25. A main lemma

> For each requirement $\mathcal{R}_\Theta$, if $\mathcal{R}_\Theta$ is finitely injured then it is finitely injuring.

• First notice — above each $\mathcal{R}_\Theta$ $\mathcal{M}_\sigma$ imposes at most one restraint on some $\sigma^{*_t} = \tau$, which is never cancelled; and $\mathcal{K}_{\sigma'}$ defines just one level of protected arguments of $*$.

• Assume each $\mathcal{R}_\gamma$ above $\mathcal{R}_\Theta$ is at most finitely injuring, and let $\widetilde{s}$ be s.t. all restraints and protected arguments originate prior to stage $\widetilde{s}$, and no $\mathcal{R}_\gamma$ above $\mathcal{R}_\Theta$ requires attention at a stage $s+1 > \widetilde{s}$.

•• <u>Then</u> — $\mathcal{R}_\Theta$ *can only require attention finitely often in relation to a given potential witness* x:

• Such an x can be assumed to be appointed through (1) at some stage $s_x + 1 > \widetilde{s}$ — and if cancelled x can never be reappointed.

- $\mathcal{R}_\Theta$ cannot require attention via (3), part I, in relation to x more than once —

- Since otherwise at a stage $s+1 > \widetilde{s}$ we would implement an f_{s+1}, g_{s+1}-matching ψ with γ-state $(\,\cdot\,)$ (only $\mathcal{P}_\beta$ being relevant), and restrain $f_{t+1}, g_{t+1} \supseteq f_{s+1}, g_{s+1} \supseteq f_{s_i}, g \restriction \vartheta(x)[s_j]$, respy, at all later stages, with $f_{s_i}(x) \neq \Theta^{g_{s_j}}(x)$ —

- So precluding $\mathcal{R}_\Theta$ from requiring attention at a subsequent stage $t+1$, other than through (4) or (5).

- And any such attention can only have the effect of consolidating a $*_{t+1}$-matching, and of preserving this situation.

- Assume $\mathcal{R}_\Theta$ to require attention at infinitely many stages through (3), subcase II, and that $g \restriction \vartheta(x)[s_i]$, $i \geq 1$, is a list of all strings realised in relation to x, where, by choice of x, no such realised string is ever cancelled.

- To get a contradiction, one needs to find a stage $s+1$ at which $\mathcal{R}_\Theta$ satisfies the conditions for attention through (3), part I in relation to x:

- That is — a stage $s+1$ at which there exist distinct $i, j \geq 1$, such that the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$ at stage $s+1$.

- — and to do this, one needs to verify the following facts:

(A) There are only finitely many isomorphism types possible for the stratified inner $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured ranks of each f_{s_j} and g_{s_j} (respy) above $\mathcal{R}_\Theta$, from which it follows that there is a limiting value for the isomorphism types applying to the inner $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured ranks of f_t, g_t, respy, above $\mathcal{R}_\Theta$ at stages $t+1$.

(B) At all stages $s+1 \geq s_j+1$ at which $\mathcal{R}_\Theta$ requires attention via (3), the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_j} matches the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$, each $j \geq 1$.

(C) For all $i, j \geq 1$ — if at stage $s{+}1 > s_x{+}1$ the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$ at stage $s+1$, then the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$ at stage $s+1$.

There are only finitely many isomorphism types possible for the stratified inner $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured ranks of each f_{s_j} and g_{s_j} (respy) above $\mathcal{R}_\Theta$ ...

- <u>Using key</u>:

◇ denotes f_s

□ for a string $\subseteq \sigma$

○ for other strings

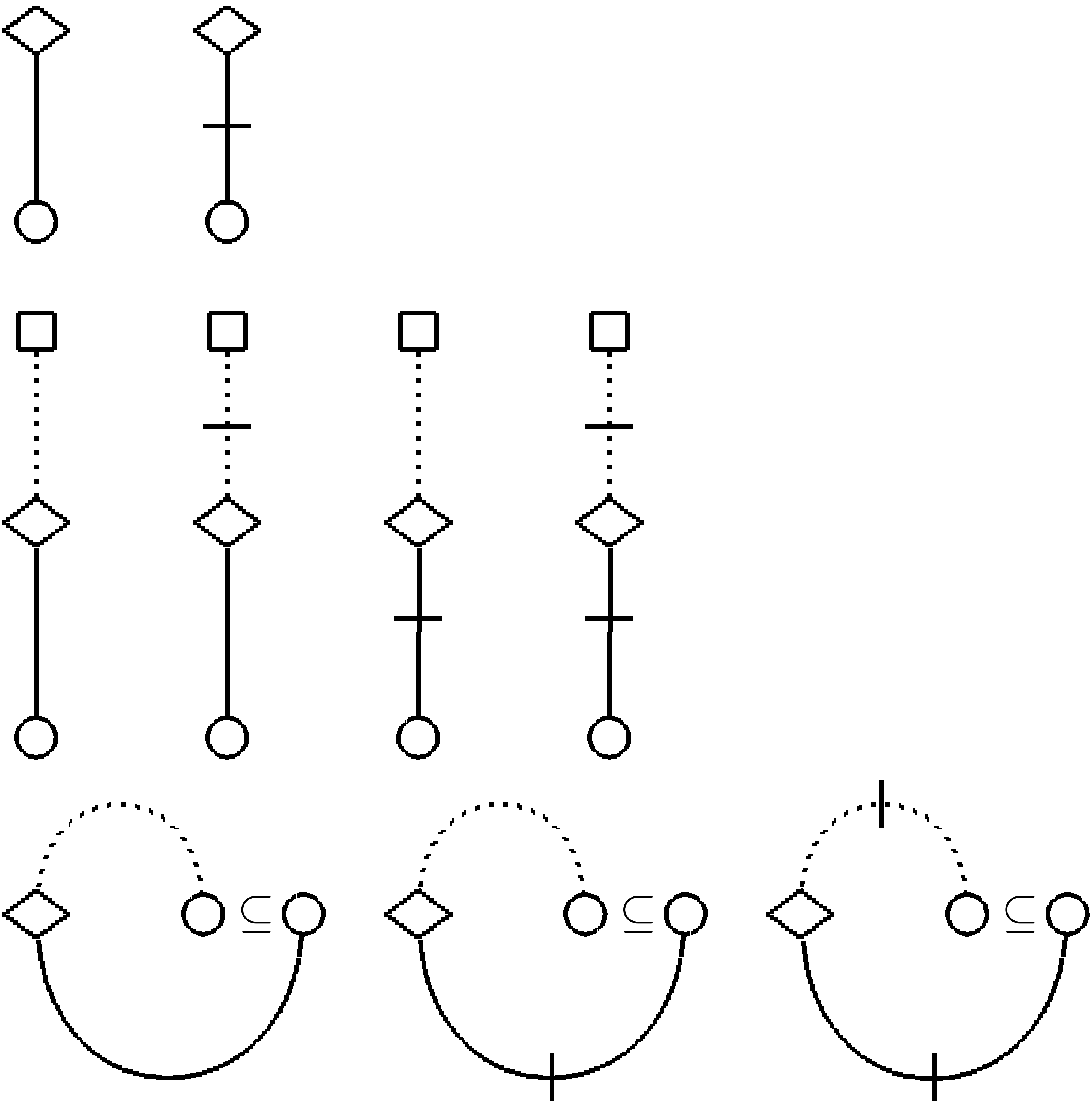
— for a generic protected argument of $*_s$

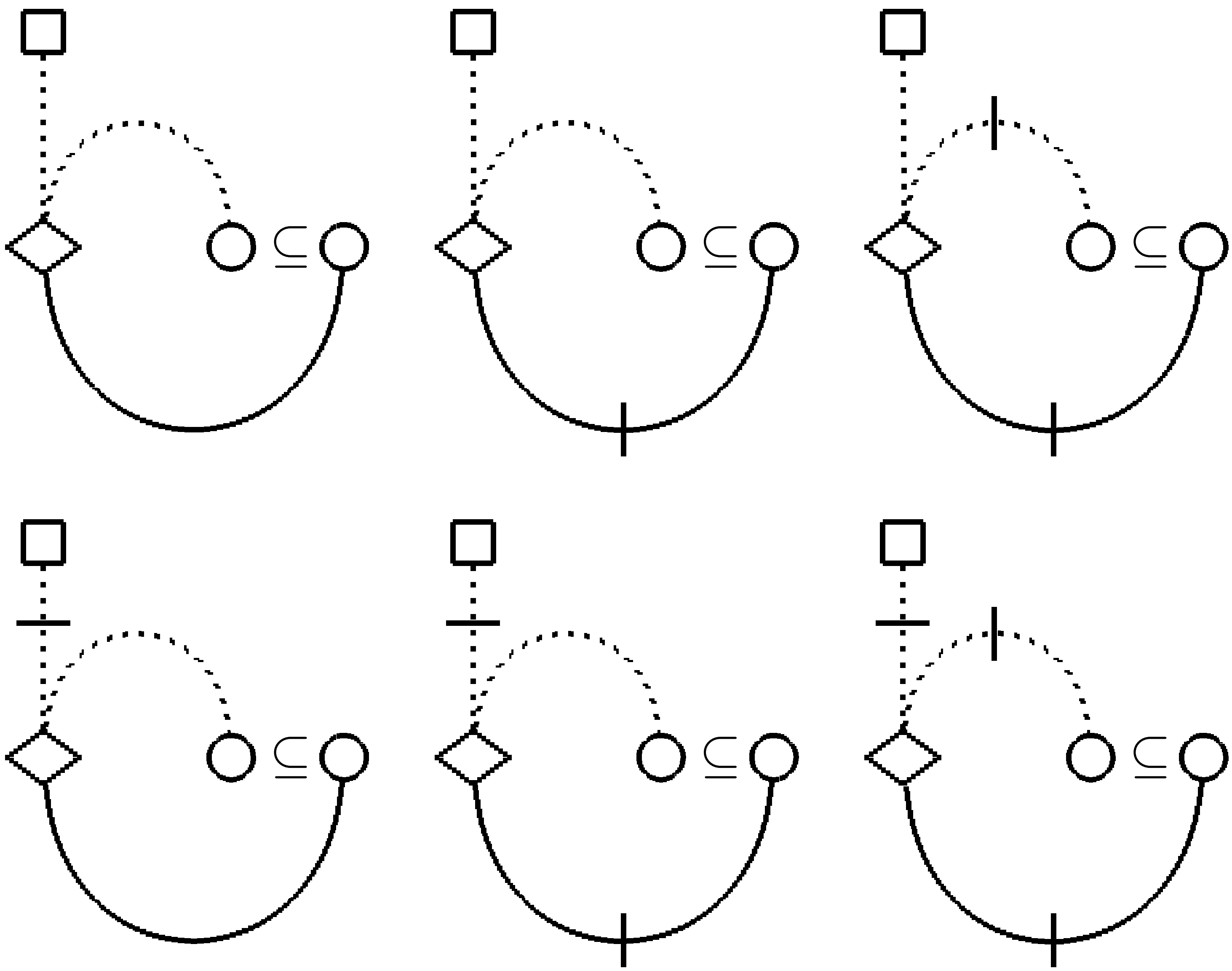
———————— for a Φ_α relationship

.... for a Φ_β relationship

<u>CAN LIST</u> the possible isomorphism types for $\overline{\overline{\varepsilon}}_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_s)$ (allowing for a restraint on σ^{*_s}):

Isomorphism types (all originating along the true paths):





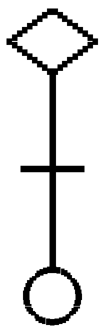
Notes:

- Some minor variations involving $\bigcirc \longrightarrow \square$ etc omitted.

- Can re-interpret the above schemes to get the list of possibilities for $\overline{\overline{\varepsilon}}_s^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_\Theta, g_s)$.

. . . from which it follows that there is a limiting value for the isomorphism types applying to the inner $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured ranks of f_t, g_t, respy, above $\mathcal{R}_\Theta$ at stages $t+1$.

• Then notice — any sufficiently long chain of appreciations in stratifed inner preconfigured ranks via (2) eventually results in relevant limiting isomorphism type



— this being dependent on the non-return to succeeding isomorphism types. □

At all stages $s+1 \geq s_j+1$ at which $\mathcal{R}_\Theta$ requires attention via (3), the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_j} matches the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$, each $j \geq 1$.

•• <u>Just need to revise</u> the previous argument for modules 1 and 2:

• Already have this for each $s = s_j$, $j \geq 1$.

• Can trace any breakdown in the matching at a stage $s+1 > s_j+1$ to new axioms for Φ_α, Φ_β, $\widehat{\Phi}_{\bar{\alpha}}$ or $\widehat{\Phi}_{\bar{\beta}}$.

• All such new axioms for Φ_α, Φ_β must be relevant to ${}^{+}\varepsilon_{s+1}^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{s+1})$.

• And any new axiom '$\sigma = \Phi_\alpha^\tau$' or '$\sigma = \Phi_\beta^\tau$' implemented for Φ_α or Φ_β, respy, at a stage $t+1$ only involves $\sigma, \tau \in \mathrm{Dom}(*_{t+1})$, and any corresponding new axiom for $\widehat{\Phi}_{\bar{\alpha}}$ or $\widehat{\Phi}_{\bar{\beta}}$ at a stage $s+1 \geq t+1$ is of the form '$\sigma^{*_{s+1}} = \widehat{\Phi}_{\bar{\alpha}}^{\tau^{*_{s+1}}}$' or '$\sigma^{*_{s+1}} = \widehat{\Phi}_{\bar{\beta}}^{\tau^{*_{s+1}}}$'.

- So all new axioms for $\widehat{\Phi}_{\bar{\alpha}}$ or $\widehat{\Phi}_{\bar{\beta}}$ at stage $s+1$ must be relevant to ${}^{+}\varepsilon_{s+1}^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_{\Theta}, g_{s+1})$.

- There is no non-trivial reoganisation of $*_t$ at any stage $t+1$ with $s_j < t+1 \leq s+1$, due to $\mathcal{R}_{\Theta}$ requiring attention, and no $\mathcal{R}_{\gamma}$ above $\mathcal{R}_{\Theta}$ requires attention at such a stage.

- And at each such stage $t+1$ there is a restraint of priority that of $\mathcal{R}_{\Theta}$ on π^{*_t}, each $(\pi, \pi') \in \varepsilon_t^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_{\Theta}, f_{s_j})$ — and no hatted functionals involved in the $*_t$-matching of the ranks of f_{s_j}, g_{s_j} appear in $\varepsilon_t^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_{\Theta}, f_{s_j})$.

- Let ψ be the $*_s$-matching of the ranks of f_{s_j}, g_{s_j}, with corresponding augmentation ${}^{+}\varepsilon_s^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_{\Theta}, g_{s_j})[\widehat{\Phi}'_{\bar{\beta}}, \widehat{\Phi}'_{\bar{\alpha}}]$ of ${}^{+}\varepsilon_s^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_{\Theta}, g_{s_j})$.

