself. (There are other solutions to the problem, but these can usually be expressed in such terms.) And one would expect to arrive at such a description by identifying an appropriately close relationship between E_t and the known natural laws. The relationship between E_t, the boundary conditions, and the relevant laws, is assumed to be governed by the rule that all causal sequences permitted by the logical properties of the system are allowed to occur. The empirically familiar 'coarse grained' structure of reality, even at the quantum level, will follow from the appropriate choice of logical framework. Of course, if one has no theoretical explanation of the genesis of natural laws, then one is restricted to a sort of super-sophisticated taxonomy of laws, with all its Humean uncertainties. As Guth [1997] says in relation to the inductive extension of quantum-like behaviour to cosmogony:

> If the creation of the universe can be described as a quantum process, we would be left with one deep mystery of existence: What is it that determined the laws of physics?

— which leads us to an appropriate version of Penrose's [1987] 'strong determinism' (according to which, pp. 106-107, "all the complication, variety and apparent randomness that we see all about us, as well as the precise physical laws, are all exact and unambiguous consequences of one single coherent mathematical structure"), and the theoretical problem of how the relations of a physical system are subject to immanent definability. Such concerns are not new, elements of which can be traced back to to Leibniz's relational, as opposed to Newtonian, view of the Universe, and which find a particular expression (via a view[11] of the primacy of observation over fact) in C. S. Peirce (*The Architecture of Theories*, 1891) [1931-58]:

> To suppose universal laws of nature capable of being apprehended by the mind and yet having no reason for their special forms, but standing inexplicable and irrational, is hardly a justifiable position. Uniformities are precisely the sort of facts that need to be accounted for. Law is par excellence the thing that wants a reason. Now the only possible way of accounting for the laws of nature, and for uniformity in general, is to suppose them results of evolution.

We will argue below that the Turing model not only gives a logically concise description of the development of the Universe relative to its laws, but also of the form taken by those laws, resulting in a meaningful description of the level of determinism applying in the material universe and a bypassing of such epistemological difficulties. Whatever the relevance of the Turing model, it is surely an important but neglected task (of computability theory) to ascertain as closely as possible the computational complexity of the existing level of determinism of our Universe. Between the Laplacian 'clockwork universe' and the trivially deterministic one of Russell, there must lie a mathematically precise bound related to Russell's function.

The laws of nature

As we have seen, mathematical considerations suggest that mechanisms are as basic as material particles, and that at the subatomic level noninvariance may possibly be related to such ill-defined phenomena as arise from wave/particle confusion. But invariance appears to be an attribute of the (empirically detectable) fundamental forces of nature and the algorithmically describable laws constraining them. In any case, the abundance of Turing definable relations at the local level, but not of singletons, leads one to emphasise the link between natural laws and invariance, giving a more secure foundation for the notion of causal connection. Such connections of the

[11] cf. the Peircian maxim ([1931-58], vol. 5, paragraph 412): "if one can define accurately all the conceivable experimental phenomena which the affirmation or denial of a concept could imply, one will have a complete definition of the concept, and there is *absolutely nothing more* in it".

more fundamental kind have a clearly conceived, if mathematically sophisticated, genesis as materialisations of Turing defined mechanisms. While this link persists at the classical level even where no apparent algorithmic content to the causal connections exists. The difficulty in establishing empirically a relation so conceived puts the continuing debate about how to precisely describe what one means by a law (in particular, how to add a concise notion of *necessity* to Humean conjunctions of events) in a revealing context — and, as we shall see later, throws light on the relationship between *theory* and *observation*. However, the link between natural laws and invariance does not explain the *genesis* of *specific* laws. The existing mathematical theory of mechanism is as yet unadapted to deal with such basic uncertainties. Moreover (cf. the speculative cosmogony of, for example, Andrei Linde [1991], combining inflation and quantum theory), an element of arbitrariness in the definitions of the basic parameters of our Universe is consistent with a global determinism, in that one cannot exclude causally unconnected components of material existence, of which our Universe may be just one.

We notice that this model of causality is very much in the spirit of such Humean solutions as those of F. P. Ramsey [1978] (revived by David Lewis [1973]) in attempting to derive the precise counterpart to the intuitive content of the notion of a law via its context as a generalisation in some ideal systematisation of knowledge. Making the necessary translation between consequences of a deductive system and relations invariant in automorphic models, this is apparent from Ramsey (p. 138):

> ... [laws are] consequences of those propositions which we should take as axioms if we knew everything and organised it as simply as possible in a deductive system,

or Lewis (p. 73):

> ... a contingent generalization is a law of nature if and only if it appears as a theorem (or axiom) in each of the true deductive systems that achieves a best combination of simplicity and strength.

Moreover, the actual form of the Turing explanation provides explication of the notions of necessity of such rejecters of the Humean tradition as Armstrong [1983], Dretske [1977] and Tooley [1977] — and successfully addresses the criticisms (eg. Tooley [1977]) of the Ramsey–Lewis approach.

Of course, implicit in all this is a confirmation of the role of 'nomic necessity', particularly in the sense of Swoyer [1982], for whom laws express non-contingent relations. Earman [1986], p. 105, points out that:

> ... if there is nomic necessitation, its ultimate springs are most likely hidden from our view. The ultimate laws of nature, whatever they may be, will most likely involve universals whose instancings corre-

> spond to states of affairs which are not directly observable and which are thus knowable only inferentially.

But this is wholly in line with the mathematical model.

See also Skyrms [1980] and Honderich [1988].

The picture which emerges from the theory of Turing automorphisms is one in which at neither the material nor the epistemological level can the basic causal relations guarantee a uniquely defined reality, but within which one can, from a realist perspective, explain the condensation of a so-called 'quasi-classical domain' in terms of Turing invariance. While the epistemological dimension is viewed as an integral part of the deterministic Universe, firmly based on the physical dimension, and algorithmically related to it via familiar proof-theoretic structure, but admitting qualitatively similar processes of systemic input as described for the material case. Intrinsic to this unity is the acceptance of human descriptions of the universe in a standard language as finitary approximations to limiting versions (existing in an appropriate infinitary language), no different in formal role to the rationals in relation to the descriptions of real numbers as infinitary decimals. Such a comprehensive relationship of model to that modelled is of course complex, but has the advantage of capturing intuitively based hierarchical structure independently of phenomenalist confusion. If such a scenario is accepted, there are a number of important consequences for science and for what are widely perceived to be its current crises.

