Computability, Enumerability, and the Structure of Information

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Basic reference:

• S. Barry Cooper, Computability Theory, Chapman & Hall/CRC, 2004

Topics

• Post’s Programme extended to the Ershov hierarchy
• Randomness - new kinds of information content
• Enumeration reducibility: A different model of relative computation
• Turing definability in the real world
A convincing model of computability

- 1936 - Turing’s machines appear
- Provide a model of algorithmic natural processes within structures which are countably presented

What does this Turing machine compute?

\[
q_0 10q_1 \quad \text{subroutine for deleting 1’s}
\]
\[
q_1 0Rq_0 \quad \begin{cases}
\text{move right in search of another 1,} \\
\text{in preparation for return to “delete” subroutine}
\end{cases}
\]

What does this Turing machine compute?

\[
\varphi_T(n) = 0 = O(n) \quad \text{for all } n \in \mathbb{N}.
\]
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1936 - Turing's machines appear
Provide a model of algorithmic natural processes within structures which are countably presented
But - techniques for presenting machines give the Universal Turing machine - and incomputable objects

The Universal Turing machine -
Can computably code, and so list, the Turing programs, giving:

\[ \varphi^{(k)}_e = \text{the } k\text{-place partial function computed by } P_e, \]

\[ \varphi_e = \varphi^{(1)}_e = \text{the } e\text{-th partial computable (p.c.) function} \]

There exists a Turing machine \( U \) — the Universal Turing Machine — which if given input \( (e,x) \) simulates the \( e\text{-th} \) Turing machine with input \( x \).
That is, \( \varphi^{(2)}_U (e,x) = \varphi_e (x) \).

A convincing model of computability

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Some incomputable objects:

- The Halting problem for the Universal TM
- The set of theorems of Peano arithmetic
- (Church's Theorem) The set of logically valid sentences of first-order logic
- And these all give computably enumerable (c.e.) sets essentially the same as:

\[ K = \{ x \mid x \in W_x \} \]
The Turing landscape

- 1939 - Turing's oracle Turing machines appear
- Provides a model of algorithmic content of structures which are based on the reals
- 1944 - Post defines the degrees of unsolvability as a classification of reals in terms of their relative computability
- ... giving a landscape one can zoom in or out of

Oracle Turing machines -

- Turing's idea - Allow programs with query quadruples, which can ask for finite information from the 'real world', given by an 'oracle' -

And the Turing degree structure -

**Definition**

1. We write $A \equiv_T B$ if $A$ and $B$ are Turing equivalent.
2. We define the Turing degree — or degree of unsolvability — of $A \subseteq \mathbb{N}$ to be
   $$\deg(A) = \text{dom} \{ X \subseteq \mathbb{N} \mid X \equiv_T A \}.$$
3. We write $D$ for the collection of all such degrees, and define an ordering $\leq$ induced by $\leq_T$ on $D$ by:
   $$\deg(B) \leq \deg(A) \iff B \leq_T A.$$

Relativisation

$\Phi_e^A$ will denote the — possibly partial — 0–1 valued function computed by $\hat{P}_e$ with oracle $A$.

We say $B$ is computably enumerable in $A$ — or just $A$-e. — if we can computably enumerate the members of $B$ with help of an oracle for $A$.

$$W_e^A = \text{dom} \Phi_e^A$$

1. We define the jump $A'$ of a set $A$ to be
   $$A' = \text{dom} \{ (x, y) \mid x \in W_e^A \} = K_0^A.$$
2. The $(n+1)^{th}$ jump of $A$ is defined to be $A^{(n+1)} = \text{dom} (A^{(n)})'$. 
Mapping the landscape

- Look more closely at distinctive features of the Turing landscape...
- And extract the underlying information content
- Intuition: Particular types of information are associated with distinctive causal structure

The arithmetical hierarchy

1. $\Sigma_0 = \Pi_0 = \Delta_0$ = all the computable relations. And for $n \geq 0$:
   2. $\Sigma_n = \Pi_n = \Delta_n$ = all relations of the form $(x, y) \rightarrow R(x, y)$, with $R \in \Pi_n$.
   3. $\Pi_{n+1} = \cup \Sigma_n \cup \Delta_n$.
   4. $\Delta_{n+1} = \Sigma_n \cup \Pi_n$.

R is arithmetical if $R \in \cup_{n \geq 0} (\Sigma_n \cup \Pi_n)$.