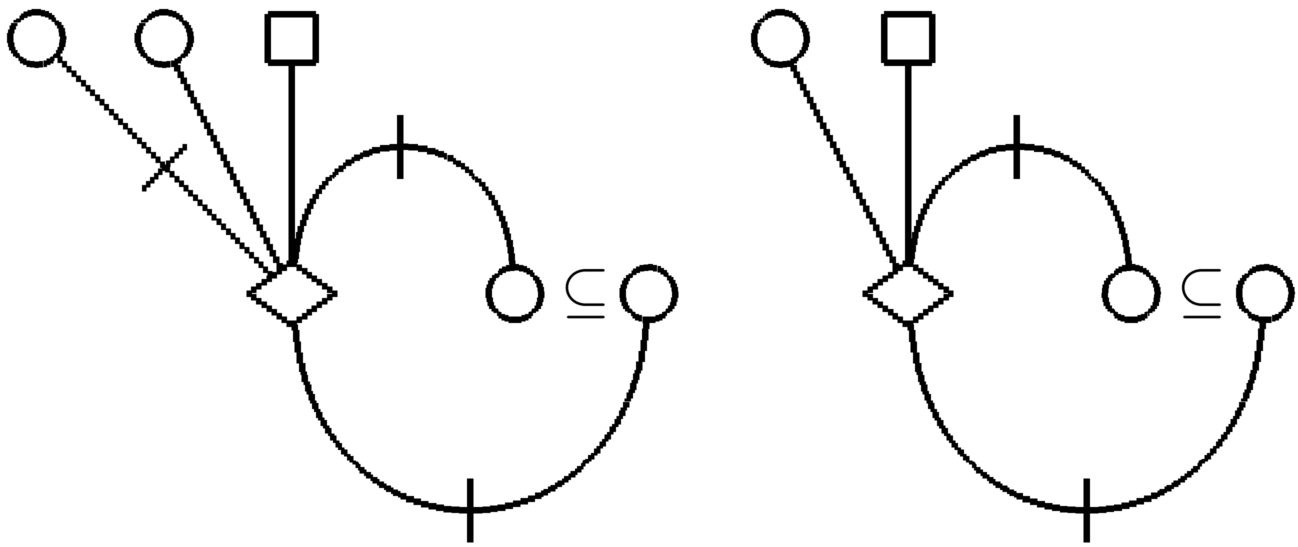
- Then — by the way in which new axioms for Φ_α, Φ_β, $\widehat{\Phi}_{\bar\alpha}$ or $\widehat{\Phi}_{\bar\beta}$ are stipulated and implemented — any change in ${}^{+}\overline{\overline{\varepsilon}}_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{s_j})$ at stage $s+1$ must be associated with a change in ${}^{+}\overline{\overline{\varepsilon}}_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{s+1})$ —

- Which must be then *either* reflected in the appropriate augmentation of ${}^{+}\overline{\overline{\varepsilon}}_{s+1}^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_\Theta, g_{s+1})$, and hence in ${}^{+}\overline{\overline{\varepsilon}}_{s+1}^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_\Theta, g_{s_j})[\widehat{\Phi}'_{\bar\beta}, \widehat{\Phi}'_{\bar\alpha}]$ — *or* leads to an appreciation in $\varepsilon_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_s)$ and, via (2), a consequent appointment of a new prepared inner preconfiguration at $\mathcal{R}_\Theta$ (contradicting the choice of x).

- And, any change in ${}^{+}\overline{\overline{\varepsilon}}_s^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_\Theta, g_{s_j})[\widehat{\Phi}'_{\bar\beta}, \widehat{\Phi}'_{\bar\alpha}]$ arising from a new axiom for $\widehat{\Phi}_{\bar\alpha}$ or $\widehat{\Phi}_{\bar\beta}$ must originate in ${}^{+}\overline{\overline{\varepsilon}}_{s+1}^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{s_j})$. $\square$

For all $i, j \geq 1$ — if at stage $s+1 > s_x+1$ the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$ at stage $s+1$, then the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$ at stage $s+1$.

- Trivial — referring, e.g., to possible isomorphism types:



for ${}^{+}\overline{\overline{\varepsilon}}_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{s_i})$ and ${}^{+}\overline{\overline{\varepsilon}}_s^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_\Theta, g_{s_j})$ —

- <u>Notice that</u> $^{+}\overline{\overline{\varepsilon}}_s^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_\Theta, g_{s_j})$ cannot present any problems (by the defn. of f_{s_i}, g_{s_j}-matching).

- And those presented by $^{+}\overline{\overline{\varepsilon}}_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{s_i})$ are resolved by suitable choice of augmentation $^{+}\overline{\overline{\varepsilon}}_s^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_\Theta, g_{s_j})[\widehat{\Phi}'_{\bar{\beta}}]$ — giving, in the absence of protected arguments, a $*_s$-matching of the appropriate ranks. $\square$

• So — from (A) and (B) one eventually gets a matching of the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of some f_{s_i} by the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of some g_{s_j} above $\mathcal{R}_\Theta$.

• And then (C) gives a full f_{s_i}, g_{s_j}-matching — so $\mathcal{R}_\Theta$ cannot require attention infinitely often via (3) part II, as required.

• Finally — to see that $\mathcal{R}_\Theta$ cannot require attention via (5) infinitely often, one just needs to observe that for an application of (4) to lead to (5) at some stage $s+1$, one needs an appreciation in $\overline{\overline{\varepsilon}}_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_s)$ or $\overline{\overline{\varepsilon}}_s^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_\Theta, g_s)$ at stage $s+1$, which is then retained as in (3) part I — a process, by (A), not infinitely iterable.

•• So — as required — $\mathcal{R}_\Theta$ can only require attention finitely often in relation to a given x.

- <u>But</u> — assuming $\mathcal{R}_\Theta$ to be finitely injured — Only infinitely many appreciations of inner preconfigured rank of f_s or g_s above $\mathcal{R}_\Theta$, resulting in the defn. of new prepared inner preconfigurations, can result in infinitely many potential witnesses for $\mathcal{R}_\Theta$.

- Result follows by (A) if one can be sure that no appreciation in such rank is subsequently reversed, other than through activity on higher priority $\mathcal{R}$-requirements.

- But arguing as before, the addition of new axioms (hatted or unhatted) can only lead to further appreciations.

- While if f_{s+1}, g_{s+1} are defined anew via (3) part II, the current inner preconfigurations are determined by that part of f_{s+1}, g_{s+1} restrained at $\mathcal{R}_\Theta$ and respected by the new defns., and so are common to f_{s+1}, g_{s+1}.

- And — finally — No $*_s$-reorganisation due to an $\mathcal{R}_\gamma$ below $\mathcal{R}_\Theta$ can degrade the current prepared inner preconfigurations at $\mathcal{R}_\Theta$ because of the existing restraints accompanying them. $\square$

26. Towards the full module

The limits $*$, f, g of $\{*_s\}_{s\in\omega}$, $\{f_s\}_{s\in\omega}$, $\{g_s\}_{s\in\omega}$ exist, and there is a bijective $\widetilde{*}$ induced by $*$, with $f^{\widetilde{*}} = g$.

All the requirements are satisfied.

- Notice — No problem now in deriving the satisfaction of $\mathcal{M}_\sigma$ and $\mathcal{K}_{\sigma'}$ — and of all the $\mathcal{R}$-requirements below these.

- But — Full verification of (iii) needs the satisfaction of *all* the $\mathcal{K}$-, $\mathcal{L}$-, $\mathcal{M}$- and $\mathcal{N}$-requirements — which again is no problem given a suitable extension of the Main Lemma (above).

- And then the satisfaction of $\mathcal{P}_\Phi$ and $\mathcal{P}_{\Phi'}$ follow much as before.

- *So a final ‘ $\Box$ ’ awaits the general setting …*

27. Residual questions

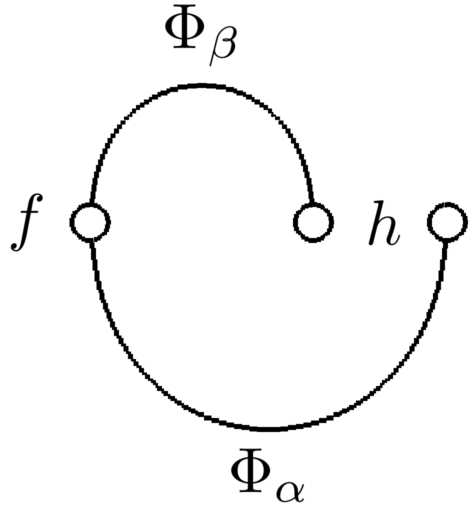
(1) Notice how the choice of α, β in relation to module 4 brought out the role of the inner preconfigurations — <u>but</u>:

What happens to Φ_β *(in a different context) if, despite appearing on a potential true path at some stage, it does not qualify for inner preconfiguration?* (Does it matter that $\mathcal{P}_{\Phi'}$ may not then be satisfied?)

A. Of course, when $*_s$-reorganisation takes place relative to inner preconfigurations which ignore the role of axioms for Φ_β and $\widehat{\Phi}_{\bar{\beta}}$, one must cancel such axioms for $\widehat{\Phi}_{\bar{\beta}}$ which may prevent future axioms for $\widehat{\Phi}_{\bar{\beta}}$ consistently reflecting Φ_β.

• And one can accommodate finitely many such cancellations relative to a given $h \in \omega^\omega$ with $h' = \Phi_\beta^h$, say — so long as there is some neighbourhood of a $\pi \subset h$ avoiding strings appearing in such axioms.

• This may not be possible if, in the limit, h is Turing configured relative to f with an index $\alpha \prec \beta$:



• But then the role of $\widehat{\Phi}_{\bar{\beta}}$ is taken over by that of some $\widehat{\Phi}_{\bar{\alpha}'}$ vskip 1mm

• But also note — the real reason for inner preconfigurations — how if $\beta \prec \alpha$ one cannot then delegate to $\mathcal{P}_\Phi$ the responsibilities of $\mathcal{P}_{\Phi'}$.

• Because there may then be many different candidates $\mathcal{P}_\alpha$ below $\mathcal{P}_\beta$ for such a role, with no eventual fulfilment.

(2) *Are there any serious problems arising from allowing infinitely many $\mathcal{K}$-, $\mathcal{L}$-, $\mathcal{M}$- and $\mathcal{N}$-requirements?*

A. None not accommodated via the appropriate cancellations — given the eventual verification that the activity is finite injury ...

(3) Only very simple (inner) preconfigurations have appeared in modules up to now.

So have these brought out all the basic ingredients required for a fully comprehensive module?

A. Almost — i.e., apart from a small — but important — feature ...

- Module 3 — but not module 4 — involved multiple applications of (3) part I.

- The Main Lemma (for module 4) depended on protection of f_{s_i}, g_{s_i}-matchings, and this was also a necessary feature of the verification for module 3.

- For module 3, this was achieved trivially.

- For module 4, this was achieved in relation to lower priority nontrivial $*_s$-reorganisation via restraints of priority that of $\mathcal{R}_\Theta$.

- But for the full module, in relation to further nontrivial $*_s$-reorganisation of priority that of $\mathcal{R}_\Theta$, one needs to extend — in a straightforward way — the argument that disruption of such matchings can only acompany appreciations of inner preconfigurations above $\mathcal{R}_\Theta$.

Module 5: **Action on the $\mathcal{R}$-requirements in the context of all the other requirements**

Background activity and assumptions:

- $*_s$, $*_s^{-1}$ respect $\subseteq$ on strings, with $*_s$ routinely extended at stage $s+1$.
- Have $f_s^{*_s} = g_s$ at stage each $s+1$. And, except as specified by the module, have $f_{s+1}, g_{s+1} \supset f_s, g_s$, respy, with $f_{s+1} \restriction |f_s|$, $g_{s+1} \restriction |g_s|$ $*_s$-independent.
- The stages $s+1 > 0$ are partitioned into alternate *receptive* and *reactive* stages.
- The only nontrivial activity at receptive stages is the implementation of new axioms for unhatted functionals, the appointment of new restraints and prepared inner pre-configurations, and the possible subsequent application of phases (4) – (5) of the module to an appropriate highest priority $\mathcal{R}_\Theta$ —

- Any other nontrivial activity takes place at the reactive stages.

- For each $\mathcal{M}_\sigma$ (or $\mathcal{N}_\sigma$), if σ^{*_s} ($\sigma^{*_s^{-1}}$, respy) is defined $= \pi$, say, at the receptive stage $s{+}1$, then $\sigma^{*_{s+1}}$ ($\sigma^{*_{s+1}^{-1}}$, respy) is restrained, with priority that of $\mathcal{M}_\sigma$ (or $\mathcal{N}_\sigma$ respy), $= \pi$ at stage $s+1$ —

- In which case say $\mathcal{M}_\sigma$ (or $\mathcal{N}_\sigma$) *receives attention* at stage $s+1$.

- The results of all activity on requirements below one receiving attention at stage $s+1$ are *cancelled*.

- Also, for each $\mathcal{K}_\sigma$ (or $\mathcal{L}_\sigma$) above $\mathcal{R}_\Theta$ (the current $\mathcal{R}$-requirement requiring attention at a reactive stage $s+1$), in reorganising $*_s$ at stage $s+1$, ensure that for each $h \in \omega^\omega$ with $h \supset \sigma$ there is some τ, $h \supset \tau \supseteq \sigma$, with $\tau \in \mathrm{Dom}(*_{s+1})$ ($\tau \in \mathrm{Range}(*_{s+1})$, respy) — each such τ being a *protected argument* of $*_{t+1}$ ($*_{t+1}^{-1}$ respy) at each stage $t{+}1 > s{+}1$.

- All axioms for Φ_α or $\widehat{\Psi}_\alpha$, $\alpha \in \mathfrak{T}_f$, or for Ψ_β or $\widehat{\Phi}_\beta$, $\beta \in \mathfrak{T}_g$, stipulated at a stage $t+1 \leq s+1$ only involve beginnings of strings in $\mathrm{Dom}(*_{t+1})$ or $\mathrm{Range}(*_{t+1})$, respy, and, if on a true path at stage $t+1$, are immediately implemented — where:

- An axiom '$\sigma = \Xi_\gamma^\tau$', with Ξ_γ some hatted or unhatted Φ_γ or Ψ_γ, some $\gamma \in \mathfrak{T}_f$ or $\mathfrak{T}_g$, as appropriate, is *on a true path at stage* $s+1$ iff

either $\ulcorner\gamma = \Xi$ and σ, τ are on a potential true path via $\langle\gamma\rangle$, $\langle\gamma^-\rangle$, respy, relative to f_s or g_s (as appropriate) at stage $s+1$,

or $\ulcorner\gamma = \Xi^{-1}$ and σ, τ are on a potential true path via $\langle\gamma^-\rangle$, $\langle\gamma\rangle$, respy, relative to f_s or g_s (as appropriate) at stage $s+1$,

with a basis of the form $\Phi_{\alpha_1,s},\dots,\Phi_{\alpha_k,s}$ or (correspondingly) $\Psi_{\beta_1,s},\dots,\Psi_{\beta_k,s}$.

- Given an axiom '$\sigma = \Xi^{\tau}$' or '$\sigma = \widehat{\Xi}^{\tau}$' to be implemented at stage $s+1$, one ensures that for each σ' maximal $\subseteq \sigma$, τ' minimal $\supseteq \tau$, with $\sigma', \tau' \in \mathrm{Dom}(*_{s+1})$, one defines an axiom '$\sigma' = \Xi^{\tau'}$' for Ξ at stage $s+1$.