Quantum versus classical reality

The first of these consequences is in relation to the search for a realist interpretation of quantum theory and a wider theoretical cohesion (encompassing, for example, what we know of gravity and the theory of general relativity). There is a conceptual difficulty in that the description of the constituent elements of the Universe in terms of reals involves an acceptance of *information content* (beyond those incidental details arising from the details of the specific presentation chosen) as the essential ingredient correlating with the underlying reality. And that in the quantum context there may be no counterpart to everyday experience leading one to expect that distinctions between entities can be achieved by reduction to a suitably basic level of information content. This brings brings Leibniz's formulation of the *identity of indiscernibles* (which roughly translates as the claim that objects must differ in some intrinsic, non-relational way to be distinct) into sharp focus — which in a letter to Samuel Clarke (see Alexander [1956]) is derived from another principle basic to realist interpretations of quantum theory, namely the *principle of sufficient reason*, which according to Leibniz himself (see Leibniz [1714], sections 31, 32) says that:

> ... there can be found no fact that is true or existent, or any true proposition, without there being a sufficient reason for its being so and not otherwise, although we cannot know these reasons in most cases.

The implication is that a well-developed cache of information is a prerequisite for any infrastructure which is of significance to a Universe whose laws are based on finitary transfer of data. In choosing the representation of a particular particle (assuming some form of atomism), one must require it to reflect its information content in relation to its global context. One may choose (consistently with the presentation) to represent information about the state of a larger body, in such a way that there will be a simple relationship between this and the conflated information content of its constituent parts. Ideally, one has a predilection for presentations of the Universe which reflect that information content in what we might describe as an *honest* way, relating to the boundary conditions in such a way that a particles actual evolved information content, relative to other constituent parts of the Universe, is essentially that of its representation. This is what Laplacian determinism would have provided, under fairly reasonable assumptions. Although the relativistic trend of physical theory first suggests one looks for presentations defined in terms of *relations* on the structure (cf. Smolin [1991]), this ignores the essentially hierarchical nature of information content. This is not to say that the information content does not exist independently of its history, but it does not seem possible to retrieve it, locally notated, without it.[12] So one looks to a Universe evolving according to processes based on definite mathematical principles. This means that the role of the causal structure in generating such information content requires that any presentation respects it, so that the range of allowable presentations is restricted by the available Turing automorphisms (as determined by Slaman and Woodin's finite automorphism base). The existence of nontrivial automorphisms indicate that there is no such thing as the hoped for honest presentation. This can be interpreted not just as indicating the existence of many indistinguishable but essentially different presentations, but as referring to possible material manifestations of the Turing model for basic causal structure in that matter can be regarded as a possible framework of labels for it. Since, in accordance with the principle of sufficient cause, nature appears to allow all histories not systemically excluded by the mathematical model, the observed effect is that of physically existing (although

[12] There is of course an extensive literature relating to historical determinism, reversibility of computation, and its consequences for the particulars of modelling scientific laws (to which we will not try to add). See, for example, Berlin [1969], Gandy [1980], Fredkin and Toffoli [1982], Margolus [1984] and Toffoli [1980].

the status of such existence varies according to different interpretations) parallel alternatives.

The particular form of the causal structure at the quantum level (ignoring for the moment the basal information content) will depend on the local structure of the Turing universe. This structure is not easily accessible (see for example Lerman [ta]). But as noted in section 4, the infinitary nature of Aut($\boldsymbol{\mathcal{D}}$) translates, via known definabilities and automorphism bases, into that for Aut($\boldsymbol{\mathcal{E}}$) and Aut($\boldsymbol{\mathcal{D}}(\leq \mathbf{0}')$) (corresponding to the theoretically predicted, and empirically confirmed, multiplicity of available histories at the quantum level), while the definability of all levels of the Kleene-Post hierarchy corresponds to the structural regularities of theory. One does need to ask how much information content need be materially manifest for the theory to apply. Is it possible for nonrigidity to break down for particular local substructures of the Turing universe? If one restricts oneself to *invariant* substructures (and it is hard to see how any naturally occurring structure generated according to specific mathematical laws could be otherwise) one concludes that any nontrivial level of information content is likely to be accompanied by nonrigidity. Certainly any such structure containing $\boldsymbol{\mathcal{E}}$ qualifies, as (by Ambos-Spies [ta]) do a number of basic definable substructures of $\boldsymbol{\mathcal{E}}$. Although, at least in a formal sense, the existence or otherwise of *naturally realisable* substructures of $\boldsymbol{\mathcal{E}}$ with pathological automorphism groups is still not known, there is no evidence that the material universe does not exhibit a full richness of noncomputable computably enumerable phenomena (relative to its initial conditions). In other words, it seems that according to all reasonable projections *one can view the material universe as forming an automorphism base for the Turing universe*. Present techniques seem sufficient to eliminate the possibility of definable computably enumerable singletons other than $\mathbf{0}$ or $\mathbf{0}'$, consistent with the association of the collapse of the wave function with entanglement with the classical level. Whether all such singletons are automorphism bases is not known.

The resulting explanation of Heisenberg's uncertainty principle in terms of systemic imperfections of invariance, extends to natural phenomena theoretically based on uncertainty, such as quantum fluctuations and the matter/anti-matter dichotomy. This limits the cosmogonical role of the former, in that invariance/noninvariance can only determine the form of an existing structure or phenomenon, suggesting that quantum fluctuations depend on a failure of definability of already present empty space as a basic prerequisite. So any idea (originating with Tryon [1973]) of creation of the Universe as *purely* such a quantum event must represent a victory of style over substance.

Returning to the question of how, specifically, the fundamental forces of nature emerged, speculation closely follows the current view provided by physics. What impresses itself upon us is that in the absence of *appropriate* (in a sense to be elaborated on) information content (in the very early universe, say) there would be insufficient basis for the differentiation of more than *one* such causal agent, manifesting itself according to the $\aleph_0$ available algorithmic materialisations and Turing automorphic images available. While a more developed information content would be expected to lead to a collapse in possibilities according to the emergent invariance, perceived as physically related to an underlying breakdown in symmetry. Further crystallisation of basic forces might emerge by a similar process, until, ultimately, all ambiguities are squeezed out of the system by the weight of growing organisation of information content, and the familiar structure of nature is in place.