Post’s Theorem

Let $A \subseteq \mathbb{N}$ and $n \geq 0$. Then:
- $(a)$ $\emptyset^{(n+1)}$ is $\Sigma_{n+1}$-complete.
- $(b)$ $A \in \Sigma_{n+1} \iff A$ is c.e. in $\emptyset^{(n)}$.
- $(c)$ $A \in \Delta_{n+1} \iff A \leq_T \emptyset^{(n)}$.

\[ A \in \Delta_2 \iff A \leq_T \emptyset' \]

1944: Post’s Problem

- February 26, 1944 - Post addresses the New York meeting of the American Mathematical Society

- QUESTION: Is there a non-recursive r.e. set of strictly lower degree of unsolvability than $K$ with respect to arbitrary recursive reducibility?
The approach? - Look for a natural example

- Or, as we would ask for the property now: Is there a non-computable c.e. set of strictly lower Turing degree than $0'$?
- Of course - (Friedberg-Muchnik Theorem) - “YES”

Post’s approach: Look at classes of sets naturally definable - if possible, in the lattice $E$ of computably enumerable sets

For many-one reducibility - simple sets have the property
For truth-table reducibility - hypersimple sets have it

Post’s Programme (narrow version):
Find a definable property of c.e. sets which delivers a non-computable c.e. set of strictly lower Turing degree than $0'$

Gave rise to a range of immunity properties which one can use to say “the complement of $A \in E$ avoids the c.e. sets”
Post’s Programme

- Post’s Programme (narrow version):
  - Find a definable property of c.e. sets which delivers a non-computable c.e. set of strictly lower Turing degree than $0'$
  - Gave rise to a range of immunity properties which one can use to say “the complement of $A \in \mathcal{E}$ avoids the c.e. sets”
  - Strongest immunity property - $\hat{A}$ is cohesive if it is not split by any c.e. set into two infinite parts - in which case we say $A \in \mathcal{E}$ maximal

- Remember - the Turing jump $A'$ is what one gets from the halting problem for a Turing machine with oracle $A$ ...
  - ... and $A$ is high if $A' = 0''$ (and low if $A' = 0'$)

The High/Low Hierarchy

1. The high/low hierarchy is defined by
   
   \[ \text{High}_n = \{ a \leq 0' | a^{(n)} = 0^{(n+1)} \}, \quad \text{Low}_n = \{ a \leq 0' | a^{(n)} = 0^{(n)} \}, \]

   for each $n \geq 1$.

2. If $\text{deg}(A) \in \text{High}_n$ we say $A$ and $\text{deg}(A)$ are high$_n$. We similarly define the low$_n$ sets and degrees.

3. For $n = 1$ we often drop the subscript — $A$ and $\text{deg}(A)$ are low if $A' \in 0'$, and high if $A' \in 0''$. 
Remember - the Turing jump $A'$ is what one gets from the halting problem for a Turing machine with oracle $A$ ...

... and $A$ is high if $A' = 0''$ (and low if $A' = 0'$)

Martin’s Theorem (1966): Not only are there maximal sets in $0'$ (Yates, 1965) - but the degrees of maximal sets are exactly those which are high - so high is lattice invariant

Post’s Programme (broader and deeper version): discover new relationships between natural information and computability-theoretic structure

What came later ...

Post’s remark in this paper that Hilbert’s tenth problem “begs for an unsolvability proof” had a major influence on my own work.

Martin Davis
PROBLEM: Given any polynomial equation in one or more variables, with integer coefficients, find a solution consisting entirely of integers — that is, solve any given Diophantine equation.

HILBERT’S TENTH PROBLEM: To find a general way of telling effectively whether a given Diophantine equation has a solution or not.

A set $A \subseteq \mathbb{N}$ is Diophantine if

$$A = \{x \in \mathbb{N} \mid (\exists y_1, \ldots, y_n \in \mathbb{N})[p_A(x, y_1, \ldots, y_n) = 0]\}$$

for some polynomial $p_A(x, y_1, \ldots, y_n)$ (with integer coefficients).

Julia Robinson, Davis, Hilary Putnam (1960): For Davis’ strategy, just need to show one exponentially increasing set is diophantine.

Enter some more old mathematics: Fibonacci sequences are exponentially increasing, e.g., $\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n$ approximates $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots, a_n, a_{n+1}, a_{n+2} = a_{n+1} + a_n, \ldots$

Yuri Matiasevich (1970): The Fibonacci sequence is diophantine.

COROLLARY (Davis, Matiasevich, Putnam, Robinson)

(i) Every computably enumerable set is Diophantine.
(ii) There is no positive solution to Hilbert’s Tenth Problem.

$A = \{1, 3, 4, 5, 7, 8, 9, 11, \ldots\}$
Then $A = \{x \in \mathbb{N} \mid x = y_1^2 - y_2^2$, some $y_1, y_2 \in \mathbb{N}\}$. So $A$ is Diophantine with $p_A(x, y_1, y_2) = y_1^2 - y_2^2 - x$.