- And — if one has an implemented axiom '$\sigma = \Phi_{\alpha}^{\tau}$' for Φ_{α}, or an axiom '$\sigma = \Psi_{\beta}^{\tau}$' for Ψ_{β}, on a true path at stage $s+1$, an axiom '$\sigma^{*_{s+1}} = \widehat{\Phi}_{\bar{\alpha}}^{\tau^{*_{s+1}}}$ or '$\sigma^{*_{s+1}^{-1}} = \widehat{\Psi}_{\bar{\beta}}^{\tau^{*_{s+1}^{-1}}}$, respy, is stipulated at stage $s+1$.

- Say there is an axiom '$\sigma = \widehat{\Psi}^{\tau}_{\alpha}$' for $\widehat{\Psi}_{\alpha}$ at stage $s+1$, some $\alpha \in \mathfrak{T}_f$, with τ $\mathcal{P}[\Phi'_{\alpha_1}, ..., \Phi'_{\alpha_k}], \widehat{\mathcal{Q}}$-preconfigured above $\mathcal{R}_{\Theta}$ at stage $s+1$ according to $\langle \alpha^- \restriction_{\geq i} \rangle$ or $\langle \alpha \restriction_{>i} \rangle$, say, with α^- or $\alpha \restriction_{\leq i}$, respy, $\mathcal{Q}_{\Psi}$- relevant, relative to f_s, and that τ is inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured (possibly with a different corresponding basis $\Phi''_{\alpha_1}, ..., \Phi''_{\alpha_k}$, say) relative to f_s above $\mathcal{R}_{\Theta}$ at stage $s+1$ according to $\langle \hat{\alpha}^- \restriction_{\geq j} \rangle$, say, — and that σ is not inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured relative to f_s above $\mathcal{R}_{\Theta}$ at stage $s+1$ according to $\langle \ulcorner \alpha \rangle$.

  Then each such axiom is *cancelled* by being removed (as appropriate) from $\widehat{\Psi}_{\alpha}[s+1]$ or $\widehat{\Phi}_{\beta}[s+1]$.

- And similar cancellations in relation to such axioms '$\sigma = \widehat{\Phi}^{\tau}_{\beta}$' for $\widehat{\Phi}_{\beta}$ at stage $s+1$, a $\beta \in \mathfrak{T}_g$.

The module at a reactive stage $s+1$:

- Assume $\mathcal{R}_\Theta$ requires attention at stage $s+1$ according to the activity described below (in which case, f_{s+1}, g_{s+1} are said to be $\mathcal{R}_\Theta$-*determined*) — and such activity relative to each lower priority $\mathcal{R}_\gamma$ is *initialised*.

(1) *Choose* a *potential witness* $x = |f_s|$ for $\mathcal{R}_\Theta$ — new, and greater than any existing restraint on f — and *proceed* to (3), part II.

(2) *Wait* for $g_s \restriction \vartheta_s(x)$ to be realised — i.e., $\Theta^{g_s}(x) \downarrow = f_s(x)$ and $s = s_k$, say — and *proceed* to (3), following the initiation of the process of stage-by-stage updating of the prepared inner preconfigurations at $\mathcal{R}_\Theta$, namely:

- Say at some subsequent receptive stage $t+1$

$$\overline{\overline{\varepsilon}}^{\mathcal{P},\widehat{\mathcal{Q}}}_{t+1}(\mathcal{R}_\Theta, f_{t+1}) \not\cong \overline{\overline{\varepsilon}}^{\mathcal{P},\widehat{\mathcal{Q}}}_{t-1}(\mathcal{R}_\Theta, f_{s_k})$$

($\varepsilon^{\mathcal{P},\widehat{\mathcal{Q}}}_t(\mathcal{R}_\Theta, f_{s_k})$ *appreciates* at stage $t+1$).

- Then *cancel* any existing potential witness for $\mathcal{R}_\Theta$ at stage $t+1$, and *define* and *restrain* (with priority that of $\mathcal{R}_\Theta$) $f_{s_k}, g_{s_k} \subseteq f_{u+1}, g_{u+1}$, respy, at all later stages $u+1$ (prior to any injury of the restraint through activity on an $\mathcal{R}$-requirement of higher priority than that of $\mathcal{R}_\Theta$) $\geq t+1$, *restrain* $\rho^{*u+1} = \rho^{*t+1}$ for all ρ inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured relative to f_t above $\mathcal{R}_\Theta$ at the end of stage $t+1$, and say that $\varepsilon_{t+1}^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{s_k})$ is a new *prepared inner preconfiguration* at $\mathcal{R}_\Theta$ at all such stages.

And then *return* to (1).

(3) Let $g{\restriction}\vartheta(x)[s_i]$, $1 \leq i \leq k$, be a list of all current realised strings for $\mathcal{R}_\Theta$, where $g{\restriction}\vartheta(x)[s_i]$ became realised at stage s_i+1, each i (so $s+1 = s_k+1$ and $g{\restriction}\vartheta(x)[s_k] = g{\restriction}\vartheta(x)[s]$). If there exists a current acceptable sequence $(f,g)_{s_{i_{\cdot r}}}$ which is $\overrightarrow{\gamma}^{0}_{s_{i_{\cdot r}}}, \overrightarrow{\gamma}^{1}_{s_{i_{\cdot r}}}$-acceptable at stage $s+1$. Let $n =$ the sum of the lengths of $\overrightarrow{\gamma}^{0}_{s_{i_{\cdot r}}}$ and $\overrightarrow{\gamma}^{1}_{s_{i_{\cdot r}}}$.

Then —

- Say that this current acceptable sequence is *improvable* iff $|(f,g)_{s_{i\cdot_r}}| \geq n!\,3^n$.

- If $(\overrightarrow{\gamma}^0, \overrightarrow{\gamma}^1) \succ (\overrightarrow{\gamma}^0_{s_{i\cdot_r}}, \overrightarrow{\gamma}^1_{s_{i\cdot_r}})$, then any $\overrightarrow{\gamma}^0, \overrightarrow{\gamma}^1$-acceptable subsequence of $(f,g)_{s_{i\cdot_r}}$ *potentially improves* $(f,g)_{s_{i\cdot_r}}$ – and if of length $\geq n'!\,3^{n'}$, $n' =$ the sum of the lengths of $\overrightarrow{\gamma}^0$ and $\overrightarrow{\gamma}^1$, then it is an *improvement* of $(f,g)_{s_{i\cdot_r}}$.

- If a potential improver of $(f,g)_{s_{i\cdot_r}}$ of the form $\{(f_{s_{i\cdot_r}}, g_{s_{i\cdot_r}}),\ (f_{s_{i\cdot_{r'}}}, g_{s_{i\cdot_{r'}}})\}$ materialises at a stage at which an f_{s_i}, g_{s_j}-matching ψ, say, above $\mathcal{R}_\Theta$ is currently being implemented, then it is said to be ψ*-related*.

Assume given *current* values for all potential improvements $(f,g)_{s_{i\cdot_{r_u}}}$, representing each possible $\overrightarrow{\gamma}^0_{s_{i\cdot_{r_u}}}, \overrightarrow{\gamma}^1_{s_{i\cdot_{r_u}}}$-acceptable sequence, $\overrightarrow{\gamma}^0_{s_{i\cdot_{r_u}}}, \overrightarrow{\gamma}^1_{s_{i\cdot_{r_u}}} \succ \overrightarrow{\gamma}^0_{s_{i\cdot_r}}, \overrightarrow{\gamma}^1_{s_{i\cdot_r}}$.

- *Then ask:* Is the current acceptable sequence improvable?

Case I. 'Yes':

First — Update the potential improvements:

- Say the current $*_s$-matching of the preconfigured ranks of f_s, g_s above $\mathcal{R}_\Theta$ contains a matching ψ', say, previously implemented by $\mathcal{R}_\Theta$ via (3) part I (and possibly modified via (5) at a subsequent stage) — and there is a new ψ'-related potential improver of $(f,g)_{s_{i \cdot r}}$ of the form $\{(f_{s_{i \cdot r}}, g_{s_{i \cdot r}}),\ (f_{s_{i \cdot r'}}, g_{s_{i \cdot r'}})\}$. Then if $\{(f_{s_{i \cdot r}}, g_{s_{i \cdot r}}),\ (f_{s_{i \cdot r'}}, g_{s_{i \cdot r'}})\}$, or $(f,g)_{s_{i \cdot r_u}} \cup \{(f_{s_{i \cdot r'}}, g_{s_{i \cdot r'}})\}$, or $(f,g)_{s_{i \cdot r_u}} \cup \{(f_{s_{i \cdot r}}, g_{s_{i \cdot r}})\}$ extends $(f,g)_{s_{i \cdot r_u}}$, and is also a potential improvement of $(f,g)_{s_{i \cdot r}}$, revise the current value of $(f,g)_{s_{i \cdot r_u}}$ accordingly — and ψ' is no longer *needed to be implemented* by $\mathcal{R}_\Theta$.

And — Update the current acceptable sequence by using any newly available improvement.

- The *available part* of $(f,g)_{s_{i_{\cdot r}}}$ is the set of all pairs $(f_{s_{i'}}, g_{s_{i'}}) \in (f,g)_{s_{i_{\cdot r}}}$ for which no matching above $\mathcal{R}_\Theta$ implemented while $(f,g)_{s_{i_{\cdot r}}}$ is current involves an $f_{s_{j'}}$ or $g_{s_{j'}}$ with $j' \leq i'$.

- *Then:* Look for distinct i', j', with $k \geq i', j' \geq 1$, and $f_{s_{i'}}, g_{s_{j'}} = f_{s_{i_{\cdot r}}}, g_{s_{i_{\cdot r'}}}$ appearing in the *available part* of the current acceptable sequence, such that there is an $f_{s_{i'}}, g_{s_{j'}}$-matching ψ (say) above $\mathcal{R}_\Theta$ at stage $s+1$, *and either*

  (i) $\Theta^{g_s}(x) = f_s(x)$, *or*

  (ii) there is no ψ' contained in the current $*_s$-matching of the f_s and g_s ranks above $\mathcal{R}_\Theta$ needed to be implemented by $\mathcal{R}_\Theta$ at stage $s+1$.

If ψ exists, take ψ to be such a matching, with r, r' maximal, and then ψ of maximal γ-state. Take action under the following headings, and then *proceed* to (4) (at subsequent receptive stages). Otherwise, return to (3).

(a) Revision of f and g. *Choose* strings $\pi \supset f_{s_i}$, $\rho \supset g_{s_j}$, with the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of π matching the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of ρ above $\mathcal{R}_\Theta$.

Define $f_{s+1} = \pi$, $g_{s+1} = \rho$, and restrain at all later stages.

(b) Revision of $*$. *ψ needs to be implemented by $\mathcal{R}_\Theta$*, until x is cancelled, or a ψ-related potential improvement is incorporated into a current $(f, g)_{s_{i_{r_u}}}$.

Define $\sigma^{*_{s+1}} = \psi(\sigma)$ for each $\sigma \in \mathrm{Dom}(\psi)$ — and $\sigma^{*_{s+1}} = \sigma^{*_s}$ for each σ with $\psi(\sigma), \psi^{-1}(\sigma^{*_s})\uparrow$, whenever consistent with $\subseteq$ on strings.

(c) Definition of axioms for $\widehat{\Psi}_\alpha$, $\widehat{\Phi}_\beta$. As in background activity.

Case II. 'No': Then progress the search for such an improvable sequence:

- *Choose* (minimal) $*_s$-independent strings $\pi, \rho \supset f_s \restriction (x-1), (f_s \restriction x-1)^{*_s}$, respy, with $\pi(x) \neq \Theta^{g_{s_i}}(x)$ or $f_{s_i}(x)$ each i, $1 \leq i \leq k$.

Define $f_{s+1} = \pi$, $g_{s+1} = \rho$, $f_{s+1}^{*_{s+1}} = g_{s+1}$, set up a *restraint*, of priority that of $\mathcal{R}_\Theta$, on ρ^{*_u}, each $(\rho, \rho') \in \varepsilon_u^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{s_i})$, each stage $u+1 > s$. And —

- Let the current *acceptable sequence* $(f,g)_{s_{i_r}}$ be the longest $\overrightarrow{\gamma}^0, \overrightarrow{\gamma}^1$-acceptable subsequence of $(f,g)_{s_i}$, some $\overrightarrow{\gamma}^0, \overrightarrow{\gamma}^1$ defined, respy, on the nonconfiguring $\mathcal{Q}$- and $\mathcal{P}$-nodes above $\mathcal{R}_\Theta$ —

- And *return* to (2) at the next stage.

<u>The module at a receptive stage $s+1$</u>:

- Carry out any appropriate background implementation of axioms stipulated for unhatted functionals — Then the $\mathcal{R}_\Theta$ requiring attention is the highest priority $\mathcal{R}_\gamma$ qualifying for phase (5) below (with, f_{s+1}, g_{s+1}, as usual, said to be $\mathcal{R}_\Theta$-*determined*).

(4) *Ask:* Does the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank of f_s above $\mathcal{R}_\gamma$ $*_s$-match the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_s, respy, above $\mathcal{R}_\gamma$ at stage $s+1$? And does the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank of each f_{s_i} above $\mathcal{R}_\gamma$ $*_s$-match the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_i}, respy, above $\mathcal{R}_\gamma$ at stage $s+1$?

'No': *Proceed* directly to (5).

'Yes': *Return* to (4) at the next receptive stage.

(5) *Choose* an $f_s, g_s \,/\, f_{s_i}, g_{s_i}$-matching ψ at stage $s+1$ which maximally respects $*_s$ (that is, so that the witness to ψ not being a $*_s$-matching occurs at a minimal level of $\mathfrak{T}_f$ or $\mathfrak{T}_g$), and *reorganise* $*$ along the lines of part (b) of (3), subcase I.

Return to (3).

- And then appoint any new restraints needed.

<u>Need to verify:</u>

(✠) Each phase of module 5 can be implemented.

(i) For each $\mathcal{R}_\Theta$, the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank of f_s above $\mathcal{R}_\Theta$ $*_s$-matches the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_s above $\mathcal{R}_\Theta$ at each stage $s+1$.

(ii) For each requirement $\mathcal{R}_\Theta$, if $\mathcal{R}_\Theta$ is finitely injured then it is finitely injuring — and consequently all the $\mathcal{R}$-, $\mathcal{K}$-, $\mathcal{L}$-, $\mathcal{M}$- and $\mathcal{N}$-requirements are satisfied.

(iii) The limits $*$, f, g of $\{*_s\}_{s\in\omega}$, $\{f_s\}_{s\in\omega}$, $\{g_s\}_{s\in\omega}$ exist, and there is a bijective $\widetilde{*}$ induced by $*$, with $f^{\widetilde{*}} = g$.

(iv) All the $\mathcal{P}$- and $\mathcal{Q}$-requirements are satisfied.

Each phase of the module is implementable.

• The activity for the reactive stages is relatively straightforward to implement.

• But similarly to before, one must examine how phase (5) results from new axioms and restraints being introduced (at the receptive stages) into the matched environment — *either* due to the unboundedness of the possible proper preconfigurations above $\mathcal{R}_\Theta$ — *or* from the added possibility of new axioms or restraints introducing novel enlargements of the inner preconfigured environments relative to f and g.

• <u>The context needed for this</u> is that of the inductive proof of (i).