One also obtains an extension of Penrose's 'cosmic censorship hypothesis' — "... namely the hypothesis that ... naked singularities do not occur", Penrose [1996], p. 27 — providing an explanation of the (non)-role of *any* mathematical singularities in nature. As Smolin [1997] reminds us: "Many people who work on quantum gravity have faith that the quantum theory will rescue us from the singularities" (proved by Penrose [1965] and Hawking and Penrose [1970] to be inevitably associated, under fairly simple and broad assumptions, with sufficiently large systems behaving according to the theory of general relativity). Echoing the Hartle-Hawking [1983] 'no boundary' cosmological model (best known for ingeniously avoiding the infinite regress of cause and effect familiar from most arguments for the existence of God), one is able, without discarding causality itself, to exclude *any* naturally occurring singularity by relating the level of definition of natural laws to the projected information content in closed systems. To do this, one needs to look more closely at the likely information content in the early universe, or more exactly, very close to the big bang. Leaving aside spontaneous creation (there appears to be little one can say about that, even when it comes disguised by a boundary-free context), the scenario is either a more structured creational one (which does not seem to be what one is dealing with), or extreme disorder (but with significant systemic regularities) inherited from a feature of a previous (or at least, logically connected) phase of a more extensively conceived universe. In view of what our own universe seems capable of passing on (ignoring more speculative projections), 'feature' will be interpreted as 'collapse'. Even within this refined context there is considerable scope for speculation — for example the cosmological evolutionary proposal of Smolin [1992], summarised in [1997], p. 88:

> What we are doing is applying [the] bounce hypothesis, not to the universe as a whole, but to every black hole in it. If this is true, then we live not in a single universe, which is eternally passing through the same recurring cycle of collapse and rebirth. We live instead

in a community of "universes", each one of which is born from an explosion following the collapse of a star to a black hole.

Potentially, an analysis of information content around a 'bounce' (with its corresponding quantum cosmology) can theoretically by-pass the extreme disorder implicit in the standard view of the 'big crunch', so putting such speculations on a firmer footing. But for the avoidance of space-time singularities, one need only extrapolate from the consequences of the inevitable homogenisation and local ramdomisation of information content near the big bang, with its consequent dissolution of the invariant relations on which the fundamental forces of nature depend. What is significant is not so much *loss* of information content in the vicinity of an incipient singularity (this is hard to quantify), but the decay of Turing invariant relations (or at the physical level, of the formations of nature) on which higher-order structure must be based. This can be formalised in terms of invariance within the class of random degrees (cf. Kučera [1990]), noting for instance that the 2-random degrees are known to avoid the cone above any nonzero degree below $\mathbf{0}'$. The eventual dismantling of the motive force for a singularity cannot avoid extreme compression and homogenisation of the antecedent information content, and this is necessary for the regularity of structure on which a new phase will depend. One seems to require at least the occupancy of a single atomic jump class. The picture that then emerges is of successive phases, involving structurally similar development, but potentially more *complex* in that it may be based on cumulatively more developed boundary conditions determined by previous phases. Theoretically, the underlying reason for this increase in complexity of the ensuing universe is to be found in disproofs of homogeneity in the Turing universe (see section 4). Further, the potential *relevance* of the increase is emphasised by Shore's [1981] demonstration that there are many Turing degrees $\mathbf{a}, \mathbf{b}$ for which $\boldsymbol{\mathcal{D}}[\mathbf{a}, \mathbf{a}']$ and $\boldsymbol{\mathcal{D}}[\mathbf{b}, \mathbf{b}']$ are not elementarily equivalent (for example $\boldsymbol{\mathcal{D}}(\leq \mathbf{0}')$ is not elementarily equivalent to $\boldsymbol{\mathcal{D}}[\mathbf{0}', \mathbf{0}'']$). While any analysis of the information content passed on by a collapsing universe depends on a much better understanding of the consequences of a failure of Church's thesis in a non-discrete physical system for the level its informational entropy.[13] (See David Layzer [1990] for a discussion of some of the issues involved.)

Another reassuring consequence of the Turing model is the theoretical counterpart to the classical appearance of reality above the quantum level provided by results on definability and rigidity in $\boldsymbol{\mathcal{D}}$. The definability of the jump (and so in particular, of $\mathbf{a}'$ in $\boldsymbol{\mathcal{D}}(\geq \mathbf{a})$, where $\mathbf{a}$ can be taken to be a member of the atomic jump class determined by the initial conditions of the

[13] The popular equation of 'more entropy' with 'less information' (see for example Ferris [1997], p. 92) must of course refer to *availability* of the information (according to the classic formulation of Shannon [1948]).

Universe), with its canonical relationship to the most basic naturally generated incomputability, provides a gateway to classical structure, entered via the background subatomic information content inherited from the early universe and intensified quantum contingency. Despite the apparent nonrigidity of $\boldsymbol{\mathcal{D}}(\geq \mathbf{a}')$ in $\boldsymbol{\mathcal{D}}(\geq \mathbf{a})$, the acquired information content implicit in the phase-transition from non-invariance to invariance can be relied on to lift the consequent phenomenological structure to the (rigid in $\boldsymbol{\mathcal{D}}(\geq \mathbf{a})$) cone above $\mathbf{a}''$, thereby removing any possibility of a return to quantum ambiguity. (The phase-transition depends on the fact that any increment in local information content is *relative* to the context, and is expressed via a presentation which is *globally* modified by individual changes.) The apparent nonexistence of definable computably enumerable singletons other than $\mathbf{0}$ and $\mathbf{0}'$ is consistent with this picture.

How does the explanation of quantum nonlocality in terms of variable levels of underlying Turing invariance compare with existing, physically specific, 'realistic' interpretations? Of these the front-runner currently is the theory of 'decoherence' particularly associated with Gell-Mann and Hartle [1990], Griffiths [1984] and Omnès [1994], with its pleasing consequences for quantum cosmology. This seeks to augment the 'many-worlds' interpretation of Everett [1957] with an explanation, in terms of the logical structure of the Universe, of the origins of our 'quasi-classical domain' and of the 'branching' accompanying measurement which creates the inaccessibility to observation of the parallel worlds (re-termed *decoherent histories*) at the classical level.

According to the theory (for which we mainly refer to the version of Gell-Mann and Hartle), our observations can be made consistent with the 'many worlds' scenario by appropriately locating us within a particular structure of decohering alternatives, but are an inadequate guide to the specific *form* of the decoherence, except in so far as we can imperfectly project from the quantum level. This is described by Gell-Mann [1994] in terms of *coarse graining* (p. 144):

> For histories of the universe in quantum mechanics, coarse graining typically means following only certain things at certain times and only to a certain level of detail. A coarse-grained history may be regarded as a class of alternative fine-grained histories, all of which agree on a particular account of what is followed, but vary over all possible behaviors of what is not followed, what is summed over.