DAVIS’ STRATEGY: Show that every computably enumerable set is Diophantine.

But say -

$$K = \{x \in \mathbb{N} \mid (\exists y_1, \ldots, y_n \in \mathbb{N})[p_K(x, y_1, \ldots, y_n) = 0]\}$$

Then no algorithm exists for the equations:

$$p_K(0, y_1, \ldots, y_n) = 0$$
$$p_K(1, y_1, \ldots, y_n) = 0$$
$$p_K(2, y_1, \ldots, y_n) = 0$$

...$

$u + w - \text{FIB}(2u) - 2 = 0$

$l - 2\text{FIB}(2u) - 2a - 1 = 0$

$l^2 - 2l - 2 = 0$

$g - bl^2 = 0$

$g^2 - gh - h^2 - 1 = 0$

$m - c(2h + g) - 3 = 0$

$m - fl - 2 = 0$

$x^2 - mxy + y^2 - l = 0$

$(d - l)l + u - x - 1 = 0$

$x - \text{FIB}(2u) - (2h + g)(l - 1) = 0$. 
Ubiquitous natural information content

- So ... diophantine sets are distributed throughout the rich structure of the computably enumerable degrees ...

- High school arithmetic is enough carry us well into the incomputable ...

- Leaving us with a puzzle -

- Someone gives us a diophantine set - how do we recognise it as being incomputable? Back to Post’s programme ...
THEOREM

Let $A \subseteq \mathbb{N}$ be infinite. Then $A$ is hyperimmune if and only if no computable function dominates $p_A$.

PROOF

$(\Rightarrow)$ Assume some computable $f$ dominates $p_A$. Say $f(n) \geq_A (n)$ for all $n \geq N$.

Then for each $n \geq N$ the finite interval $[n, f(n)] \subseteq \mathbb{N}$ contains some $x \in A$.

It is easy to choose a c.e. array from amongst these intervals — say $[N, f(N)]$, $[f(N) + 1, f(f(N) + 1)]$, etc. Since $A$ does not avoid such an array, we get $A$ not hyperimmune.

$(\Leftarrow)$ On the other hand, say $A$ fails to avoid the c.e. array $\{D_{\Phi(i)}\}_{i \geq 0}$.

Defining $f(x) = \max\{y \mid y \in D_{\Phi(x)}; \text{ some } z \leq x\}$,

we get a computable $f$ which dominates $p_A$.

Some negative and positive results ...

- (Martin, Soare, et al) Development of automorphism techniques to break link with $\mathcal{E}$ ...
- Martin: Hypersimple is not definable in $\mathcal{E}$ - can choose $\Phi$ so that $\Phi(h\text{-simple}) \subset h\text{-simple}$
- Harrington-Soare: Make $\Phi(A)$ high, given incomputable $A$

So if a lattice invariant set of c.e. degrees contains a non-zero degree, it contains a high degree.
Some negative and positive results...

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So if a lattice invariant set of c.e. degrees contains a non-zero degree, it contains a high degree

More results and a question...

- **Harrington and Soare, 1996** - Non-low is not lattice invariant - but ...
- **Harrington-Cholak, 1999** - This is an exception

Non-low$_n$ is lattice invariant for every $n > 1$

Some negative and positive results...

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- **Harrington-Soare**: Make $\Phi(A) \in \text{high}$, given incomputable $A$
- ... but use obstacle to building $\Phi(A) \in 0'$ to get a dynamic property $Q$, giving a solution to Post's problem

More results and a question...

- **Harrington and Soare, 1996** - Non-low is not lattice invariant - but ...
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$A$ is low$_2$ if $A'' = 0''$, high$_2$ if $A'' = 0'''$
More results and a question...

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- A is low\(_2\) if \(A'' = 0''\), high\(_2\) if \(A'' = 0''\)

- Lachlan, 1968, Shoenfield 1976 - The non-low\(_2\) c.e. degrees are those which contain c.e. sets with no maximal supersets

So we have a characterisation in terms of natural information of the non-low\(_2\) c.e. degrees

OPEN QUESTION: Find a natural lattice theoretic property characterising the high\(_2\) degrees

The need to find data non-monotonically

"... if a machine is expected to be infallible, it cannot also be intelligent. There are several theorems which say almost exactly that."

A.M. Turing, talk to the London Mathematical Society, February 20, 1947, quoted by Andrew Hodges in "Alan Turing - the enigma", p.361

By Arslanov's Completeness Criterion - Every complete extension of Peano arithmetic of c.e. degree is of degree \(0'\)

But - by the Low Basis Theorem - some are even low
The Ershov difference hierarchy

- **Idea**: Iterate boolean operations on c.e. sets to get a hierarchy for the $\Delta_2$ sets.