• Assume that at the beginning of the receptive stage $s+1$ one has a $*_s$-matching ψ of the $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured ranks of f_s, g_s, respy, above $\mathcal{R}_\gamma = \mathcal{R}_\Theta$, say, with corresponding augmentations ${}^+\varepsilon_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_s)[\widehat{\Psi}'_{\alpha_1}, \ldots, \widehat{\Psi}'_{\alpha_l}]$ and ${}^+\varepsilon_s^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_\Theta, g_s)[\widehat{\Phi}'_{\beta_1}, \ldots, \widehat{\Phi}'_{\beta_k}]$ (the argument with f_{s_i}, g_{s_i} in place of f_s, g_s is similar) —

• And that a new axiom '$\sigma = \Phi^{\tau}_{\alpha}$' or '$\sigma = \Psi^{\tau}_{\beta}$' on the true path at $\mathcal{P}_{\alpha}$ or $\mathcal{Q}_{\beta}$, respy, above $\mathcal{R}_{\Theta}$ is implemented at stage $s+1$, leading to an inconsistency, at some level below that of $\alpha \in \mathfrak{T}_f$ or $\beta \in \mathfrak{T}_g$, respy, with any potential $*_s$-compatible f_s, g_s-matching above $\mathcal{R}_{\Theta}$ at the end of stage $s+1$, due to an already existing hatted axiom — call such new axioms *rank altering*.

• <u>Aim</u> — To modify ψ to obtain the required f_s, g_s-matching $\hat{\psi}$, say — with corresponding augmentations

$$+\varepsilon^{\mathcal{P},\widehat{\mathcal{Q}}}_{s+1}(\mathcal{R}_{\Theta}, f_s)[\widehat{\Psi}''_{\alpha'_1}, \ldots, \widehat{\Psi}''_{\alpha'_{l'}}],$$

$$+\varepsilon^{\widehat{\mathcal{P}},\mathcal{Q}}_{s+1}(\mathcal{R}_{\Theta}, g_s)[\widehat{\Phi}''_{\beta'_1}, \ldots, \widehat{\Phi}''_{\beta'_{k'}}],$$

say — to be implemented via (5) at stage $s+1$.

•• <u>To do this</u> — First, inductively define $\hat{\psi}$ on those pairs of strings chosen from the stratified *inner* $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank of f_s above $\mathcal{R}_{\Theta}$ at the end of stage $s+1$.

- Write

$$\varepsilon_{s+1}^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_s)[\widehat{\Psi}^\natural_{\alpha'_1}, \ldots, \widehat{\Psi}^\natural_{\alpha'_{l'}}],$$

$$\varepsilon_{s+1}^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_\Theta, g_s)[\widehat{\Phi}^\natural_{\beta'_1}, \ldots, \widehat{\Phi}^\natural_{\beta'_{k'}}],$$

for the current inductively defined values of the corresponding augmentations, where each $\widehat{\Psi}^\natural_{\alpha'_i}, \widehat{\Phi}^\natural_{\beta'_j}$ will be $\subseteq$ the eventual $\widehat{\Psi}''_{\alpha'_i}, \widehat{\Phi}''_{\beta'_j}$, respy.

- Assume that at some current level of the inductive definition of inner $\mathcal{P}[\widehat{\Psi}^\natural_{\alpha'_1}, \ldots, \widehat{\Psi}^\natural_{\alpha'_{l'}}], \widehat{\mathcal{Q}}$-preconfigured relative to f_s above $\mathcal{R}_\Theta$ one encounters some $\hat{\alpha}$, and some

$$(\sigma, \tau) \in \overline{\overline{\varepsilon}}_{s+1,\hat{\alpha}}^{\mathcal{P},\widehat{\mathcal{Q}},\Phi_{\hat{\alpha}}}(\mathcal{R}_\Theta, f_s)[\widehat{\Psi}^\natural_{\alpha'_1}, \ldots, \widehat{\Psi}^\natural_{\alpha'_{l'}}],$$

with *either* $\hat{\psi}(\tau)\downarrow$, but $\hat{\psi}$ as yet undefined on σ, *or* $\hat{\psi}(\sigma)\downarrow$, but $\hat{\psi}$ as yet undefined on τ —

- And say σ (or τ, respy) is *as yet unaffected* iff every rank altering axiom so far encountered in this inductive definition has been succeeded along every path of this defn. to σ by some intervening $*_s$-compatible defn. of the form $\hat{\psi}(\pi)$.

• Inductively assume the axioms of $\widehat{\Psi}^{\natural}_{\alpha'_1}, \ldots$
$\ldots, \widehat{\Psi}^{\natural}_{\alpha'_{l'}}$ defined between as yet unaffected strings to be exactly those corresponding axioms of $\widehat{\Psi}'_{\alpha_1}, \ldots, \widehat{\Psi}'_{\alpha_l}$, where they exist; and that $\hat{\psi}(\pi) = \psi(\pi)$ on all as yet unaffected strings π for which $\psi(\pi)\downarrow$, and otherwise $\hat{\psi}(\pi)$ is defined compatibly with $*_s$ —

• And, assume a similar scenario to that being described affecting the defn. of ψ^{-1} and $\varepsilon^{\widehat{\mathcal{P}},\mathcal{Q}}_{s+1}(\mathcal{R}_\Theta, g_s)[\widehat{\Phi}^{\natural}_{\beta'_1}, \ldots, \widehat{\Phi}^{\natural}_{\beta'_{k'}}]$ —

• And if σ (the argument for τ in the second case is analagous) is an as yet unaffected string, define $\hat{\psi}(\sigma)$ etc accordingly:

•• Say $\hat{\alpha} = \alpha$, with '$\sigma = \Phi^{\tau}_{\alpha}$' an axiom already existing at stage $s+1$ (and so not rank altering).

• Then if $\psi(\sigma)\uparrow$, one is free to define $\hat{\psi}(\sigma)$ compatibly with $*_s$, and, if there is no existing axiom of $\widehat{\Phi}'_{\bar{\hat{\alpha}}}$ corresponding to '$\sigma = \Phi^{\tau}_{\alpha}$' to enumerate into $\widehat{\Phi}^{\natural}_{\bar{\hat{\alpha}}}$, one can enumerate one $*_s$-reflecting it.

•• Say $\hat{\alpha} = \alpha$, with '$\sigma = \Phi^{\tau}_{\alpha}$' a new axiom (possibly rank altering).

• For inner $\mathcal{P}[\widehat{\Psi}'_{\alpha_1}, ..., \widehat{\Psi}'_{\alpha_l}], \widehat{\mathcal{Q}}$-preconfigured σ, τ relative to f_s above $\mathcal{R}_{\Theta}$ at stage $s+1$, an instance of the above inductive step will again define $\hat{\psi}(\sigma)$ consistently with $*_s$ — in which case one can naturally $*_s$-reflect '$\sigma = \Phi^{\tau}_{\alpha}$' in an axiom for $\widehat{\Phi}''_{\bar{\hat{\alpha}}}$.

•• <u>This is because</u> — *in this case*, '$\sigma = \Phi^{\tau}_{\alpha}$' *cannot be rank altering* —

• Since the introduction of the new axiom creates no new inner preconfigurations — so ensuring that the inner preconfigured context of this new axiom is $*_s$-matched —

• And any local obstacle, presented by the new axiom, to the trivial extension of the matching must come in the form of an existing axiom of the form '$\sigma'^{*_s} = \widehat{\Phi}^{\tau^{*_s}}_{\bar{\alpha}}$' with $\sigma'^{*_s} \mid \sigma^{*_s}$ —

• And such an axiom could only exist, in the context of the matching ψ, given an existing axiom of the form '$\sigma' = \Phi^{\tau}_{\alpha}$', contradicting the consistency of Φ_{α}.

•• Assume first that τ, but not σ, is inner $\mathcal{P}[\widehat{\Psi}'_{\alpha_1}, \ldots, \widehat{\Psi}'_{\alpha_l}], \widehat{\mathcal{Q}}$-preconfigured relative to f_s above $\mathcal{R}_\Theta$ at stage $s+1$ —

• In this case, one then has freedom to define $\hat{\psi}(\sigma)$ (respecting $\subseteq$ on strings) so as to enable at index $\hat{\alpha}$ a suitable extension of the augmentation $\varepsilon^{\widehat{\mathcal{P}},\mathcal{Q}}_{s+1}(\mathcal{R}_\Theta, g_s)[\widehat{\Phi}^\natural_{\beta'_1}, \ldots, \widehat{\Phi}^\natural_{\beta'_{k'}}]$ contributing to that underlying the definition of

$$\hat{\psi} : \overline{\overline{\varepsilon}}^{\mathcal{P},\widehat{\mathcal{Q}}}_{s+1}(\mathcal{R}_\Theta, f_s)[\widehat{\Psi}''_{\alpha'_1}, \ldots, \widehat{\Psi}''_{\alpha'_{l'}}] \longrightarrow \overline{\overline{\varepsilon}}^{\widehat{\mathcal{P}},\mathcal{Q}}_{s+1}(\mathcal{R}_\Theta, g_s)[\widehat{\Phi}''_{\beta'_1}, \ldots, \widehat{\Phi}''_{\beta'_{k'}}].$$

• This is because there can be no axiom for $\widehat{\Phi}'_{\bar{\hat{\alpha}}}$ (or, by the same argument, for any similar functional) preventing such a definition, since σ is not inner $\mathcal{P}[\widehat{\Psi}'_{\alpha_1}, \ldots, \widehat{\Psi}'_{\alpha_l}], \widehat{\mathcal{Q}}$-preconfigured relative to f_s above $\mathcal{R}_\Theta$ at stage $s+1$.

•• And such a definition cannot violate a restraint on $*_s$ —

• Since *either* the restraint must have existed already (at the start of stage $s+1$), and so is already included in the matching ψ restricted to existing inner preconfigurations, contrary to assumption, *or* is a new restraint appointed after the application of phase (5).

•• On the other hand, assume that τ is not inner $\mathcal{P}[\widehat{\Psi}'_{\alpha_1}, \ldots, \widehat{\Psi}'_{\alpha_l}], \widehat{\mathcal{Q}}$-preconfigured relative to f_s above $\mathcal{R}_\Theta$ at stage $s+1$.

• Then in this case, if '$\sigma = \Phi^\tau_\alpha$' is rank altering, then τ, and possibly other strings encountered along paths of the inductive defn. of τ being currently inner $\mathcal{P}[\widehat{\Psi}^\natural_{\alpha'_1}, \ldots, \widehat{\Psi}^\natural_{\alpha'_{l'}}], \widehat{\mathcal{Q}}$-preconfigured relative to f_s above $\mathcal{R}_\Theta$ at stage $s+1$, are now *affected* by '$\sigma = \Phi^\tau_\alpha$' (i.e., are no longer 'as yet unaffected') —

• And those parts of the defn. of $\hat{\psi}$, and of the underlying augmentation, dependent on the updated information must now be *redefined* using the same freedom on appropriate affected strings encountered in the inductive definition of τ being currently inner $\mathcal{P}[\widehat{\Psi}^{\natural}_{\alpha'_1}, \ldots, \widehat{\Psi}^{\natural}_{\alpha'_{l'}}], \widehat{\mathcal{Q}}$-preconfigured relative to f_s above $\mathcal{R}_\Theta$ at stage $s+1$ to define $\hat{\psi}$ incompatibly with $*_s$.

• One can similarly deal with rank affecting axioms of the form '$\sigma = \Psi^{\tau}_{\beta}$', in relation to the stratified inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_s above $\mathcal{R}_\Theta$ at the end of stage $s+1$.

• So the isomorphism

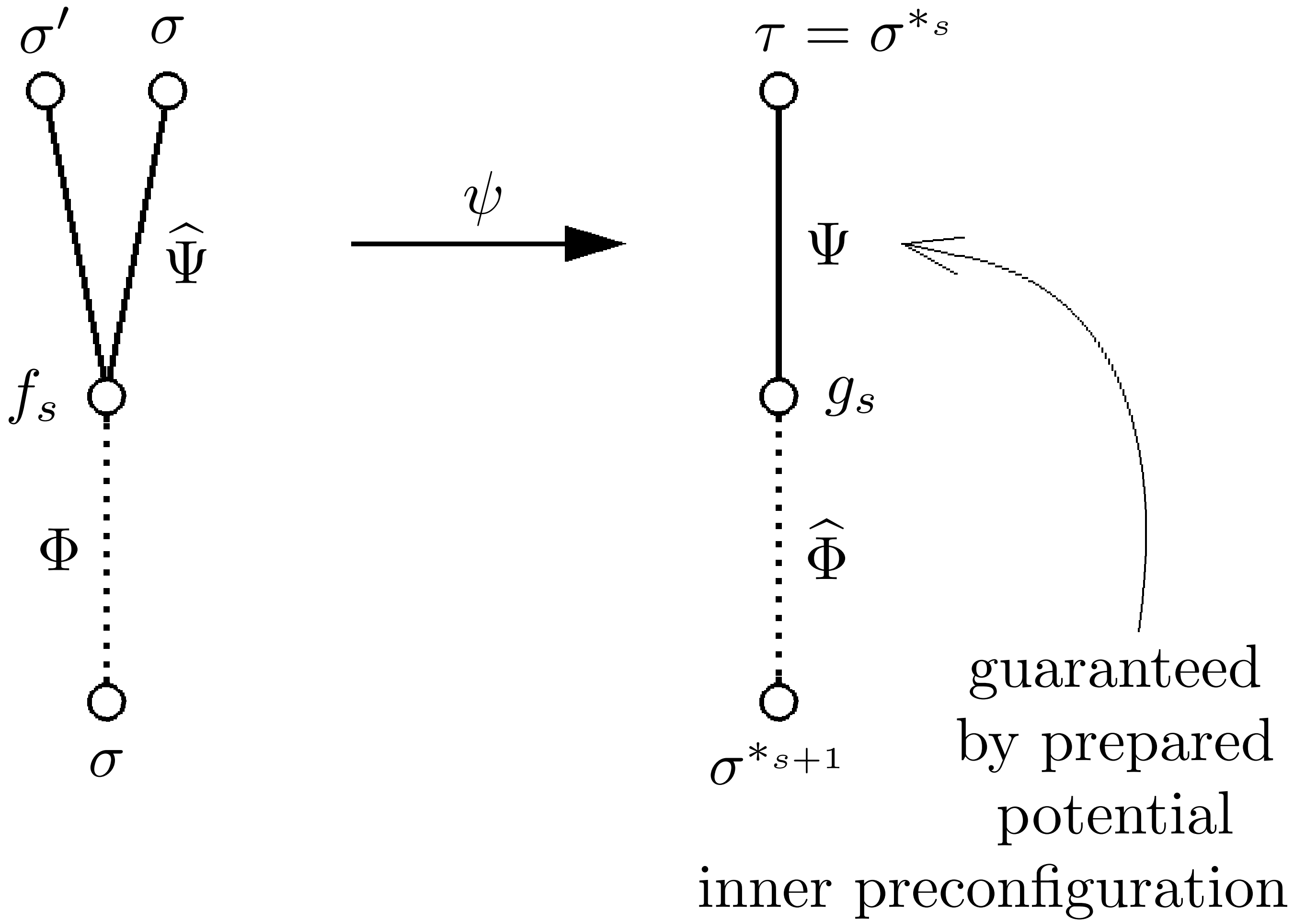
$$\hat{\psi} : \overline{\overline{\varepsilon}}^{\mathcal{P},\widehat{\mathcal{Q}}}_{s+1}(\mathcal{R}_\Theta, f_s)[\widehat{\Psi}''_{\alpha'_1}, \ldots, \widehat{\Psi}''_{\alpha'_{l'}}] \longrightarrow \overline{\overline{\varepsilon}}^{\widehat{\mathcal{P}},\mathcal{Q}}_{s+1}(\mathcal{R}_\Theta, g_s)[\widehat{\Phi}''_{\beta'_1}, \ldots, \widehat{\Phi}''_{\beta'_{k'}}]$$

follows directly from the definition of ψ.