Then (Gell-Mann and Hartle [1990], p. 445):

> As observers of the universe, we deal with coarse grainings that are appropriate to our limited sensory perceptions, extended by instruments, communication, and records, but in the end characterized by a great deal of ignorance. Yet we have the impression that the

> universe exhibits a finer-grained set of decohering histories, independent of us, defining a sort of "classical domain", governed largely by classical laws, to which our senses are adapted to dealing with only a small part of it. No such coarse graining is determined by pure quantum theory alone. Rather, like decoherence, the existence of a quasiclassical domain in the universe must be a consequence of its initial condition and the Hamiltonian describing its evolution.

A quasiclassical domain "should be a set of alternative decohering histories, maximally refined consistent with decoherence, with its individual histories exhibiting as much as possible patterns of classical correlation in time". Disturbance by quantum events ensures that "[t]here are no classical domains, only quasiclassical ones". This leads on to a general consideration of maximal sets of alternative decohering histories (that is "those for which there are no finer-grained sets that are decoherent"), but then makes the special (or otherwise) status of the quasiclassical domain of which we seem to be a part problematic (see for example Dowker and Kent [1996]). Efforts to find a way out of the resulting confusion have led to an impression of desperation. The likelihood of decoherent coarse grained histories makes explanation of our quasiclassical domain centre (a variation on the anthropic principle) on the role of the observer (abstracted by Gell-Mann and Hartle as an IGUS, or 'information gathering and utilising system'), and leads to the 'lack of economy' complained of by Penrose [1989], p. 382, in the many worlds interpretation. Added to this, the theory does not appear to provide a full solution to the measurement problem.

However, if one looks more closely at accounts of the mechanism by which decoherence breaks down, one is struck by a remarkable convergence between these and local descriptions of the imposition of Turing invariance. Histories decohere in the absence of 'entanglement', expressed in terms of the interference term between histories. According to the Turing model of the causal structure of the universe, different automorphic images become possible in the absence of definability of substructure relative to what is rigid in $\boldsymbol{\mathcal{D}}$. For the former, the emergence of a coarse grained alternative is related to cumulative entanglements affecting its corresponding fine grained components. There is no immediate reason for the cumulative breakdown of decoherence to define a unique convergent reality. This is easily recognised as an 'on the ground' description of the development, or otherwise, of invariance within a structure. Viewed at a phenomenological level, the full picture of decoherence between course grainings is complex. But empirically one is led to expect correlations — we are dealing with the *structure of nature* with all its observed regularities of behaviour. So one expects to abstract from the disparate phenomena a common mathematical framework, namely the causal structure captured by the Turing model. This holds out the possible renewal of the project based on known results of classical computability theory. From a recognition of the nature of the underlying mathematics of decoherence, one arrives at a precise explanation of why the coarse grained alternative we see has its special status, so answering what Omnès [1994], p. 504, calls "*the* problem, which is the existence of facts". What is missing from the decoherence scenario is now provided by the known features of the Turing universe, and the evidence for a quantum/classical dichotomy around the $\mathbf{0}'$, $\mathbf{0}''$ level, the rigidity in $\boldsymbol{\mathcal{D}}$ of the causal structure above $\mathbf{0}''$ giving the required unique, familiar quasi-classical domain. One can even anticipate that the nature of the probabilities intrinsic to the measurement problem can be explained via a better knowledge of the automorphism group of the Turing universe.

There is also an apparent consistency with the other main contender, the reconstructed 'hidden-variables' theory of David Bohm. Developed in a sequence of papers and books, from Bohm [1952] to Bohm and Hiley [1993], the details of the theory — the wave function ψ taken to represent a real field (avoiding the apparent ambiguity of its reality status in the Gell-Mann and Hartle proposal) whose fluctuations originate at a classical sub-quantum-mechanical level involving hidden variables, with a non-local transfer of 'active information' (augmenting the 'pilot wave' idea of de Broglie [1927]) responsible for a sort of cosmological version of the 'global village' — are not transparently related to any notion of mathematical non-locality. But within the philosophical constraints inherent in the approach, the picture developed (particularly of an *implicate* order, Bohm [1980]), becomes an informative metaphor for the more theoretically robust one. There are other proposals using hidden variables to restore exact determinism, for example that of Ghirardi, Rimini and Weber [1986], this time introducing at the quantum level a new and empirically undetected, randomly occurring, localising effect, which at macroscopic levels has the potential to fix the position of a solid object via state entanglement. So it would be too simplistic to dismiss all such theories as "a return to antiquated ideas", but the judgement (Omnés [1994], p. 401) that it

> ... seems difficult to accept that the deep and unexpected mathematical properties one found for the [operators] expressing classical properties [of] the decoherence effect do not contain a large part of truth. They rely upon so few assumptions and recover so many well-known features of reality that had remained unexplained before them that they have the kind of beauty Dirac coined as the mark of truth ... ,

is convincing. And the underpinning of the logical aspects within the Turing context of can only strengthen that impression.

Emergence and Entropy

Another area in which reductionist explanations have become augmented in recent years by those based on more global considerations is the biological sciences, with vigorous debate centering on such thorny problems as characterising the evolutionary process and the origins of life. A key concept has become that of 'emergence' in relation to the anti-entropic forces represented by life and its development, and more generally (see for example Holland [1998]). The problem is the recurring one of the breakdown of reductionism in the sciences, despite what appears to be a firm causal link between phenomena at different levels of knowledge. Recent analyses of the process whereby higher-order structure emerges via systemmic expressions of relatively well-understood local laws in biology are presaged by earlier ideas such as those of Lynn Margolis [1981] on 'mutualism', which as Steven Rose [1997] remarks, were once considered heretical, but (p. 229) have "now become the conventional wisdom of the textbooks". The key connection which has been made recently is between the emergence of order, in say the origins of life or in natural selection, and the mathematical phenomenon of attractors. See, for instance, Stuart Kauffman [1995], p. 26:

> The wonderful possibility, to be held as a working hypothesis, bold but fragile, is that on many fronts, life evolves toward a regime that is poised between order and chaos. The evocative phrase that points to this working hypothesis is this: life exists on the edge of chaos. Borrowing a metaphor from physics, life may exist near a kind of phase transition. ... Networks in the regime near the edge of chaos — this compromise between order and surprise — appear best able to coordinate complex activities and best able to evolve as well. It is a very attractive hypothesis that natural selection achieves genetic regulatory networks that lie near the edge of chaos.