- At bottom level get c.e. sets ... next level differences of c.e. sets -

- A is 2-c.e. or d.c.e. iff $A = B - C$ for some c.e. sets $B, C$

- **Dynamically**: $A$ has a sequence of finite approximations $A^s$ such that $|\{s : A^s(x) \neq A(x)\}| \leq 2$ (for $\leq n$ get n-o.e.)

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The n-c.e. Hierarchy

1. Let $A \in \Delta_2^0$ have standard approximating sequence $\{A^n\}_{n \geq 0}$. We say $A$ is n-o.e. if $A^n(x)$ changes value at most $n$ times, each $x \in \mathbb{N}$ — that is, for all $x \in \mathbb{N}$

   $$|\{s : A^s(x) \neq A^{s+1}(x)\}| \leq n.$$  

   We say $A$ is $\omega$-c.e. if there is a computable bound on the number of such changes — that is, if there is some computable $f$ such that for all $x \in \mathbb{N}$

   $$|\{s : A^s(x) \neq A^{s+1}(x)\}| \leq f(x).$$

2. A degree $a$ is d.c.e. if it contains a d.c.e. set. And for each $n \geq 2$ we write $D_n$ for the $n^{th}$ level of the corresponding n-c.e. hierarchy of n-c.e. degrees.

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Post’s Programme and the Ershov hierarchy

**QUESTION**: How does Post’s programme extend to other finite levels of the Ershov hierarchy?

- Is there an easy analogue of Martin’s Theorem?

- Are the high d.c.e. degrees exactly those which contain a cohesive d.c.e. set?

- Are the high n-c.e. degrees exactly those which contain a cohesive n-c.e. or co-n-c.e. set?
Post's Programme and the Ershov hierarchy

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The Lachlan c.e. predecessor

- **Notice:** \( A = B \setminus C \) is c.e. in \( C \).
- So either \( A \) is c.e. or \( C \) is incomputable.
The Lachlan c.e. predecessor

- Notice: \( A = B - C \)
  - is c.e. in \( C \)
  - So either \( A \) is c.e.
  - or \( C \) is incomputable

- Now - exchange \( C \) for \( E = \{(x,s) : x \in B^2 - A\} \) - \( E \) still c.e.

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A promising start ...

- **Definition**: \( X_{\infty} \) is immune iff it has no infinite c.e. subset - equivalently, iff it has no infinite computable subset

- **Definition**: \( X \) is weak truth-table reducible to \( Y \), i.e. \( X \leq \text{wtt} Y \) - iff \( X \) is computable from \( Y \) with a computable bound on the size of any oracle question asked computing from \( x \)

---

Theorem (Afshari, Barmpalias, C, Stephan): Every d.c.e. wtt-degree contains an immune d.c.e. set
Let \( \#_E(x) \) be the number of members of \( E \leq x \)

\[ B = \{(x, \#_E(x)) : x \in A\} \text{ a d.o.e. set} \]

\[ B \leq_{wtt} A \text{ since } E \leq_m A \text{ and so } E \leq_{wtt} A \]

And \( A \leq_{wtt} B \) since \( x \in A \) iff \( (x,y) \in B, \text{ some } y \leq x + 1 \)

And \( B \) is immune - since assuming \( F \subseteq B \) infinite ...

Then \( F \) is not computable, since can compute \( E \) from \( F \):

Look for a \( (y, \#_E(y)) \in F \) with \( (x,s) \leq y \), and enumerate \( E \) to see if \( (x,s) \) is in \( E \) or not

Theorem (Afshari, Balmaglas, C, Stephan) Every d.o.e.
Turing degree \( \neq 0 \) contains a hyperimmune d.o.e. set

And these results can be extended to every finite level of the \( n \)-c.e. hierarchy -

But cohesive d.c.e.
sets are all co-c.e.

Friedberg splitting theorem for incomputable c.e.
A

But for odd \( n \) we will get co- \( n \)-c.e. immune or
hyperimmune sets of the required degree
**But cohesive d.c.e. sets are all co-c.e.**

- Friedberg splitting theorem for incomputable c.e. A
- Owings splitting theorem for d.e. A-D which is not co-c.e.

**Using weaker notions than cohesive?**

- Weaker properties of sets characterising the high $\Delta_2$ degrees include hyperhyperimmune, r-cohesive and dense immune - but

  (ABCS) None of these work for n-c.e. with $n > 1$

- Infinitary iteration of the Owings splitting theorem gives:
  **Theorem:** If $A$ is n-c.e. and hh-immune then $A$ is co-c.e.

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- For the other notions the situation is more interesting
Re-enter Lachlan’s c.e. predecessor ...