Note on the underlying subinduction:

- This uses the freedom to reorganise $*_s$ to recapture discarded structure associated with the hatted functionals, and not already contributing to an appreciated prepared inner preconfiguration.
- This is ensured via the procedures of the matching process (providing matchings of the tree of potential inner preconfigurations), and the prepared inner preconfigurations.

EXAMPLE:



• Formally — The tree of potential inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigurations relative to f_s above $\mathcal{R}_\Theta$, say, consists of all (up to isomorphism) inner $\mathcal{P}[\Phi'_{\alpha_1}, \ldots, \Phi'_{\alpha_k}], \widehat{\mathcal{Q}}$-preconfigurations, with $\Phi'_{\alpha_1}, \ldots, \Phi'_{\alpha_k}$ a potential true path.

• Then need a matching of the appropriate trees of potential inner preconfigurations, where the branches relative to f_s, g_s, say, are matched as before via a matching ψ.

• And this isomorphism cannot be disturbed by new restraints appointed at stage $s+1$ — since such restraints are *either* potentially restrained before the application of (4)–(5), and so are already involved in the matching of inner preconfigured ranks, *or* derive from an appreciation of inner preconfigured rank at stage $s+1$ involving a new prepared inner preconfiguration appointed after (4)–(5), and already accounted for in the definition of $\hat{\psi}$.

•• Finally — The extension to an isomorphism

$$\hat{\psi}: {}^{+}\overline{\overline{\mathcal{E}}}{}^{\mathcal{P},\widehat{\mathcal{Q}}}_{s+1}(\mathcal{R}_\Theta, f_s)[\widehat{\Psi}''_{\alpha'_1}, \ldots, \widehat{\Psi}''_{\alpha'_{l'}}] \longrightarrow {}^{+}\overline{\overline{\mathcal{E}}}{}^{\widehat{\mathcal{P}},\mathcal{Q}}_{s+1}(\mathcal{R}_\Theta, g_s)[\widehat{\Phi}''_{\beta'_1}, \ldots, \widehat{\Phi}''_{\beta'_{k'}}]$$

follows from:

> If, at (the end of) stage $s+1$, the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank of f_s above $\mathcal{R}_\Theta$ matches the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_s above $\mathcal{R}_\Theta$, then the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank of f_s above $\mathcal{R}_\Theta$ matches the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_s above $\mathcal{R}_\Theta$ at that stage.

•• And this follows from the fact that as soon as one encounters some $\hat{\alpha}$, and some

$$(\sigma, \tau) \in^{+} \overline{\overline{\varepsilon}}^{\mathcal{P}, \widehat{\mathcal{Q}}, \Phi_{\hat{\alpha}}}_{s+1, \hat{\alpha}}(\mathcal{R}_\Theta, f_s)[\widehat{\Psi}^{\natural}_{\alpha'_1}, \ldots, \widehat{\Psi}^{\natural}_{\alpha'_{l'}}],$$

with $\hat{\psi}(\sigma) \downarrow$, but $\hat{\psi}$ as yet undefined on τ some current level of the inductive definition of $\mathcal{P}[\widehat{\Psi}^{\natural}_{\alpha'_1}, \ldots, \widehat{\Psi}^{\natural}_{\alpha'_{l'}}], \widehat{\mathcal{Q}}$-preconfigured relative to f_s above $\mathcal{R}_\Theta$, with σ not inner $\mathcal{P}[\widehat{\Psi}'_{\alpha_1}, \ldots, \widehat{\Psi}'_{\alpha_l}], \widehat{\mathcal{Q}}$-preconfigured relative to f_s above $\mathcal{R}_\Theta$ at stage $s+1$, then (given the cancellation of hatted axioms via the background activity at stage $s+1$) one has freedom to (re)define $\hat{\psi}(\sigma)$ (respecting $\subseteq$ on strings) so as to enable at an index $\hat{\alpha}$ a suitable extension of the augmentation ${}^{+}\varepsilon^{\widehat{\mathcal{P}}, \mathcal{Q}}_{s+1}(\mathcal{R}_\Theta, g_s)[\widehat{\Phi}^{\natural}_{\beta'_1}, \ldots, \widehat{\Phi}^{\natural}_{\beta'_{k'}}]$ contributing to that underlying the definition of

$$\hat{\psi} : {}^{+}\overline{\overline{\varepsilon}}^{\mathcal{P}, \widehat{\mathcal{Q}}}_{s+1}(\mathcal{R}_\Theta, f_s)[\widehat{\Psi}''_{\alpha'_1}, \ldots, \widehat{\Psi}''_{\alpha'_{l'}}] \longrightarrow {}^{+}\overline{\overline{\varepsilon}}^{\widehat{\mathcal{P}}, \mathcal{Q}}_{s+1}(\mathcal{R}_\Theta, g_s)[\widehat{\Phi}''_{\beta'_1}, \ldots, \widehat{\Phi}''_{\beta'_{k'}}].$$

• And similarly for strings σ' at subsequent levels of the definition of $\mathcal{P}[\widehat{\Psi}^{\natural}_{\alpha'_1}, \ldots, \widehat{\Psi}^{\natural}_{\alpha'_{l'}}], \widehat{\mathcal{Q}}$-preconfigured relative to f_s above $\mathcal{R}_\Theta$. □

The $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank of f_s above each $\mathcal{R}_\Theta$ $*_s$-matches the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_s above $\mathcal{R}_\Theta$ at each stage $s+1$.

• As before, let stage $s'+1$ be the first at which the matching fails, with $s' = s+1$.

•• Again the argument falls into two main parts:

[**1**] That — unless *either* $s+1$ is reactive and phase (3), part I applies, relative to some $\mathcal{R}_\gamma$ at stage $s+1$, *or* $s+1$ is receptive and (5) applies relative to some $\mathcal{R}_\gamma$ at stage $s+1$ — then the inductive assumption of a $*_s$-matching at stage $s+1$ gives a $*_{s+1}$-matching at stage $s+1$, and hence a $*_{s'}$-matching at stage $s'+1$,

and —

[**2**] And otherwise the activity at stage $s+1$ is specifically directed to the implementation of an f_{s+1}, g_{s+1}-matching above $\mathcal{R}_\gamma$ which then becomes a $*_{s+1}$-matching at stage $s'+1$.

• Following which it is straightforward to verify that this matching above $\mathcal{R}_\gamma$ gives a similar $f_{s'}, g_{s'}$-matching above $\mathcal{R}_\Theta$.

•• <u>If stage $s+1$ is receptive</u>, and phase (5) of the module does not apply, one has $*_{s+1} \supset *_s$, in which case possible factors in a breakdown of the matching at stage $s+1$ are:

1. The implementation of new axioms for the unhatted functionals at stage $s+1$, or
2. The inclusion of new strings in dom$(*_{s+1})$ or range$(*_{s+1})$ at stage $s+1$, or
3. The new definitions for f_{s+1}, g_{s+1}, or
4. The appointment of new restraints on $*_{s+1}$.

• But if f_{s+1}, g_{s+1}, $*_{s+1}$ are defined $\supset f_s$, g_s, $*_s$, respy, via the background activity, this is done in such a way that the $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured ranks of f_{s+1}, g_{s+1}, respy, above $\mathcal{R}_\Theta$ at stage $s{+}1$ are the same as those of f_s, g_s, respy, at (the beginning of) stage $s+1$ —

- In which case, the assumed $*_s$-matching of the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_s above $\mathcal{R}_\Theta$ at stage $s+1$ by the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_s will give a $*_{s+1}$-matching of the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s+1} at stage $s+1$ by the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank of f_{s+1}.

- So only factors 1 or 4 — new axioms for unhatted functionals at stage $s+1$, or new restraints at stage $s+1$ — can block the modification of the $*_{s+1}$-matching ψ, say, at stage $s+1$ to one at stage $s'+1$ —

- And that cannot happen, since the implementation of such new axioms precedes phase (4), which in turn does not lead to (5) for any $\mathcal{R}$-requirement $\mathcal{R}_\gamma$ — while, as argued above, all new restraints which potentially block the modification of ψ are added within the matching process relative to the inner preconfigurations.

•• <u>If stage $s+1$ is reactive</u>, and phase (3), part I does not apply relative to some $\mathcal{R}_\gamma$ at stage $s+1$, one again has $*_{s+1} \supset *_s$, in which case possible factors in a breakdown of the matching at stage $s+1$ are:

1. The implementation of new axioms for the hatted functionals at stage $s+1$, or
2. The inclusion of new strings in dom$(*_{s+1})$ or range$(*_{s+1})$ at stage $s+1$, or
3. The new definitions for f_{s+1}, g_{s+1}.

•• Arguing as before, if $f_{s+1}, g_{s+1} \supseteq f_s, g_s$ respy, one need only focus on factor 1 —

• Which will be done in relation to the $\mathcal{R}_\gamma$ receiving attention at stage $s+1$.

• Say an axiom of the form '$\sigma = \widehat{\Psi}^{\tau}_{\alpha}$' causes a breakdown of the matching at the end of stage $s+1$.

• Then this only occurs via the implementation of '$\sigma = \widehat{\Psi}^{\tau}_{\alpha}$', following stipulation, at stage $s+1$ —

• And since '$\sigma = \widehat{\Psi}^{\tau}_{\alpha}$' is on the true path at stage $s+1$, it will be covered by the $*_{s+1}$-matching of the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank of f_{s+1} and the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s+1} above $\mathcal{R}_{\gamma}$, at the start of stage $s+1$ — so presenting no obstacle to such a $*_{s+1}$-matching at the end of stage $s+1$.

• One can argue similarly for axioms of the form '$\sigma = \widehat{\Phi}^{\tau}_{\beta}$' newly implemented at stage $s+1$.

•• <u>And</u> if (3), part II, applies at stage $s+1$, then the $*_s$-matching of the $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured ranks above $\mathcal{R}_{\gamma}$ of f_s, g_s, respy, at stage $s+1$, gives one relative to $f_s \restriction (x-1)$, $(f_s \restriction x-1)^{*_s}$ —

• Where this is because *either* $x = |f_s|$, and $f_s \restriction (x-1) = f_s$, with the required matching given by ψ, *or* there was some earlier reactive stage $t+1 < s+1$ at which $\mathcal{R}_\gamma$ required attention at stage $t+1$ via (3) part II, and $f_t = f_t \restriction (x-1) = f_s \restriction (x-1)$ —

• And in the latter case, at no receptive stage $t'+1$, $t+1 < t'+1 < s+1$, does the inner $\mathcal{P}, \widehat{\mathcal{Q}}$- or $\widehat{\mathcal{P}}, \mathcal{Q}$- preconfigured rank of f_s or g_s, respy, above $\mathcal{R}_\gamma$ at stage $s+1$ appreciate, giving (by the restraints on $*_s$) inner $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured ranks of f_s, g_s, respy, above $\mathcal{R}_\gamma$ at stage $s+1$ = to those of $f_t = f_s \restriction (x-1)$, $(f_s \restriction x-1)^{*_s}$, respy.

• So — by a similar argument to that above extending matching of inner preconfigurations to full preconfigurations — and by choice of f_{s+1}, g_{s+1} via (3), part II, again get an appropriate $*_s$-, and hence $*_{s+1}$-, matching relative to f_{s+1}, g_{s+1} at the end of stage $s+1$, contradicting the choice of s'.

•• Assume now that $\mathcal{R}_\gamma$ requires attention at stage $s+1$ through subcase I of (3).

• Then for appropriate $i \neq j$, at stage $s+1$ the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\gamma$ of f_{s_i} matches the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\gamma$.

• And then, according to parts (a) and (b) of the consequent action, f_{s+1}, g_{s+1} and $*_{s+1}$ are defined in such a way (guided by the appropriate f_{s_i}, g_{s_j}-matching) that at stage $s+1 = s'$ the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\gamma$ of f_{s+1} $*_{s+1}$-matches the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s+1} above $\mathcal{R}_\gamma$.

• Arguing as previously, one can verify that this $*_{s'+1}$-matching is not disturbed by new hatted axioms defined at stage $s+1$, so that we again have a $*_{s'}$-matching of the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank of $f_{s'}$ above $\mathcal{R}_\gamma$ by the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of $g_{s'}$ above $\mathcal{R}_\gamma$ at the end of stage s'.

•• While if $s+1$ is receptive, and $\mathcal{R}_\gamma$ requires attention at stage $s+1$ through part (5), the argument is very similar.

• The $*_{s+1}$-matching of the $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured ranks above $\mathcal{R}_\gamma$ of f_{s+1}, g_{s+1}, respy, is again obtained at stage $s+1$ via an implementation of a matching ψ', say, —

• Which again survives the possible addition of new restraints at stage $s+1$.

•• <u>And finally</u> — it follows that for $\mathcal{R}_\Theta$ of priority greater than or equal to that of $\mathcal{R}_\gamma$, this matching still holds with $\mathcal{R}_\Theta$ in place of $\mathcal{R}_\gamma$ —

•• And, by the initialising of functionals $\widehat{\Phi}_\beta$ or $\widehat{\Psi}_\alpha$ being built by $\mathcal{P}$- or $\mathcal{Q}$- requirements of lower priority than that of $\mathcal{R}_\gamma$, and by the routine extension of $*$, $\widehat{\Phi}_\beta$ and $\widehat{\Psi}_\alpha$ at stage $s+1$, the matching also holds for lower priority (than $\mathcal{R}_\gamma$) $\mathcal{R}_\Theta$ in place of $\mathcal{R}_\gamma$. □

A useful consequence:

All the hatted functionals are consistent.

•• This is because — Each axiom for a hatted functional, $\widehat{\Psi}_\alpha$ say, implemented at a reactive stage $s+1$ derives from an axiom stipulated for it, which is currently on a true path —

• Hence — Any inconsistency involving a newly implemented axiom for $\widehat{\Psi}_\alpha$ must be in relation to an existing axiom $\sigma = \widehat{\Psi}^\tau_\alpha$, say, for $\widehat{\Psi}_\alpha$ —

• And which consequently must also be on a true path.

• But this — by the existence of the f_{s+1}, g_{s+1}-matching at stage $s+2$ — would contradict the consistency of $\Psi_{\bar{\alpha}}$. $\square$

For each requirement $\mathcal{R}_\Theta$, if $\mathcal{R}_\Theta$ is finitely injured then it is finitely injuring — and consequently all the $\mathcal{R}$-, $\mathcal{K}$-, $\mathcal{L}$-, $\mathcal{M}$- and $\mathcal{N}$-requirements are satisfied.

- Assume that each $\mathcal{R}_\gamma$ above $\mathcal{R}_\Theta$ is finitely injuring, and let $\widetilde{s}$ be s.t. no $\mathcal{R}_\gamma$ above $\mathcal{R}_\Theta$ receives attention at a stage $s+1 > \widetilde{s}$.