Or again, Rose [1997], p. 166:

> The cellular web ... has a degree of flexibility which permits it to reorganize itself in response to injury or damage. Self-organization and self-repair are its essential autopoietic properties. These properties of stability and self-organization, which Stuart Kauffman has described a 'order for free', are the key to appreciating the fundamental irreducibility of living cells. Their metabolic organization is not merely the sum of their parts, and cannot be predicted simply by summing every enzyme reaction and substrate concentration that we can measure. For us to understand them, we have to consider the functioning of the entire ensemble.

But graphic and informative as these descriptions of the emergence of new relations are, it is the lack of a secure theoretical basis which makes such speculations "fragile". However, one is struck by the close parallels between the descriptions of self-organisation in nature, and the emergence of 'order for free', and ones conceptualisation of the process of emergence of Turing invariance. Once again, it is possible to make detailed correspondences, and there *is* an explanation, in the context of the Turing model.

Epistemological relativism

Corresponding to these consequences for the structure of matter, there are analogous ones for that of knowledge of the Universe. This is strikingly adumbrated (in ways not envisaged by the author of Turing's biography) in the following quote from Andrew Hodges [1983] (in relation to Hofstadter's [1979] views on "the significance of Gödel's incompleteness and Turing undecidability for the concept of Mind"), p. 540:

> Far more significant, in my view, is the limitation of human intelligence by virtue of its social embodiment – and this is a problem relegated to a marginal place in Hofstadter's work as in so many other accounts, though I have placed it at the centre of my own.[14] The study of Alan Turing's life does not show us whether human intelligence is limited, or not limited, by Gödelian paradoxes. It does show intelligence thwarted and destroyed by its environment.

As noted previously, incomputability in the objects of scientific investigation is relevant to the process of establishing truth in science, not just in that computability or otherwise determines the manner of discovery, but in that the process of discovery itself is an analysable physical phenomenon. To quote Bohm and Hiley [1993], p. 326:

> ... the view that our theories constitute appearances does not deny the independent reality of the universe as a whole. Rather it implies that even the appearances are part of this overall reality and make a contribution to it.

However Newton [1997] has a subtle variation on the theme that nature is deterministic and that it is just observation which is ill-defined, suggesting that it is the *language which is used to describe* observation which is inadequate. But this is analogous to saying invariance and Turing definability give rise to relevant distinctions — which they do not, by Slaman and Woodin (see the previous section). Anyway, instead of a classical reality sitting uncomfortably on top of a level of ambiguously defined micro-phenomena, one is now confronted with epistemological relativism, a failure of reductionism and a concomitant disunity of scientific theory. Similarly to before the emergent levels of definition enter again into the structural development via processes of systemic observation and awareness. What is familiar from

[14] Although oracle Turing machines get little more than a passing mention.

nature is the way one is able to establish a fairly comprehensive computable framework of relationships at a basic level, but whose explanatory power dissipates as the lines of causal connections extend — descriptions of the emergent chaos being effectively incomputable, even if the exact nature of the incomputability is not precisely characterised. Sitting on top of such turbulence, new regularities emerge involving new computable parameters and dynamic relations, giving rise to a more local theory, which may be less theoretically comprehensive, but maybe more important from an everyday perspective. The parallel with the structure of the Turing model is striking. Here again one sees (suggested by recent work) an onset of noninvariance, within which new structures become defined, but relative to which there is a recurrence of the lower-level formlessness.

The 'nonrigidity' of knowledge is already implicit, in a limiting form, in the so-called Duhem-Quine thesis. Also, Popper's [1959] qualification of the 'scientific method' is surely right in essentials, dealing as it does with the practical aspects of the consequently more subtle relationship between observation, hypothesis and theory, even if one persists in detecting scientific induction in the way confidence in a given theory is consolidated by a mixture confirmatory evidence (such as realised predictions) and *absence* of negatory evidence (and Popperian falsification — see Lakatos [1970] — being conditional on a competitive induction associated with *different* theories). Many relations on the Turing universe may be definable within the global context, but may not be computable, inductive extensions of observations. Others may be even less accessible to the scientific method. Related to this is the fact that a structure of knowledge involving unpredictability (already implicit in Gödel [1931], [1934], and made more relevant via the Turing model) means that one may have no way of judging whether a particular statement has empirical consequences or not, a fact commonly ignored — for instance, relevant to the above explanation for nonlocality in quantum mechanics, one can quote van Fraassen [1980], p. 95, on 'correlations in the behaviour of particles which have interacted in the past, but are now physically separated:

> These [hidden variable theories] do not predict exactly the same correlations — this is what makes these theories interesting to physics. So far, experiments appear to support quantum theories against those rivals. But the one response which is conspicuous by its absence is that an explanation of the correlations *must be found* which fits in exactly with quantum theory and does not affect its empirical content at all. Such metaphysical extensions of the theory (if indeed possible) would be philosophical playthings only. There are only two camps to the debate as far as physics is concerned: either this non-locality makes quantum theory pre-eminently suited to the

> representation of the world (and we need to re-school our imaginations), or else quantum theory must be replaced by an empirically significant *rival*.

One also notes that observations may relate to definability in different ways, perhaps being repeatable, so inductively extendible, or having some individual significance in relation to such inductively (in the wider sense) derived frameworks. They may lead to 'theory-laden' facts, which again have an impact on the inductively based framework of theory. This means that although *reproducibility* of experiments depends on the underlying algorithmic content of the basic observations, it may be that the imaging of the phenomenon in question via the results of observation may exploit the full subtlety of the relationship between definability and observational data, with Husserl's [1913] notion of "eidetic intuition" being particularly relevant to epistemological manifestations of this relationship.