- Let $A$ be r.o.e. Then:
  - $A$ can be r-cohesive $\exists$ an r-cohesive o.e. $E \leq_T A$
- And:
  - $A$ can be dense immune $\exists$ dense immune o.e. $E \leq_T A$
- Corollary ...

Since high c.e. degrees contain r-cohesive/ dense immune co-e.e. sets

And - by C. Lempp, Watson - there are properly d.e. degrees above any incomplete o.e. degree

- Since the equality holds over the degrees of co-e.e. sets
Since all r-cohesive/ dense immune $\Delta^0_2$ sets are high.

And - Wu, Yang (2003) - there exist high $\text{d.o.e.}$ degrees all of whose c.e. predecessors are low.

So - a number of questions to answer:

- What sort of natural information content does extend to other levels of the Ershov and high/low hierarchies?
- Can one extend automorphism techniques from the computably enumerable context to break the link between computability theoretic structure and natural information content?
- Or will Harrington-Nies type codings work here?
- What about the non-low$_2$ n.o.e. degrees?
So - a number of questions to answer

- What sort of natural information content does extend to other levels of the Ershov and high/low hierarchies?

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What about the non-low\_\_ n-o.e. degrees?

Strings as data for Turing machines

**Definition:** A string, σ or τ etc., is a finite sequence of 0’s and 1’s. E.g., 00111010 is a string. Equivalently, a string σ or τ is an initial segment of a characteristic function.

We write σ(κ) = 0 or 1 according as the x + 1\textsuperscript{st} member — if it exists — of σ is 0 or 1. E.g., if σ = 0100, we get σ(1) = 1 and σ(2) = 0.

We also define the length of σ to be |σ| = def the number of entries in σ. E.g., |0100| = 4.

Note: Can consider strings as inputs and outputs of Turing machines.
Strings as data for Turing machines

**Definition** A string, $\sigma$ or $\tau$ etc., is a finite sequence of 0's and 1's, e.g., 00110101 is a string. Equivalently, a string $\sigma$ or $\tau$ is an initial segment of a characteristic function.

We write $\sigma(x) = 0$ or $1$ according as the $x+1$th member — if it exists — of $\sigma$ is 0 or 1. E.g., if $\sigma = 0101$, we get $\sigma(1) = 1$ and $\sigma(2) = 0$.

We also define the length of $\sigma$ to be $|\sigma| = \text{data}$, the number of entries in $\sigma$. E.g., $|0100| = 4$.

Note: Can consider strings as inputs and outputs of Turing machines.

...Strings as data for Turing machines...

Notions of Randomness

- **Useful intuition:** A string $\sigma \in 2^\omega$ is random if it has no shorter description than itself.

- Given universal TM $U$ define the Kolmogorov complexity of $\sigma$ to be $C_U(|\tau|)$ for shortest $\tau$ with $U(\tau) \downarrow = \sigma$.

- A TM $M$ is prefix-free if $M$ is defined on no pair $\sigma \leq \rho$.

- Can define a sensible notion of universal prefix-free $U$.

- The prefix-free complexity $K(\sigma) = K_U(\sigma) = |\tau|$ for shortest $\tau$ with $U(\tau) \downarrow = \sigma$.

- But - Martin-Löf - no such reals (we ask too much!)
Notions of Randomness

- A TM \( M \) is prefix-free if \( M \) is defined on no pair \( 0 \subseteq \rho \)
- Can define a sensible notion of universal prefix-free \( U \)
- The prefix-free complexity \( K(0) = K_U(0) = |T| \) for shortest \( T \) with \( U(T) = 0 \)

2 books available online:

- Computability and Randomness by André Nies
- Algorithmic Randomness and Complexity by Rod Downey, Denis Hirschfeldt

Alternative intuition: \( A \) is random if it avoids being “captured” by “computably small” set of reals
- So a Martin-Löf test is a “computably small” collection of reals, and \( A \) is Martin-Löf random if it passes every such test by not being in it

Robustness of notions

- Definition (Levin, Schnorr, Chaitin): \( A \) is K-random if \( K(A \upharpoonright n) \upharpoonright n \) for all \( n \)
- Schnorr: K-random \( \Leftrightarrow \) Martin-Löf random
New kinds of information content

- **Defn** (Levin, Schnorr, Chaitin): A is *K-random* if $K(A \upharpoonright n) \leq n - c$ for all $n$
- **Schnorr**: K-random $\iff$ Martin-Löf random
- **Defn**: $A \leq K B$ $\iff$ $K(A \uparrow n) \leq K(B \uparrow n) + c$ for all $n$
- A is *K-trivial* $\iff$ $A \leq K N$
- **Solovay**: There exist incomputable K-triviales

More robustness of notions ...