- So all restraints and protected arguments of priority greater than that of $\mathcal{R}_\Theta$ originate prior to stage $\widetilde{s}$.

•• <u>Then</u> — extending the earlier argument — show: (a) $\mathcal{R}_\Theta$ *can only require attention finitely often in relation to a given potential witness* x (appointed at a stage $s_x + 1 > \widetilde{s}$), and (b) $\mathcal{R}_\Theta$ *can only require attention through finitely many potential witnesses* x.

- — and to do this, one needs to verify the following facts:

(A) There are only finitely many isomorphism types possible for those stratified inner $\mathcal{P}[\widehat{\Psi}'_{\alpha_1}, \ldots, \widehat{\Psi}'_{\alpha_l}], \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}[\widehat{\Phi}'_{\beta_1}, \ldots, \widehat{\Phi}'_{\beta_k}]$-preconfigured ranks of each f_{s_j} and g_{s_j}, respy, to be matched above $\mathcal{R}_\Theta$, from which it follows that there is a limiting value for the isomorphism types corresponding to the inner $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured ranks of f_t, g_t, respy, above $\mathcal{R}_\Theta$ at stages $t+1 > s_x+1$.

(B) At all stages $s+1 \geq s_i+1, s_j+1$ at which $\mathcal{R}_\Theta$ requires attention via (3), the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$, each $i, j \geq 1$.

(C) For all $i, j \geq 1$ — if at stage $s+1 > s_x+1$ the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$ at stage $s+1$, then the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$ at stage $s+1$.

There are only finitely many isomorphism types possible for those stratified inner $\mathcal{P}[\widehat{\Psi}'_{\alpha_1}, \ldots, \widehat{\Psi}'_{\alpha_l}], \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}[\widehat{\Phi}'_{\beta_1}, \ldots, \widehat{\Phi}'_{\beta_k}]$-preconfigured ranks of each f_{s_j} and g_{s_j}, respy, to be matched above $\mathcal{R}_\Theta$ ...

• <u>Remember</u> — A typical stratified inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank of a string π above $\mathcal{R}_\Theta$ at stage $s+1$, corresponding to an augmentation $\varepsilon_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, \pi)[\widehat{\Psi}'_{\beta_1}, \ldots, \widehat{\Psi}'_{\beta_l}]$ of $\varepsilon_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, \pi)$, is of the form

$$\overline{\overline{\varepsilon}}_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, \pi)[\widehat{\Psi}'_{\beta_1}, \ldots, \widehat{\Psi}'_{\beta_l}] =$$

$$(\overline{\overline{\varepsilon}}_{s,\alpha_1}^{\mathcal{P},\widehat{\mathcal{Q}},\Phi_{\alpha_1}}(\mathcal{R}_\Theta, \pi)[\widehat{\Psi}'_{\beta_1}, \ldots, \widehat{\Psi}'_{\beta_l}], \ldots$$

$$\ldots, \overline{\overline{\varepsilon}}_{s,\alpha_k}^{\mathcal{P},\widehat{\mathcal{Q}},\Phi_{\alpha_k}}(\mathcal{R}_\Theta, \pi)[\widehat{\Psi}'_{\beta_1}, \ldots, \widehat{\Psi}'_{\beta_l}],$$

$$\overline{\overline{\varepsilon}}_{s,\beta_1}^{\mathcal{P},\widehat{\mathcal{Q}},\widehat{\Psi}'_{\beta_1}}(\mathcal{R}_\Theta, \pi)[\widehat{\Psi}'_{\beta_1}, \ldots, \widehat{\Psi}'_{\beta_l}], \ldots$$

$$\ldots, \overline{\overline{\varepsilon}}_{s,\beta_l}^{\mathcal{P},\widehat{\mathcal{Q}},\widehat{\Psi}'_{\beta_l}}(\mathcal{R}_\Theta, \pi)[\widehat{\Psi}'_{\beta_1}, \ldots, \widehat{\Psi}'_{\beta_l}])$$

where —

- The lists

$$\overline{\overline{\varepsilon}}^{\mathcal{P},\widehat{\mathcal{Q}},\Phi_{\alpha_1}}_{s,\alpha_1}(\mathcal{R}_\Theta,\pi)[\widehat{\Psi}'_{\beta_1},\dots,\widehat{\Psi}'_{\beta_l}],\dots$$
$$\dots,\overline{\overline{\varepsilon}}^{\mathcal{P},\widehat{\mathcal{Q}},\Phi_{\alpha_k}}_{s,\alpha_k}(\mathcal{R}_\Theta,\pi)[\widehat{\Psi}'_{\beta_1},\dots,\widehat{\Psi}'_{\beta_l}],$$
$$\overline{\overline{\varepsilon}}^{\mathcal{P},\widehat{\mathcal{Q}},\widehat{\Psi}'_{\beta_1}}_{s,\beta_1}(\mathcal{R}_\Theta,\pi)[\widehat{\Psi}'_{\beta_1},\dots,\widehat{\Psi}'_{\beta_l}],\dots$$
$$\dots,\overline{\overline{\varepsilon}}^{\mathcal{P},\widehat{\mathcal{Q}},\widehat{\Psi}'_{\beta_l}}_{s,\beta_l}(\mathcal{R}_\Theta,\pi)[\widehat{\Psi}'_{\beta_1},\dots,\widehat{\Psi}'_{\beta_l}]$$

comprise all such sets of higher priority than $\mathcal{R}_\Theta$, where —

- Any $\overline{\overline{\varepsilon}}^{\mathcal{P},\widehat{\mathcal{Q}},\Xi_\alpha}_{s,\alpha}(\mathcal{R}_\Theta,\pi)[\widehat{\Psi}'_{\beta_1},\dots,\widehat{\Psi}'_{\beta_l}]$ (a typical component) is the set of all $\subseteq$-maximal members of $\overline{\varepsilon}^{\mathcal{P},\widehat{\mathcal{Q}},\Xi_\alpha}_{s,\alpha}(\mathcal{R}_\Theta,\pi)[\widehat{\Psi}'_{\beta_1},\dots,\widehat{\Psi}'_{\beta_l}]$, and where —

- The sets $\varepsilon^{\mathcal{P},\widehat{\mathcal{Q}},\Phi_{\alpha_i}}_{s,\alpha_i}(\mathcal{R}_\Theta,\pi)[\widehat{\Psi}'_{\beta_1},\dots,\widehat{\Psi}'_{\beta_l}]$ and $\varepsilon^{\mathcal{P},\widehat{\mathcal{Q}},\widehat{\Psi}'_{\beta_j}}_{s,\beta_j}(\mathcal{R}_\Theta,\pi)[\widehat{\Psi}'_{\beta_1},\dots,\widehat{\Psi}'_{\beta_l}]$ consist of all (σ,τ) with σ inner $\mathcal{P}[\Phi'_{\alpha_1},\dots,\Phi'_{\alpha_k}],\widehat{\mathcal{Q}}[\widehat{\Psi}'_{\beta_1},\dots,\widehat{\Psi}'_{\beta_l}]$-preconfigured relative to π above $\mathcal{R}_\Theta$ via τ at stage $s+1$ according to $\langle \alpha_i\restriction_{\geq i'}\rangle$ or $\langle \beta_j\restriction_{\geq j'}\rangle$, respy, some basis $\Phi'_{\alpha_1},\dots,\Phi'_{\alpha_k}$ for a potential true path relative to π at stage $s+1$, —

- Where $\sigma = \Phi^{\tau}_{\alpha_i}[s]$, or $\sigma = \widehat{\Psi}'^{\tau}_{\beta_j}[s]$, with τ $\mathcal{P}[\Phi'_{\alpha_1}, \ldots, \Phi'_{\alpha_k}], \widehat{\mathcal{Q}}[\widehat{\Psi}'_{\beta_1}, \ldots, \widehat{\Psi}'_{\beta_l}]$-preconfigured relative to π above $\mathcal{R}$ at stage $s+1$, —

- *Either* according to $\langle \alpha_i^- \restriction_{\geq i'} \rangle$ or $\langle \beta_j^- \restriction_{\geq j'} \rangle$, respy, some i' or $j' \in \omega$, with, correspondingly, α_i^- or β_j^- $\mathcal{P}_{\alpha_i}$- or $\mathcal{Q}_{\beta_j}$-relevant and α_i or β_j configuring —

- *Or* according to some $\langle \alpha \restriction_{>i'} \rangle$ or $\langle \beta \restriction_{>j'} \rangle$, respy, with, correspondingly, $\alpha \restriction_{<i'}$ or $\beta \restriction_{<j'}$ being $\mathcal{P}_{\alpha_i}$- or $\mathcal{Q}_{\beta_j}$-relevant and α_i or β_j non-configuring, and —

(∗) τ is inner $\mathcal{P}, \widehat{\mathcal{Q}}[\widehat{\Psi}'_{\beta_1}, \ldots, \widehat{\Psi}'_{\beta_l}]$-preconfigured relative to π above $\mathcal{R}_\Theta$ at stage $s+1$ according to some $\langle \hat{\alpha}^- \restriction_{\geq j''} \rangle$, where the corresponding basis $\Phi''_{\alpha_1}, ..., \Phi''_{\alpha_k}$ of the inner $\mathcal{P}, \widehat{\mathcal{Q}}[\widehat{\Psi}'_{\beta_1}, \ldots, \widehat{\Psi}'_{\beta_l}]$-preconfiguration of each π', with index $\preccurlyeq \alpha$, occurring in the definition to τ is compatible with $\Phi'_{\alpha_1}, \ldots, \Phi'_{\alpha_k}$.

- From this, one can verify —

SUBLEMMA Over all $\gamma =$ some such α_i or β_j one has that at any stage $s+1$ the set of all corresponding pairs (σ, τ) contributing to such typical stratified inner $\mathcal{P}[\Phi'_{\alpha_1}, \dots, \Phi'_{\alpha_k}]$, $\widehat{\mathcal{Q}}[\widehat{\Psi}'_{\beta_1}, \dots, \widehat{\Psi}'_{\beta_l}]$-preconfigured ranks has a uniform bound determined by the number of restraints and levels of protected arguments above $\mathcal{R}_\Theta$, and of the potential outcomes at each node of higher priority than that of $\mathcal{R}_\Theta$.

•• Proof by induction —

• Assume given $\gamma = \alpha_i$ or β_j as above.

•• First notice —

There are at most finitely many levels of a given path of the inductive definition of inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfiguration within which pairs can enter either $\varepsilon_{s,\alpha_i}^{\mathcal{P},\widehat{\mathcal{Q}},\Phi_{\alpha_i}}(\mathcal{R}_\Theta, \pi)[\widehat{\Psi}'_{\beta_1}, \dots, \widehat{\Psi}'_{\beta_l}]$ *or* $\varepsilon_{s,\beta_j}^{\mathcal{P},\widehat{\mathcal{Q}},\widehat{\Psi}'_{\beta_j}}(\mathcal{R}_\Theta, \pi)[\widehat{\Psi}'_{\beta_1}, \dots, \widehat{\Psi}'_{\beta_l}]$.

• This is because —

•• <u>At level 1</u>, pairs are only contributed via the $\mathcal{P}, \widehat{\mathcal{Q}}$-configurations above $\mathcal{R}_\Theta$ at a stage $s+1$ according to some α' of higher priority than that of $\mathcal{R}_\Theta$, relative to either π, or a string restrained above $\mathcal{R}_\Theta$, at stage $s+1$ —

•• <u>And say</u> (σ, τ) is newly contributed at level $n+1$, say, > 1 via some τ_0 —

•• <u>Then</u> — no new (σ', τ') can be contributed — due to σ' being inner $\mathcal{P}[\Phi''_{\alpha_1}, \ldots, \Phi''_{\alpha_k}]$, $\widehat{\mathcal{Q}}[\widehat{\Psi}'_{\beta_1}, ..., \widehat{\Psi}'_{\beta_l}]$-preconfigured relative to π above $\mathcal{R}_\Theta$ at stage $s+1$ according to any $\langle \alpha_i \restriction_{\geq i''} \rangle$ or $\langle \beta_j \restriction_{\geq j''} \rangle$, respy, any basis $\Phi''_{\alpha_1}, \ldots, \Phi''_{\alpha_k}$ for a potential true path relative to π at stage $s+1$ — via a τ'' appearing in a pair (σ'', τ'') contributed at a point on a path of the inductive defn. of inner $\mathcal{P}, \widehat{\mathcal{Q}}[\widehat{\Psi}'_{\beta_1}, \ldots, \widehat{\Psi}'_{\beta_l}]$-preconfiguring below that at which τ occurred —

• Since otherwise, the functionals $\Phi''_{\alpha_1}, \ldots, \Phi''_{\alpha_k}$ could not be compatible with the relevant axioms from the basis $\Phi'_{\alpha_1}, \ldots, \Phi'_{\alpha_k}$. $\square$

• But this means that the length of any given path of the defn. of inner $\mathcal{P}, \widehat{\mathcal{Q}}[\widehat{\Psi}'_{\beta_1}, \ldots, \widehat{\Psi}'_{\beta_l}]$-preconfigured relative to π above $\mathcal{R}_\Theta$ is limited by the number of indices of the form $\alpha_{i'}$ or $\beta_{j'}$.

• So as required, there is a bound on the lengths of the paths of the relevant inductive definition of inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured, where —

•• *This bound is determined by the number and character of the requirements of higher priority than that of* $\mathcal{R}_\Theta$ — and so — *is uniform over all stages* $s+1>0$.

•• Also —

The number of strings figuring in the relevant stratifications at a particular branching of the inductive definition any one stage is uniformly bounded.

• This is because —

- The strings added at level $n+1$ of the induction can only arise from downward Turing relationships from previously included strings —

- Those new strings arising from entries taken from the finite number of strings of outcomes of higher priority than that of $\mathcal{R}_{\Theta}$ —

- Any one such entry at a particular branch giving rise (by consistency) to a single new stratified string. $\Box$

- The first part of (A) follows immediately from the sublemma, in relation to the possible isomorphism types of the stratified inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured ranks of each f_{s_j}, to be matched (relative to the corresponding augmentations) above $\mathcal{R}_{\Theta}$.

- A similar argument applies to the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured ranks (again relative to the corresponding augmentations). $\Box$

… from which it follows that there is a limiting value for the isomorphism types corresponding to the inner $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured ranks of f_t, g_t, respy, above $\mathcal{R}_\Theta$ at stages $t+1 > s_x + 1$.

•• <u>Assume</u> that at some (least) stage $t+1 > \widetilde{s}$ one has

$$\overline{\overline{\varepsilon}}_{t+1}^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{t+1}) \not\cong \overline{\overline{\varepsilon}}_{t}^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_t).$$

Then $t+1$ is receptive, and

$$\overline{\overline{\varepsilon}}_{t-1}^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{t-1}) \not\cong \overline{\overline{\varepsilon}}_{t+1}^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{t+1}).$$

• <u>And hence</u> —

•• $\varepsilon_t^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_t)$ appreciates at stage $t+1$, involving a restraint on f_{u+1} at all stages $u+1 > t+1$ —

• Where — any such appreciation must involve the identification, and subsequent preservation, of an acretion of new incompatibilities of strings within the structure of the stratified inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigurations relative to f_t above $\mathcal{R}_\Theta$ at stage $t+1$, arising from new axioms for hatted or unhatted functionals at stage $t+1$.