Particularly relevant to the extension of the model to the epistemological domain is Quine's [1953] cogently argued dismissal of any "fundamental cleavage between truths which are *analytic*, or grounded in meanings independently of matters of fact, and truths which are *synthetic*, or grounded in fact". For Quine (p. 44–45):

> Physical objects are conceptually imported ... as irreducible posits comparable, epistemologically, to he gods of Homer. ... Moreover, the abstract entities which are the substance of mathematics ... are another posit in the same spirit. Epistemologically these are myths on the same footing with physical objects and gods, neither better nor worse except for differences *in the degree to which they expedite our dealings with sense experiences*. (added italics)

While the exact mythological (or otherwise) status of the observations whereby objects are 'imported' is tangential to the discussion (although, for instance, one tends to be sceptical about mathematics as a cultural construct), it is clear that there is a conceptual convergence between this and the present picture, in that different categories of knowledge are associated with levels of irreducibility (independent of any particular atomist assumptions) based on qualitatively similar formation processes. As a result one can even speculate on the relevance of nonrigidity to the epistemological status of such well-known undecided statements of mathematics as the continuum hypothesis (CH). Even Gödel's [1964] approach to CH, ostensibly Platonist, seemed to admit a degree of relativism in that it involved contemplation of alternative models for ZFC, in as much as his description of the search for intuitively acceptable new axioms capable of deciding CH suggests an underlying consideration of new models in which such statements are true and which seem to realistically extend what is already believed to be true. It is a trivial observation that whatever the absolute status of mathematics, models are *theoretically* integral, so that our universe is symbiotically related to mathematical truth, in so far as it exists, via its modelling role. There arises the possibility that widely dispersed but apparently analogous phenomena may in fact be subject to a fundamental unifying principle.

So a first consequence of such nonrigidity is a partial confirmation of current relativistic views of human knowledge, but in a form which limits the extent of this relativism, and undermines its wilder inductive extensions. There is a difference between the conventionalism of Poincaré, for instance, and the more recent anti-realist views of scientific knowledge (see for example van Fraassen [1980]). There is a level of consistency of the former with Einstein's view of the theoretical results of science being in some sense *determined* (even if one does have underdetermination of theory by data to contend with), and anyway, the seeking out of conventions is not necessarily arbitrary and unrooted in an objective reality (cf. Newton [1997], p. 15). Again, one can differentiate between the pragmatism of say Peirce (the mathematics reflecting the subtlety of the argument), and the more radical, apparently anti-epistemological, versions of Schiller [1907] or Rorty [1979], [1982]. But while postmodernism can be termed (Gross and Levitt [1994], p. 87) an "instrument of revenge" against the tendency, derived from the earlier dominance of the logical positivism (the classical statement of which being A. J. Ayer [1936]), to constrain the humanities to the methodology of the empirical sciences, one cannot entirely dismiss the intuitions which underpin such analyses. Beyond the routine cataloguing of the dislocations of reality and discourse, and the tentative attempts (see for example Katherine Hayles [1990]) to enlist Gödel's theorem and chaos to explain the mismatch, there is potentially a solid theoretical explanation in terms of Turing nonrigidity and corresponding failures of definability in the real world. Implicit in the failure of bi interpretability is a fractured relationship between language and observation — there seems to be no alternative characterisation of the Turing definable relations available, while even at the local level (see Cooper [1997] and Nies, Shore and Slaman [1996]) definability in arithmetic and invariance are not comparable notions. On the other hand, it is difficult to envisage any consensus about what is certain, and what is culturally created (a contemporary battlefield as described by Gross and Levitt [1994]) without some sort of theoretical limitations on the inductive projections indulged in on both sides of the argument.

Secondly, one finds via the Turing model an echo of current doubts about the validity of reductionism in science, related to, for instance, the instrumentalism of Nancy Cartwright [1983] (there is no guarantee that received definitions of fundamental laws correspond exactly to their invariant status) or, more relevantly, the "epistemological pluralism" of Dupré [1993]. The Turing analysis does seem to give some credence to the disunity of science from a realist perspective. Reductionism is concisely described by Oppenheim and Putnam [1958], who

> ... propose the following levels: elementary particles, atoms, molecules, living cells, multicellular organisms, and social groups. ... Reduction consists in deriving the laws at each higher (reduced) level from the laws governing the objects at the next-lower (reducing) level. Such reduction, in addition to the knowledge of the laws at both the reducing and reduced levels, will also require so-called bridge principles (or bridge laws) identifying the kinds of objects at the reduced level with particular structures of the objects at the reducing level. Given the transitivity of such deductive derivation, the end point of this program will reveal the whole of science to have been derived from nothing but the laws of the lowest level and the bridge principles.

The problem is that a closer look at our *experience* of science does not entirely confirm this picture. Dupré (pp. 88–89), for instance, acknowledges reductionism's "great successes", in particular noting that: "Many of the greatest achievements of science depend essentially on insight into the structure of objects", but goes on to point out that "it does not follow that *all* scientific enquiry has to do with elucidation of structures". Consequently science, and human knowledge generally, often appears compartmentalised into diverse disciplines each with its own specific assumptions and technical basis. As David Bohm [1957], p. 133, observes:

> Any given set of quantities and properties of matter and categories of laws that are expressed in terms of these quantities and properties is in general applicable only within limited contexts, over limited ranges of conditions and to limited degrees of approximation, these limits being subject to better and better determination with the aid of further scientific research. Indeed, both the very character of the empirical data and the results of a more detailed logical analysis show that ... the possibility is always open that there may exist an unlimited variety of additional properties, qualities, entities, systems, levels, etc., to which apply correspondingly new kinds of laws of nature.

The resulting distinction between more basic and relatively local theories is very much in accord with the theoretically originating demarcation lines growing out of areas of chaos, with newly defined parameters emerging across boundaries, to be incorporated in new fields of study.

So it is possible to view the hierarchical nature of scientific theory as itself a response to the incomputabilities generated at successive levels, along with the surprising but theoretically explicable reattainment of algorithmic content (provided by a less fundamental, possibly, 'local' theory) at each level (comparable to the emergence of 'strange attractors' in chaos theory). The accumulation of *information* content at a particular level introduces new definabilities which become the raw material for the *algorithmic* content at the succeeding level. This description is closely echoed in Stephen Weinberg's [1992] quote (p. 61) from a talk by James Gleick:

> ... there are fundamental laws about complex systems, but they are new kinds of laws. They are laws of structure and organisation and scale, and they simply vanish when you focus on the individual constituents of a complex system — just as the psychology of a lynch mob vanishes when you interview individual participants.