- **Defn**: $A \leq LR B$ $\iff$ every B-random real is A-random
- (Kučera-Terwijn) Say A is **low for random** $\iff A \leq LR N$
- A low for random $\iff$ A is K-trivial
- **Notice**: $\leq LR$ is an extension of Turing reducibility, giving rise to the structure of the *LR-degrees* (see Nies, Barmpalias, Lewis, Miller ...)

A novel solution to Post’s problem

- **Nies**: K-triviales $\leq$ low - in fact $\leq$ superlow ...
  - giving a very different solution to Post’s problem

A novel solution to Post’s problem

- **Nies**: K-triviales $\leq$ low - in fact $\leq$ superlow ...
  - giving a very different solution to Post’s problem

A set $A$ is called superlow if $A' \equiv_{wtt} \emptyset'$. 
A novel solution to Post’s problem

- Nies: K-trivals \subseteq \text{low} - in fact \subseteq \text{superlow}...
- ... giving a very different solution to Post’s problem
- Nies: There is a low\textsubscript{2} c.e. degree \geq \text{all K-trivals}!

**Question:** Can this degree be low? (Nies - “no”, but by Kucera-Slaman, can be low and \Delta\textsubscript{2})

Real world randomness

- Quantum randomness is a familiar experimental and theoretical phenomenon
  - It passes all reasonable statistical properties of randomness
- Cris Calude: It is Turing incomputable

Whether a \text{U}_{238} nucleus will emit an alpha particle in a given interval of time is “random”. If we collapse a wave function, what it ends of being is “random”. Which slit the electron went through in the double slit experiment, again, is “random”.

Is there any sense to say that “random” in the above sentences means “truly random”? When we flip a coin, whether it’s heads or tails looks random, but it’s not truly random. It’s determined by the way we flip the coin, the force on the coin, the way force is applied, the weight of the coin, air currents acting on it, and many other factors. This means that if we calculated all these values, we would know if it was heads or tails without looking. Without knowing this information—and this is what happens in practice—the result looks as if it’s random, but it’s not truly random.

Is quantum randomness “truly random”? Our working model of “truly random” is “algorithmic randomness” in the sense of Algorithmic Information Theory.

Open question: How random is quantum randomness?
Post’s real anticipation

- Early recognition of the importance of the relation between information content and computability theoretic structure
- ... leading to Hartley Rogers’ seminal focus on globally definable relations
- ... giving rise to a Programme which remains full of interest, surprises, and open questions
Revisiting relative computability

- Compute using an oracle - i.e., call deterministically on values of a total function
- Compute using emergent or enumerated information - i.e., call non-deterministically on values of a partial function

Enumeration reducibility

\[ b_0 \ b_1 \ \cdots \ \in B \text{ in some order} \]
\[ a_0 \ a_1 \ \cdots \ \in A \text{ enumerated in any order} \]

\[ n \in \Psi_i^A \iff \text{defn } (\exists \text{ a finite } D \subseteq A)[\langle n, D \rangle \in \Psi]. \]
THEOREM
Let \( f, g \) be partial functions. Then
\[
 f \leq_{NT} g \iff \text{Graph}(f) \leq_{e} \text{Graph}(g).
\]
If \( g \) is total, we have
\[
 f \leq_{NT} g \iff f \leq_{T} g.
\]

The link with NT-reducibility

Define \( A \equiv_{e} B \iff \text{defn } A \leq_{e} B \land B \leq_{e} A \).

The enumeration degree — or \( e \)-degree, written \( \text{deg}_e(A) \) — of \( A \) is
\[
 \text{deg}_e(A) = \text{defn } \{ X \mid X \equiv_{e} A \}.
\]
We define \( \text{deg}_e(A) \leq_{e} \text{deg}_e(B) \iff \text{defn } A \leq_{e} B \).
We write \( \mathcal{D}_e = \text{defn } \) the set of all \( e \)-degrees with the ordering \( \leq \).

The partial degree of a partial function \( f \) is
\[
 \text{deg}(f) = \text{defn } \{ g \mid \text{Graph}(f) \equiv_{e} \text{Graph}(g) \} = \{ g \mid f \equiv_{NT} g \}.
\]
We write \( \mathcal{P} = \text{defn } \) the set of all partial degrees, with ordering \( \leq \) defined by
\[
 \text{deg}(f) \leq_{\text{deg}(g)} \iff \text{defn } \text{Graph}(f) \leq_{e} \text{Graph}(g) \iff f \leq_{NT} g.
\]
We say that an \( e \)-degree \( a_e \) is total if there is a total function \( f \) with \( \text{Graph}(f) \in a_e \).
We write \( \text{TOT} = \text{defn } \) the set of total \( e \)-degrees.
The Natural Embedding

PROPOSITION 0.3.19
The e-jump agrees with the natural embedding of the Turing jump — that is, for each $A \subseteq \mathbb{N}$ we have $\iota(\text{deg}(A')) = \text{deg}(\text{J}_e(\chi_A))$.