•• Then — No such acretion can be reversed at a later stage —

• Since by choice of $\widetilde{s}$, and the activity at later stages, the restraint of $f_{t+1} \subseteq f_{u+1}$, all $u+1 \geq t+1$, is permanent.

• But — by the first part of (A) — one cannot have an infinite sequence of distinct isomorphism types for $\overline{\overline{\varepsilon}}_{u+1}^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{u+1})$, $u+1 \geq t+1$. $\square$

At all stages $s+1 \geq s_i+1, s_j+1$ at which $\mathcal{R}_\Theta$ requires attention via (3), the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$, each $i, j \geq 1$.

• Similarly to the argument for f_s, g_s, it follows that —

At all stages $s+1 \geq s_j+1$ at which $\mathcal{R}_\Theta$ requires attention via (3), the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_j} matches the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$, each $j \geq 1$.

•• Inductively assume (B) at stage s —

• Moreover — by the way in which the hatted axioms are implemented at stage $s+1$, one can assume that at the last previous stage at which appreciation of the inner preconfigured rank occurred relative to $\mathcal{R}_\Theta$, all relevant axioms from the augmentation involved had corresponding axioms for hatted functionals implemented —

• Making all such matchings at later stages *augmentation independent*, so long as (B) is maintained.

•• But this means that any breakdown in (B) at stage $s+1$ must involve appreciation (since no augmentation can provide such a breakdown independently of any new axioms implemented on a true path) —

• So precluding $\mathcal{R}_\Theta$ from requiring attention via (3) at stage $s+1$. $\square$

For all $i, j \geq 1$ — if at stage $s+1 > s_x + 1$ the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$ at stage $s+1$, then the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$ at stage $s+1$.

• Let ψ be an f_{s_i}, g_{s_j}-matching of the corresponding inner $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$- preconfigured ranks above $\mathcal{R}_\Theta$ at stage $s+1 > s_x + 1$ —

• Where, say, ψ is an isomorphism

$$\overline{\overline{\varepsilon}}_s^{\mathcal{P}, \widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{s_i})[\widehat{\Psi}'_{\alpha_1}, \ldots, \widehat{\Psi}'_{\alpha_l}] \longrightarrow \overline{\overline{\varepsilon}}_s^{\widehat{\mathcal{P}}, \mathcal{Q}}(\mathcal{R}_\Theta, g_{s_j})[\widehat{\Phi}'_{\beta_1}, \ldots, \widehat{\Phi}'_{\beta_k}].$$

•• Inductively define an f_{s_i}, g_{s_j}-matching $\widetilde{\psi}$ of the full $\mathcal{P}, \widehat{\mathcal{Q}}$- and $\widehat{\mathcal{P}}, \mathcal{Q}$- preconfigured ranks above $\mathcal{R}_\Theta$ at stage $s+1$, with corresponding augmentations ${}^{+}\varepsilon_s^{\mathcal{P}, \widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{s_i})[\widehat{\Psi}''_{\alpha'_1}, \ldots, \widehat{\Psi}''_{\alpha'_{l'}}]$, ${}^{+}\varepsilon_s^{\widehat{\mathcal{P}}, \mathcal{Q}}(\mathcal{R}_\Theta, g_{s_j})[\widehat{\Phi}''_{\beta'_1}, \ldots, \widehat{\Phi}''_{\beta'_{k'}}]$ —

•• <u>Where</u> — *The inductive definition of* $\widetilde{\psi}$ *closely follows that of* $\hat{\psi}$ *in the proof of* (✠) *above.*

•• <u>Specifically</u> — First, define $\widetilde{\psi} = \psi$ on those pairs of strings chosen from the stratified inner $\mathcal{P}[\widehat{\Psi}'_{\alpha_1}, \ldots, \widehat{\Psi}'_{\alpha_l}], \widehat{\mathcal{Q}}$-preconfigured rank of f_{s_i} above $\mathcal{R}_\Theta$ at stage $s+1$ —

• And with those axioms of $\widehat{\Psi}^\natural_{\alpha'_1}, \ldots \;\; \ldots, \widehat{\Psi}^\natural_{\alpha'_{l'}}$ defined on such pairs being those corresponding axioms of $\widehat{\Psi}'_{\alpha_1}, \ldots, \widehat{\Psi}'_{\alpha_l}$, where —

• One writes

$$+\varepsilon^{\mathcal{P},\widehat{\mathcal{Q}}}_{s+1}(\mathcal{R}_\Theta, f_{s_i})[\widehat{\Psi}^\natural_{\alpha'_1}, \ldots, \widehat{\Psi}^\natural_{\alpha'_{l'}}],$$
$$+\varepsilon^{\widehat{\mathcal{P}},\mathcal{Q}}_{s+1}(\mathcal{R}_\Theta, g_{s_j})[\widehat{\Phi}^\natural_{\beta'_1}, \ldots, \widehat{\Phi}^\natural_{\beta'_{k'}}],$$

for the current inductively defined values of the corresponding augmentations, where each $\widehat{\Psi}^\natural_{\alpha'_i}, \widehat{\Phi}^\natural_{\beta'_j}$ will be $\subseteq$ the eventual $\widehat{\Psi}''_{\alpha'_i}, \widehat{\Phi}''_{\beta'_j}$, respy.

• <u>And then</u> —

• Assume that at some level of the inductive definition of $\mathcal{P}[\widehat{\Psi}^{\natural}_{\alpha_1'}, \ldots, \widehat{\Psi}^{\natural}_{\alpha_{l'}'}], \widehat{\mathcal{Q}}$-preconfigured relative to f_{s_i} above $\mathcal{R}_{\Theta}$ one encounters some $\widetilde{\alpha}$, and some

$$(\tau, \sigma) \in^{+} \overline{\overline{\varepsilon}}^{\mathcal{P}, \widehat{\mathcal{Q}}, \Phi_{\hat{\alpha}}}_{s+1, \hat{\alpha}}(\mathcal{R}_{\Theta}, f_{s_i})[\widehat{\Psi}^{\natural}_{\alpha_1'}, \ldots, \widehat{\Psi}^{\natural}_{\alpha_{l'}'}],$$

with *either* $\widetilde{\psi}(\tau)\downarrow$, but $\widetilde{\psi}$ as yet undefined on σ, *or* $\widetilde{\psi}(\sigma)\downarrow$, but $\widetilde{\psi}$ as yet undefined on τ —

•• Then — noting that the latter does not occur, by the defn. of inner preconfiguration and the cancellation of axioms for hatted functionals at stage $s+1$ —

•• Can complete the defn. of $\widetilde{\psi}$, using the fact that if σ is not inner $\mathcal{P}[\widehat{\Psi}'_{\alpha_1}, \ldots, \widehat{\Psi}'_{\alpha_l}], \widehat{\mathcal{Q}}$-preconfigured relative to f_{s_i} above $\mathcal{R}_{\Theta}$ at stage $s+1$, then (given the cancellation of hatted axioms via the background activity at stage $s+1$) one has freedom to define $\widetilde{\psi}(\sigma)$, so as to enable at index $\hat{\alpha}$ a suitable extension of the augmentation ${}^{+}\varepsilon^{\widehat{\mathcal{P}}, \mathcal{Q}}_{s+1}(\mathcal{R}_{\Theta}, g_{s_j})[\widehat{\Phi}^{\natural}_{\beta_1'}, \ldots, \widehat{\Phi}^{\natural}_{\beta_{k'}'}]$. $\square$

●● To get a contradiction to (a) — assume $\mathcal{R}_\Theta$ to require attention infinitely often in relation to some potential witness x —

● Where x is appointed through (1) at the reactive stage $s_x + 1 > \widetilde{s}$ — and if cancelled is never reappointed.

● Then — having required attention via (2) in relation to x, $\mathcal{R}_\Theta$ cannot again require attention via (2) for x, without a prior application of (3), part II for x —

● Other than through the appreciation of a prepared inner preconfiguration at $\mathcal{R}_\Theta$ — and the cancellation of x.

●● Aim — To get a contradiction, by verifying the existence of a stage $s + 1$ at which $\mathcal{R}_\Theta$ satisfies the conditions for attention through (3), part I in relation to x, resulting in the implementation of a matching ψ which needs to be implemented by $\mathcal{R}_\Theta$ at all later stages:

• <u>That is</u> — a stage $s+1$ at which there exist distinct $i, j \geq 1$, such that the $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$ at stage $s+1$, with a maximal γ-state, and where no ψ-related potential improvement is ever incorporated into a current $(f, g)_{s_{i_{r_u}}}$.

•• <u>First</u> — Assume that $\mathcal{R}_\Theta$ requires attention at infinitely many stages through (3), subcase II, and —

• Let $g\lceil \vartheta(x)[s_i]$, $i \geq 1$, be a list of all strings realised in relation to x, where, by choice of x, no such realised string is ever cancelled.

• Then by (B), at all stages $s+1 \geq s_i+1, s_j+1$ at which $\mathcal{R}_\Theta$ requires attention via (3), the inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured rank above $\mathcal{R}_\Theta$ of f_{s_i} matches the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured rank of g_{s_j} above $\mathcal{R}_\Theta$, each $i, j \geq 1$.

• And then (C) gives a full f_{s_i}, g_{s_j}-matching, each $i, j \geq 1$.

• From which it follows that, at sufficiently large stages, there exist arbitrarily large suitably $\overrightarrow{\gamma}^0, \overrightarrow{\gamma}^1$-acceptable subsequences of $(f,g)_{s_i}$ —

• Giving an improvable $(f,g)_{s_{i_r}}$ at some stage $s+1 > s_x+1$ — so excluding $\mathcal{R}_\Theta$ from attention via (3) part II at any subsequent stage.

•• <u>But then</u> — Having required attention via (3), part I, in relation to x, $\mathcal{R}_\Theta$ cannot again require attention via (3), part I for x, unless its corresponding ψ ceases to need implementing by $\mathcal{R}_\Theta$ —

• Which can only happen if some ψ-related potential improvement of the current acceptable sequence $(f,g)_{s_{i_{\cdot r}}}$, say, appears, giving rise to a revision of a current potential improvement of the form $(f,g)_{s_{i_{\cdot r_u}}}$.

- There are only finitely many such $(f,g)_{s_{i\cdot r_u}}$, and each can only be enlarged finitely often —

- *Either* resulting in an actual improvement of the current acceptable sequence — *or* leading to the exhaustion of the available part of $(f,g)_{s_{i\cdot r}}$.

- In the former case, the bound on the number of available γ-states above $\mathcal{R}_\Theta$ providing one on the possible number of improvements to the current acceptable sequence —

- And the latter possibility being excluded by the inductive assumption that $(f,g)_{s_{i\cdot r}}$ is of length $\geq n'!\,3^{n'}$ —

- Since there are n' successors of $(\overrightarrow{\gamma}^0_{s_{i\cdot r}}, \overrightarrow{\gamma}^1_{s_{i\cdot r}})$ of the form $(\overrightarrow{\gamma}^0, \overrightarrow{\gamma}^1) = (\overrightarrow{\gamma}^0_{s_{i\cdot r}} - \beta, \overrightarrow{\gamma}^1_{s_{i\cdot r}})$ or $(\overrightarrow{\gamma}^0_{s_{i\cdot r}}, \overrightarrow{\gamma}^1_{s_{i\cdot r}} - \alpha)$, with α, β nonconfiguring $\mathcal{P}$- or $\mathcal{Q}$-nodes, respy, above $\mathcal{R}_\Theta$ —

• And any revision through (3), part I, of a potential improvement of $(f,g)_{s_{i \cdot r}}$ must entail one of a current $(f,g)_{s_{i \cdot r_u}}$ which is appropriately $\overrightarrow{\gamma}^0, \overrightarrow{\gamma}^1$-acceptable.

• Then — each revision of such a potential improvement arises from the implementation of a ψ which detracts at most two pairs from the available part of $(f,g)_{s_{i \cdot r}}$ —

• So that there will be a current potential improvement $(f,g)_{s_{i \cdot r_u}}$, following the exhaustion of the available part of $(f,g)_{s_{i \cdot r}}$, of size at least $(n'!\,3^{n'})/3n' = (n'-1)!\,3^{n'-1}$.

•• Finally — assuming $\mathcal{R}_\Theta$ to require attention via (5) infinitely often — one first observes that no application of (4) to leading to (5) can involve an appreciation in $\overline{\overline{\overline{\varepsilon}}}_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_s)$ or $\overline{\overline{\overline{\varepsilon}}}_s^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_\Theta, g_s)$ at stage $s+1 > s_x+1$, since this would then be retained via (2), leading to cancellation of x.

• Assume that $s' + 1 > s_x + 1$ is such that $\mathcal{R}_\Theta$ requires attention through (3) at no stage $t+1 > s'+1$, and let ψ be the matching needed to be implemented by $\mathcal{R}_\Theta$ at all such stages $t+1$.

• Then — arguing as for module 3 — sufficient applications of (5) at such stages will eventually lead to a potential improvement of $(f, g)_{s_{i \cdot r}}$, giving rise to a revision of some $(f, g)_{s_{i \cdot r_u}}$ via part (3).

•• So — as required — $\mathcal{R}_\Theta$ can only require attention finitely often in relation to a given x.

•• On the other hand — By the choice of $\widetilde{s}$, the appointment — and cancellation — of infinitely many potential witnesses for $\mathcal{R}_\Theta$, entails $\mathcal{R}_\Theta$ requiring attention at infinitely many stages via (2), or via (5) in relation to some $f_{s_i}, g_{s_i} \not\subseteq f_s, g_s$.

- But if an application of (4) to leads to $\mathcal{R}_\Theta$ requiring attention via (5) at some stage $s+1$, resulting in the cancellation of some potential witness x for $\mathcal{R}_\Theta$, one needs an appreciation in some $\overline{\overline{\mathcal{E}}}_s^{\mathcal{P},\widehat{\mathcal{Q}}}(\mathcal{R}_\Theta, f_{s_i})$ or $\overline{\overline{\mathcal{E}}}_s^{\widehat{\mathcal{P}},\mathcal{Q}}(\mathcal{R}_\Theta, g_{s_i})$ at stage $s+1$, and an implementation of an f_{s_i}, g_{s_i}-matching ψ, say, above $\mathcal{R}_\Theta$ as in (3) part I.

- And by (B), this appreciation is retained via (2) as a new prepared inner preconfiguration.

•• <u>So</u> — assuming $\mathcal{R}_\Theta$ to be finitely injured — Only infinitely many definitions of new prepared inner preconfigurations via (2), can result in infinitely many potential witnesses for $\mathcal{R}_\Theta$.

- Result follows by (A) if one can be sure that no appreciation in such rank is subsequently reversed, other than through activity on higher priority $\mathcal{R}$-requirements.

- But arguing as before, the addition of new axioms (hatted or unhatted) can only lead to further appreciations.

•• So — as required — If a requirement $\mathcal{R}_\Theta$ is finitely injured then it is finitely injuring.

• And it immediately follows (remembering from the above that an $\mathcal{R}_\Theta$-requirement can only nontrivially complete its cycle of attentions with an implementation of an f_{s_i}, g_{s_j}-matching, $i \neq j$) that all the $\mathcal{R}$-, $\mathcal{K}$-, $\mathcal{L}$-, $\mathcal{M}$- and $\mathcal{N}$-requirements are satisfied. $\square$

The limits $*$, f, g of $\{*_s\}_{s\in\omega}$, $\{f_s\}_{s\in\omega}$, $\{g_s\}_{s\in\omega}$ exist, and there is a bijective $\widetilde{*}$ induced by $*$, with $f^{\widetilde{*}} = g$.