So for example, the structure of material things and of our accompanying knowledge of them diminishes the *overall* importance of the search for a 'grand unified theory' (GUT) for the subatomic level, and exposes the inappropriateness of the term 'theory of everything' (TOE). There is also a partial confirmation of Dupré's argument for selective relativism (whereby one avoids the inductive extension of observed scientific inadequacy in certain contexts to an attack on the entire scientific project), in that there may well be areas in which the nature of the basic relations is associated with a very rapid accretion of complexity, so that the well-understood atomic dynamics tell one very little. Of course, one would argue that the information content (or chaos) of such contexts *does* define new parameters, which may become part of a higher level of analysis to which empiricism and logic *are* relevant, and that the definability of the jump does provide a link between the regularities at the different levels. But notice that the chaotic barrier to reduction (if the Turing model is accepted) is not just a practical one. The parameters at the succeeding level are basic to the resulting theory, but are computationally, in a fundamental sense, divorced from the lower level. (We are dealing with emergent Turing definabilities.)

Despite all the above, one notes that there is of course a reduction process at work — there is no breakdown in determinism — but it involves systemic, or globally originating, processes, so is imperfectly available to us epistemologically. It is also worth noting that natural laws may be related to definability in a less than precise way, in that the phenomenon is only approximated by the defined relationship (e.g. the laws of quantum mechanics enable bizarre exceptions, of vanishingly low probability, to higher level laws).

There is also (although this will get less attention than its real-world importance merits) an epistemological level corresponding to that of the material quasi-classical domain. Although contained within a relativistic context there is a potential convergence of agreed knowledge of the universe — a comprehensive world view is not just an aid to erroneous certainty.

However, on a technical note, it may be that one should distinguish mathematically between questions of epistemological *structure* and *practice*. Whereas parallel realities in nature, or differing but coexisting world views governing certain areas of human experience, may be acceptable, scientific activity depends on a more convergent framework. The computable basis for processes relative to historically emergent *partial* information can only be modelled successfully in the Turing context by allowing indeterministic oracle machines, and this is usually formulated (see Rogers [1967]) in terms of *enumeration reducibility*. A is said to be *enumeration reducible to* B ($A \leq_e B$) if and only if there is an algorithmic description of how to enumerate the information (i.e. the members) in A (in some order) given *any* enumeration of that in B. The resulting structure (captured in the *enumeration degrees* $\boldsymbol{\mathcal{D}}_e$) forms a natural extension of the Turing model (under trivial notational adjustments). See Sorbi [1997] for a recent survey of results and techniques, or Cooper [1990] for the historical background. Recent work suggests that not only is the picture previously outlined, including the model of the familiar quasi-classical domain, to be found naturally embedded in the extended one, but there is a consistency of theory in that nonrigidity extends to $\boldsymbol{\mathcal{D}}_e$ (although many other local and global characteristics may be very different). Moreover, there is a strong structural continuity in that (Sorbi [ta]) $\boldsymbol{\mathcal{D}}$ forms an automorphism base for the enumeration degrees.

It would be hard to pass on without mentioning the relationship between free will and determinacy, and the traditional philosophical debates around the existence or otherwise of *origination* (the creation of new causal chains by free human choices), and the compatibility of determinism and moral freedom. Of course the renewed mathematical model of determinism rules out origination, including arguments[15] for it based on conjectural physical links between brain processes and quantum state vector reduction (cf. Penrose [1989], [1994], Stapp [1993]). Given determinism, other discussions are little changed in essentials, although determinism conceived less mechanistically has the effect of swinging the argument away from the incompatibilists (in regard to freedom and determinism) and the hard determinism of William James and his followers, according to which moral responsibility is illusory. Determinism can be viewed as assisting a complex and creative process of developing information content in which we and our judgements (perceived as free choices) are full participants. What is different is that, unlike in a clockwork universe, our participation does substantively change things. Moreover, the human mind involved in forming such judgements can be seen as involving a huge accretion of causal structure, with a subjective experience of free will based on the emergent form

[15] Anticipated in philosophical writings of Arthur Eddington [1928].

rather than on those of complexity and incipient disorder. The parallel between the way definability derives from complexity of information content, and our experience of mental creativity, suggests a process whereby great complexity and confusion, by a noncomputable causal connection, gives rise to unforseen structure. A concomitant of the functional complexity is the intimate relationship with a physical Universe of which our minds are microcosmic reflections (an aspect of mind central to the arguments of Dreyfus [1979] against the claims of classical AI). So the incidence of revolutionary new ideas depend not so much on more efficient (computable) mental processes but on their functional relationship to extreme confusion of mental content, and the level of systemic intolerance of formlessness. The lesson may be — avoiding the vagueness of, say, Bergson's [1907] *élan vital* underpinning his anti-scientism — that many evolutionary processes not only entail jumps in complexity of information content, but are essentially *creative*, in that the very same kind of collapse of alternative realities familiar from quantum theory are implicit in evolutionary selection. And it is this, even without direct physical links between quantum phenomena and brain processes, which may lead to recognition of the extraordinary aptness of the intuitions of Penrose and Stapp.

The existence of free will (as in the 'Free Will Defence') is intrinsic to particular responses to objections to theism (about which one might expect mathematics to have even less to say). But the basis of formal creation (viewed as a structure of invariance) in an informational contexture of hierarchical complexity can be interpreted theodistically. The aim of creation and development of a formal ideal provides a teleological context in which one can approach the problem of evil (prominent in both Western and Eastern philosophical discussions) in terms of necessity of complexity of structure (see, for example, Swinburne [1979], chs. 9–11, or John Hick [1978]). The teleological assumption is fairly minimal, given that, ultimately, the empty teleological context is not an option. In contrast, what makes the so-called 'anthropic principle' (the need for which the Turing model very successfully releases one from) so suspect is the strength and essential arbitrariness of the teleological premise.

6. Turing nonrigidity as paradigm shift

Not only does the Turing model, relating definability (in terms of inductively expanded observational data) and empirically based theories, provide a sympathetic context for Kuhn's description [1962] of how scientific knowledge (not a term Kuhn himself would have chosen) emerges, but the revision of 'the scientific paradigm' implicit in the model's acceptance can itself be seen as a Kuhnian paradigm shift.