Extending the Jump Operator

DEFINITION 0.3.16
Let $K_A = \{x \mid x \in \Psi_{e}^A\}$. Then the e-jump of a set $A$ is $J_e^A = \text{defn} A \oplus K_A$. And the jump of an e-degree $a = \text{deg}_e(A)$ is defined to be $a' = \text{deg}_e(A \oplus K_A) = \text{deg}_e(J_e^A)$.

We iterate the jump in the usual way to obtain the $n$th jump $a^{(n)}$ of $a$.

PROOF
The most important case is $n = 1$, telling us that:

$$\Delta_1(\leq \emptyset_e) = \text{the set of all } \Sigma_2 \text{ e-degrees}$$

Local information content

THEOREM
For each $A \subseteq \mathbb{N}$, $n \geq 0$ we have $\text{deg}_e(A) \leq \emptyset_e^{(n)} \iff A \in \Sigma_{n+1}$.
**Theorem 0.4.3**

For each \( A \subseteq \mathbb{N}, n \geq 0 \) we have \( \deg_e(A) \leq 0_1^{(n)} \iff A \in \Sigma_{n+1} \).

The most important case is \( n = 1 \), telling us that:

\[ D_{\leq 0_1'} = \text{the set of all } \Sigma_2 \text{ e-degrees} \]

**Some key questions**

- Characterise the local structure \( D_{\leq 0_1'} \) of the enumeration degrees.
- Are the Turing degrees definable - locally or globally - in the enumeration degrees.

**Note:** The natural embedding of the computably enumerable Turing degrees is a proper subclass of the total e-degrees - the \( \Pi_1 \) enumeration degrees.

**Question:** Are the computably enumerable Turing degrees definable in \( D_{\leq 0_1'} \)?
Useful structure - Splittings

A splitting of $a$ over $b$

Ahmad and Lachlan: There exist non-splittable $\Delta_2$ enumeration degrees over $0_e$

Question: What about splittings of total or $\Pi_2$ enumeration degrees in $\mathcal{D}_e(0_e')$?

Arslanov, C, Kalimullin (2003): Results on local distribution of total e-degrees and of degrees of semirercursive sets used in natural embedding to get results on embedding of diamond lattices etc. - e.g. a simple derivation of a generalisation of the Ahmad Diamond Theorem

Some known results ...

- Arslanov, C, Kalimullin (2003): Results on local distribution of total e-degrees and of degrees of semirercursive sets used in natural embedding to get results on embedding of diamond lattices etc. - e.g. a simple derivation of a generalisation of the Ahmad Diamond Theorem

- M. Soskova (2007): There is a (properly) $\Sigma_2$ e-degree $a$, over which $0_e'$ is not splittable.

- Arslanov and Sorbi (1999): One can always split $0_e'$ over any $\Delta_2$ enumeration degree $< 0_e'$.

Some known results ...

- A positive result ...

  $0_e'$

  $h$ high total

  $\exists b$ low total

  $a$ low

  Theorem 1 (Arslanov, C, Kalimullin, Soskova): If $a < h \leq 0_e'$, $a$ is low and $h$ is total and high, then there is a low total e-degree $b$ such that $a \leq b < h$. 

  $\exists b$ low total
A positive result ...

Corollary: Let \( a < h \leq o_e' \) be a high total e-degree, \( a \) be a low e-degree. Then there are \( \Delta_2 \) e-degrees \( c, d < h \) such that \( a = c \cap d \) and \( h = c \cup d \).


... and a negative one

Theorem 2: There is a \( \Pi_1 \) e-degree \( a \) and a \( 3\text{-c.e.} \) e-degree \( b < a \) such that \( a \) is not splittable over \( b \).

Borel’s Infinite Monkey Theorem

“If you put an infinite number of monkeys at typewriters, eventually one will bash out the script for Hamlet.”

But - even if the observable universe were filled with monkeys typing for all time, their total probability to produce a single instance of Hamlet would still be less than one in \( 10^{143,800} \).

So no intelligence needed, just a lot of time, or monkeys...
Could a Computer Write “Hamlet”? 

A: Maybe

(Need randomness, and a lot of computers, or a lot of time)

Outline:

- Turing, and the Laplacian model for a computable world
  - The mathematical pointers to the inadequacies of modern computers ...
  - ... and what do minds do, that computers don’t?
  - Emergence, and more - Turing and the human brain ...
  - Artificial intelligence and natural computing
Outline:

- Turing, and the Laplacian model for a computable world
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  - emergence, and more - Turing and the human brain ...
  - Artificial intelligence and natural computing

The Algorithmic Content of the World

- Galileo and Newton onwards - overarching aim of science became the extraction of the algorithmic content of the world - a ‘clockwork universe’
Galileo and Newton onwards - overarching aim of science became the extraction of the algorithmic content of the world - that is, capturing Nature in computer programs.