•• The existence of the limit $*$ of $\{*_s\}_{s\in\omega}$ immediately follows from the satisfaction of the $\mathcal{K}$-, $\mathcal{L}$-, $\mathcal{M}$- and $\mathcal{N}$-requirements.

•• The existence of $\mathrm{Lim}_s\, f_s(z)$, $\mathrm{Lim}_s\, g_s(z)$ for each z follows from the construction, and the finiteness of injury of each $\mathcal{R}_\Theta$.

• And — given the bijective $\widetilde{*} : \omega^\omega \longrightarrow \omega^\omega$ induced by $*$, it will follow that $\mathrm{Lim}_s\, g_s = f^{\widetilde{*}}$.

•• <u>It remains to be shown</u> that there is a bijective mapping $\widetilde{*} : \omega^\omega \longrightarrow \omega^\omega$ which satisfies

$$H^{\widetilde{*}} = G \Leftrightarrow (\exists\{(\sigma_i, \tau_i)\}_{i\in\omega} \subset *)(\forall i) \\ [\sigma_i \subset \sigma_{i+1} \subset H \,\&\, \tau_i \subset \tau_{i+1} \subset G] \qquad (\ddagger)$$

for all $H, G \in \omega^\omega$ —

|| $\{(\sigma_i, \tau_i)\}_{i\in\omega} \subset *$ is said to be a $*$-*tower* iff $(\forall i)[\sigma_i \subset \sigma_{i+1} \,\&\, \tau_i \subset \tau_{i+1}]$. ||

- If $\{(\sigma_i, \tau_i)\}_{i\in\omega}$ is a $*$-tower, let

$$F_{(\sigma,\tau)} = \cup\{\sigma_i \mid i \in \omega\}, \quad G_{(\sigma,\tau)} = \cup\{\tau_i \mid i \in \omega\}.$$

•• <u>One needs to verify that</u>:

(1) If $\{(\sigma_i, \tau_i)\}_{i\in\omega}$, $\{(\sigma'_i, \tau'_i)\}_{i\in\omega}$ are $*$-towers, then

$$F_{(\sigma,\tau)} = F_{(\sigma',\tau')} \Leftrightarrow G_{(\sigma,\tau)} = G_{(\sigma',\tau')},$$

and

(2) For each $\hat{F} \in \omega^\omega$ (or $\hat{G} \in \omega^\omega$) there exists a $*$-tower $\{(\sigma_i, \tau_i)\}_{i\in\omega} \subset *$ with $F_{(\sigma,\tau)} = \hat{F}$ (or $G_{(\sigma,\tau)} = \hat{G}$, respy).

•• For (1), let $\{(\sigma_i, \tau_i)\}_{i\in\omega}$, $\{(\sigma'_i, \tau'_i)\}_{i\in\omega}$ be $*$-towers, and assume that $G_{(\sigma,\tau)} = G_{(\sigma',\tau')}$.

- But for each $i, j \in \omega$, if $\tau_i \approx \tau'_j$, one must have $\sigma_i \approx \sigma'_j$ — since each $*_s$ respects string inclusion — giving $F_{(\sigma,\tau)} = F_{(\sigma',\tau')}$.

•• For (2), let $\hat{F} \in \omega^\omega$, and assume that $\{(\sigma_0, \tau_0), \ldots, (\sigma_i, \tau_i)\} \subset *$ has been defined with $\sigma_0 \subset \ldots \subset \sigma_i \subset \hat{F}$ and $\tau_0 \subset \ldots \subset \tau_i$.

• Let $\widetilde{s}$ be a stage such that for all $s+1 > \widetilde{s}$ one has $\sigma_j^{*_s} = \tau_j$, each j between 0 and i.

• Then — by the satisfaction of the $\mathcal{K}$- and $\mathcal{L}$-requirements (and the $\mathcal{M}$- and $\mathcal{N}$-requirements), there exist infinitely many σ with $\sigma_i \subset \sigma \subset \hat{F}$ such that $\sigma^* \downarrow$ — and $\sigma^* \supset \tau_i$.

• So there exists a (σ, τ) with $\sigma_i \subset \sigma \subset \hat{F}$, $\tau_i \subset \tau$ and $\sigma^* = \tau$ — from which the inductive step in the definition of an appropriate $*$-tower $\{(\sigma_i, \tau_i)\}_{i \in \omega}$ follows immediately. $\square$

All the $\mathcal{P}$- and $\mathcal{Q}$- requirements are satisfied.

•• Let $\mathcal{P}_\Phi$ be a typical $\mathcal{P}$-requirement, and let $\hat{s}$ be a stage after which no requirement of priority greater than that of $\mathcal{P}_\Phi$ requires attention.

• Given $X, Y \in \omega^\omega$ with $X = \Phi^Y$ — one needs to choose a suitable $\widehat{\Phi}_\alpha$ from which to extract a computable $\widehat{\Phi}$ with $X^{\widetilde{*}} = \widehat{\Phi}^{Y^{\widetilde{*}}}$.

Say $h_0 \in \omega^\omega$ is *persistently* $\mathcal{P}, \widehat{\mathcal{Q}}$*-configured*, or *persistently* (*inner*) $\mathcal{P}, \widehat{\mathcal{Q}}$*-preconfigured*, relative to $h_1 \in \omega^\omega$ above $\mathcal{R}_\gamma$ according to $\langle\alpha\rangle$ if and only if there exist infinitely many beginnings σ, π of h_0, h_1, respy, such that σ is $\mathcal{P}, \widehat{\mathcal{Q}}$-configured, or (inner) $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured, respy, relative to π above $\mathcal{R}_\gamma$ according to $\langle\alpha\rangle$ at some stage $s+1$.

• <u>Then</u> —

•• Given any $h \in \omega^\omega$ there exists an α such that h is persistently $\mathcal{P}, \widehat{\mathcal{Q}}$-(pre)configured relative to f above each sufficiently low priority $\mathcal{R}_\gamma$ according to $\langle \alpha \rangle$.

•• <u>Let α be such that</u>:

| $X = \Phi_\alpha^Y$ — and *either* X is persistently $\mathcal{P}, \widehat{\mathcal{Q}}$-configured relative to f above some $\mathcal{R}_\gamma$ according to $\langle \alpha \rangle$, *or* Y is persistently $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured, relative to f above some $\mathcal{R}_\gamma$ according to $\langle \alpha \rangle$, but is not persistently inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured, relative to f above any $\mathcal{R}_\gamma$ — and $\alpha' \prec \alpha$ for no other such index α'. |

•• <u>Then</u> —

|| One can choose the stage $\hat{s}$ such that at no stage $s+1 > \hat{s}$ is an axiom '$\sigma = \widehat{\Phi}_\alpha^\tau$' cancelled with $\tau \subset Y$. ||

•• <u>This is because</u> — If an axiom for $\widetilde{\Phi}_{\bar{\alpha}}$ is cancelled at stage $s+1$, then *either* —

(1) Some requirement $\mathcal{R}_\gamma$ of higher priority than that of $\mathcal{P}_\Phi$ requires attention at stage $s+1$ — resulting in $\widehat{\Phi}_{\bar{\alpha}}$ being initialised, *or* —

(2) There is an axiom '$\sigma = \widehat{\Phi}^{\tau}_{\bar{\alpha}}$' for $\widehat{\Phi}_{\bar{\alpha}}$ at stage $s+1$, with τ $\widehat{\mathcal{P}}, \mathcal{Q}[\Psi'_{\beta_1}, ..., \Psi'_{\beta_k}]$-preconfigured above some $\mathcal{R}_\Theta$ at stage $s+1$ according to $\langle \bar{\alpha}^- \restriction_{\geq i} \rangle$ or $\langle \bar{\alpha} \restriction_{>i} \rangle$, say, with α^- or $\alpha \restriction_{\leq i}$, respy, $\overline{\mathcal{P}}_\Phi$-relevant, relative to g_s, and that τ is inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured (some corresponding basis $\Psi''_{\beta_1}, ..., \Psi''_{\beta_k}$, say) relative to g_s above $\mathcal{R}_\Theta$ at stage $s+1$ according to $\langle \hat{\beta}^- \restriction_{\geq j} \rangle$, say, — and that σ is not inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured relative to g_s above $\mathcal{R}_\Theta$ at stage $s+1$ according to $\langle \ulcorner \bar{\alpha} \rangle$, so that '$\sigma = \widehat{\Phi}^{\tau}_{\bar{\alpha}}$' is removed from $\widehat{\Phi}_{\bar{\alpha}}[s+1]$ at stage $s+1$.

• By the initial choice of $\hat{s}$, only (2) needs considering —

•• <u>Say α is nonconfiguring</u> —

• Then Y is not persistently inner $\mathcal{P}, \widehat{\mathcal{Q}}$-preconfigured relative to f above any $\mathcal{R}_\gamma$, for any index $\hat{\alpha}$.

• But if an axiom '$\sigma = \widehat{\Phi}^{\tau}_{\bar{\alpha}}$' is cancelled at a stage $s+1$ —

• Then $\mathrm{req}(\alpha) = \mathcal{P}_\alpha$ is not of higher priority than that of $\mathrm{req}(\hat{\beta})$, some such $\hat{\beta}$ as in (2), with τ $\widehat{\mathcal{P}}, \mathcal{Q}[\Psi'_{\beta_1}, ..., \Psi'_{\beta_k}]$-preconfigured relative to g_s above some $\mathcal{R}_\Theta$ at stage $s+1$.

• Since there are only finitely many $\hat{\beta} \prec \bar{\alpha}$, one need only choose $\hat{s}$ such that there is no $\tau \subset Y$ newly inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfigured relative to g_s above $\mathcal{R}_\Theta$ at any stage $s+1$, with an index $\hat{\beta} \prec \alpha$.

•• <u>Say α is configuring</u> —

• In this case, if an axiom '$\sigma = \widehat{\Phi}^{\tau}_{\bar{\alpha}}$' is cancelled at a stage $s+1$ —

• Then the potential inner $\widehat{\mathcal{P}}, \mathcal{Q}[\Psi''_{\beta_1}, ..., \Psi''_{\beta_k}]$-preconfiguration of σ relative to g_s above $\mathcal{R}_\Theta$ at stage $s+1$ must involve a basis inconsistent with one relevant to the inner $\widehat{\mathcal{P}}, \mathcal{Q}$-preconfiguration of some τ', which has index $\prec \alpha$.

• But again, the number of such indices is finite, and, by choice of α, there can be no persistent inner preconfigurations corresponding to such indices which lead to infinitely many such cancellations — again leading to a suitable choice of $\hat{s}$.

•• Inductively define a functional $\widetilde{\Phi}_{\bar{\alpha}}$ by taking

$$\widetilde{\Phi}_{\bar{\alpha}}[\hat{s}+1] = \widehat{\Phi}_{\bar{\alpha}}[\hat{s}+1],$$

and for each $s+1 > \hat{s}$ letting

$\widetilde{\Phi}_{\bar{\alpha}}[s+1] = \widetilde{\Phi}_{\bar{\alpha}}[s] \cup \{$'$\sigma = \widetilde{\Phi}^{\tau}_{\bar{\alpha}}$' $|$ '$\sigma = \widehat{\Phi}^{\tau}_{\bar{\alpha}}$' is an axiom for $\widehat{\Phi}_{\bar{\alpha}}[s+1]$, & is consistent with $\widetilde{\Phi}_{\bar{\alpha}}[s]\}$.

•• Then —

|| $\widetilde{\Phi}_{\bar{\alpha}}$ is a computable, consistent functional for which $X^{\widetilde{*}} = \widetilde{\Phi}_{\bar{\alpha}}^{Y^{\widetilde{*}}}$. ||

• The computability and consistency of $\widetilde{\Phi}_{\bar{\alpha}}$ follow immediately from the construction and from the definition of $\widetilde{\Phi}_{\bar{\alpha}}$.

•• Need to verify — Given $\hat{\sigma}, \hat{\tau} \subset X, Y$, there exists a pair $\sigma, \tau \subset X, Y$, respy, with $\sigma, \tau \supseteq \hat{\sigma}, \hat{\tau}$ respy, for which $\sigma^* = \widetilde{\Phi}_{\bar{\alpha}}^{\tau^*}$.

• Satisfaction of the $\mathcal{K}$- and $\mathcal{L}$-requirements means one can assume $\hat{\sigma}, \hat{\tau} \in \mathrm{Dom}(*_u)$ for all sufficiently large u.

• By choice of α, have σ_0, τ_0 such that $X, Y \supset \sigma_0, \tau_0 \supseteq \hat{\sigma}, \hat{\tau}$, respy, and an axiom '$\sigma_0 = \Phi_\alpha^{\tau_0}$' is stipulated for Φ_α at some stage $s+1 > \hat{s}$ —

• Where one can assume — For all sufficiently large $u \geq s$ '$\sigma_0 = \Phi_\alpha^{\tau_0}$' is on a true path at stage $u+1$ and there exist strings $\pi, \rho \supseteq \sigma_0, \tau_0 \supseteq \hat{\sigma}, \hat{\tau}$, respy, with $\pi, \rho \in \mathrm{Dom}(*_u)$ —

•• <u>Then</u> — If not currently implemented, '$\sigma_0 = \Phi_\alpha^{\tau_0}$' is implemented along the paths X, Y at such stages $u+1$ by the choosing of appropriate σ' with $\sigma_0 \supseteq \sigma' \supseteq \hat{\sigma}$, and τ' with $\tau_0 \subseteq \tau' \subseteq \rho$, and the adding of an axiom '$\sigma' = \Phi_\alpha^{\tau'}$' to Φ_α.

• So — at some such reactive stage $u+1$ an axiom '$\pi^{*_{u+1}} = \widehat{\Phi}_{\bar{\alpha}}^{\rho^{*_{u+1}}}$' for $\widehat{\Phi}_{\bar{\alpha}}$ with $X, Y \supset \pi, \rho \supseteq \hat{\sigma}, \hat{\tau}$, respy, and $\pi^{*_{u+1}}, \rho^{*_{u+1}} = \pi^*, \rho^* \subset X^{\widetilde{*}}, Y^{\widetilde{*}}$, respy, is stipulated, this stipulation being retained at all later stages.

• And further stipulation of revised axioms at the end of any such reactive stage $u+1$ is guaranteed by prerequisite $*_{u+1}$-matchings and corresponding augmentations.

•• <u>Then</u> — for large enough $u+1$ — the axiom '$\pi^{*_{u+1}} = \widehat{\Phi}_{\bar{\alpha}}^{\rho^{*_{u+1}}}$' is implemented along $X^{\widetilde{*}}, Y^{\widetilde{*}}$ via the enumeration of '$\pi^{*_{u+1}} = \widehat{\Phi}_{\bar{\alpha}}^{\rho^{*_{u+1}}}$' into $\widehat{\Phi}_{\bar{\alpha}}$ — and is never cancelled by choice of $\hat{s}$.

• The argument is similar in relation to a typical $\mathcal{Q}$-requirement $\mathcal{Q}_\Psi$. $\square$