Following on from the philosophical foundations laid down by Descartes, Galilei and others, it was the discoveries of Newton which established the scientific outlook which so dominated western society for the next two and a half centuries and still molds the thinking of countless numbers of working scientists. Having achieved the status of a paradigm (and so, according to Kuhn, 'declared invalid only if an alternate candidate is available to take its place'), it became more basic than any of the previously challenged examples described by Kuhn, its inadequacy reflected in many longstanding philosophical controversies, and eventually becoming a key ingredient in the current crisis in the way science is popularly regarded — much to the consternation and bemusement of the professional practitioners of science (see for instance Gross and Levitt [1994] or Newton [1997]). Constituent assumptions included the belief that observation (potentially) gives a *clear image* of a real world, and that a precisely defined reality is a normal state of affairs. In regard to the former, Kuhn (p. 121, third edition) talks of the need for, and lack of, "a viable alternative to the traditional epistemological paradigm ... initiated by Descartes and developed at the same time as Newtonian dynamics", observing how:

> Today research in parts of philosophy, psychology, linguistics, and even art history, all converge to suggest that the traditional paradigm is somehow askew.

As we have seen, the latter assumption presents us with an intractable anomaly involving our everyday experiences of the classical reality and the very successful descriptions of the quantum world. The Turing model both gives a framework within which to describe what nature achieves (via noninvariance) and one to structure (via definability) the process by which we try to gain knowledge of that achievement, and (Kuhn, p. 121) "... though the world does not change with a change of paradigm, the scientist afterward works in a different world."

Within computability theory there is already in progress what amounts to a minor paradigm shift. Nies, Shore and Slaman [1996] describe the development of coding techniques (for $\mathcal{E}$) as a vehicle for exploiting complexity of structure, and how "the ultimate expression of such coding procedures" became "embodied in the [bi interpretability] conjecture that crystallized the new paradigm of complexity as a route to characterization". In relation to this one should quote Kuhn (pp. 180–181, third edition):

> A revolution is for me a special sort of change involving a reconstruction of group commitments. But it need not be a large change, nor need it seem revolutionary to those outside a single community, consisting perhaps of fewer than twenty-five people.

This limited change associated with the discovery of Turing automorphisms is partly based on technical factors with their own associated Kuhnian incommensurabilities, but more significantly it provides further evidence of the extent to which the scientific paradigm has subtly infiltrated our culture. While the nature of the past twenty years' technical activity, the sweeping successes achieved within the bi interpretability programme of research, and the huge investment of time and energy, almost all on the side of definability, provide the infrastructure of the more restricted paradigm, the basic perceptions of the material world implicit in the larger one must inevitably colour one's expectations of its abstract counterpart, even if the link is not experienced in a particularly direct way. How typical (according to the Kuhnian scenario) of the establishment of a paradigm and its period of crisis that the research activity in the intervening period should have been so notable for its unconcern with foundational, philosophical and (even) applicational issues, and for its convergent technical direction.

Early local degree theory was motivated by the aim of characterising $\boldsymbol{\mathcal{D}}$ and its more important substructures. The revelation of of what seemed increasingly pathological structure gave an air of hopelessness to this project, until renewed by Simpson [1976]. The resulting bi interpretability 'paradigm' supported an attempt, increasingly desperate, to relate this pathology to appropriate, more familiar, complexity of (mathematical) structure, and in the process to reduce the characterisation of the results of local degree theory to a small range of such results.[16] But the existence of nontrivial Turing automorphisms — not anticipated conceptually or technically — makes the nature of Turing definability *problematic*; one becomes even more aware of local degree theory as being *about* definability, rather than a reductionist tool in the struggle with pathology; while at the same time, the much wider ramifications connect the renewed project with very basic considerations of internal self-determination in the real world. The suggestion that "it is time for a new paradigm" (Nies, Shore and Slaman [1996]) becomes valid in a wider context.

Returning to the larger picture, one notes that radical as is the conceptual revision implicit in the logical explanation of the dichotomy between quantum and classical reality and of the associated appearance of breakdown in determinism, it is the parallel consequences for how we *know* the world (or otherwise) which entail a more basic renewal of thinking. Scientists (and mathematicians) have continued to be trapped in a clockwork universe, largely due to the dependence on the scientific method and the

[16] Of course, the commitment of researchers in the field to aspects of this programme varied. G. E. Sacks' well-known reaction to degree theoretic pathology was to focus on the importance of *techniques*, based on the intuition that there was something fundamental and relevant about the area in a wider sense. P. Odifreddi (private communication) pointed out (referring to the example of the many-one degrees $\boldsymbol{\mathcal{D}}_m$, see Odifreddi [ta]) that even if the Turing structure *were* to be completely characterised — which seemed impossible — there would be no obvious consequences.

lack of a theoretical framework which is both consistent with the fruits of empiricism and able to extend to admit less well defined sensory evidence of limited reproducibility. Without such a framework, the sort of confusion surrounding the relationship of local to global, particularly in the arts and humanities, appears reminiscent of the alchemical confusion of the sciences before Newton. While the work of post-modernists, (post-)structuralists, and epistemological relativists of various shades, bears more than a passing resemblance to to the process of haphazard intuition and Aristotelian inference which then passed for 'natural philosophy'.

To quote Kuhn again, from the closing pages of his book, this time on the extent of epistemological relativism and the validity of a scientific enterprise based on the *nature of the world* (p. 173):

> What must nature, including man, be like in order that science be possible at all? Why should scientific communities be able to reach a firm consensus unattainable in other fields? Why should consensus endure across one paradigm change after another? And why should paradigm change invariably produce an instrument more perfect in any sense than those known before? ... It is not only the scientific community that must be special. The world of which that community is a part must also possess quite special characteristics, and we are no closer than we were at the start to knowing what these must be. That problem — What must the world be like in order that man may know it? — was not, however, created by this essay. On the contrary, it is as old as science itself, and it remains unanswered.

One answer, or at least the basis for an answer, has already been outlined above. Whether this can provide the basis for an update to the scientific paradigm, only time can tell. But there can be no doubt that the need for a reorganisation of the conceptual basis of our epistemological universe is no less pressing than that accompanying earlier crises in relation to the physical universe. Being realistic about such change, even when overdue, one can quote Darwin's [1872], p. 389 (in 1996 reprinted edn.), well-known comments from the concluding section of "The Origin of Species":

> Although I am fully convinced of the truth of the views given in this volume ..., I by no means expect to convince experienced naturalists whose minds are stocked with a multitude of facts all viewed, during a long course of years, from a point of view directly opposite to mine. ... [B]ut I look with confidence to the future, — to young and rising naturalists, who will be able to view both sides of the question with impartiality.

Or on a more pessimistic note, Max Planck [1949], pp. 33–34:

> ... a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its oppo-

nents eventually die, and a new generation grows up that is familiar with it.

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