Einstein [p.54, `Out of My Later Years', 1950]: “When we say that we understand a group of natural phenomena, we mean that we have found a constructive theory which embraces them.”

Laplace’s Predictive ‘Demon’ as model

“Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situations of the beings who compose it - an intelligence sufficiently vast to submit these data to analysis - it would embrace in the same formula the movements of the greatest bodies and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes.”

from P. S. de Laplace [1819], “Essai philosophique sur les probabilités”

But prediction as computation?

To be, or not to be?

Hilbert’s Programme

“For the mathematician there is no Ignorabimus, and, in my opinion, not at all for natural science either. ... The true reason why [no one] has succeeded in finding an unsolvable problem is, in my opinion, that there is no unsolvable problem. In contrast to the foolish Ignorabimus, our credo avers: We must know; we shall know.”

- David Hilbert’s opening address to the Society of German Scientists and Physicians, Königsberg, September 1930
Mathematical answers to complex questions

- **Procedure** - Clarify real-world concepts and questions by producing mathematical models
- **Develop** the consequent mathematical theory
- And compare these models with the real world to get very general answers out of the theory
- **Adjust** the models to better agree with expectations

The Turing model at last

- **1936** - Turing’s machines appear
- Provide a model of computable natural processes within a wide range of contexts

But - techniques for presenting machines give the **Universal Turing machine** - and incomputable objects

Natural phenomena as discipline problem

- **Successful reduction** of “natural” examples to the Turing model - e.g. quantum computation (David Deutsch)
Natural phenomena as discipline problem

- Successful reduction of "natural" examples to the Turing model - e.g. quantum computation (David Deutsch)

I am sure we will have [conscious computers], I expect they will be purely classical, and I expect that it will be a long time in the future. Significant advances in our philosophical understanding of what consciousness is, will be needed.

Question and Answers with David Deutsch, on New.Scientist.com News Service, December, 2006

Natural phenomena as discipline problem

- Martin Davis versus the hypercomputationalists (Jack Copeland et al.) -

The great success of modern computers as all-purpose algorithm-executing engines embodying Turing's universal computer in physical form, makes it extremely plausible that the abstract theory of computability gives the correct answer to the question 'What is a computation?', and, by itself, makes the existence of any more general form of computation extremely doubtful.

Martin Davis [2004], The myth of hypercomputation, in Alan Turing: Life and legacy of a great thinker (C. Teuscher, ed.), Springer-Verlag

But back in the real world ...

- Persistence of problems of predictability - quantum uncertainty, emergent phenomena, chaos and strange attractors, relativity and singularities (black holes)

- Renewed interest in analog and hybrid computing machines leading to: "... the classical Turing paradigm may no longer be fully appropriate to capture all features of present-day computing."


Co-operative phenomena

1970 - Georg Kreisel proposes a collision problem related to the 3-body problem, which might result in "an analog computation of a non-recursive function"
Mathematical analogues of chaos

- Growth of Chaos theory, generation of informational complexity via very simple rules, accompanied by the emergence of new regularities - e.g. Robert Shaw's dripping tap [1984]

- Link between structures in nature and mathematical objects, such as the Mandelbrot and Julia sets

- Penrose, Smale - computability of Mandelbrot, Julia sets?

Now we witnessed ... a certain extraordinarily complicated looking set, namely the Mandelbrot set. Although the rules which provide its definition are surprisingly simple, the set itself exhibits an endless variety of highly elaborate structures.

Roger Penrose
in "The Emperor's New Mind", Oxford Univ. Press, 1994

Is the Mandelbrot set computable?

Emergence occurs everywhere ...

Cat's Eye Nebula

Emergence of patterns in Nature

1950s - Alan Turing proposes a simple reaction-diffusion system describing chemical reactions and diffusion to account for morphogenesis, i.e., the development of form and shape in biological systems.

From website of the Biological Modeling and Visualization research group, Department of Computer Science at the University of Calgary:

See http://www.swintons.net/jonathan/turing.htm
Big Claims -

Emergence is often invoked in an almost mystical sense regarding the capabilities of behavior-based systems. Emergent behavior implies a holistic capability where the sum is considerably greater than its parts. It is true that what occurs in a behavior-based system is often a surprise to the system’s designer, but does the surprise come because of a shortcoming of the analysis of the constituent behavioral building blocks and their coordination, or because of something else?


A Test for Emergence

1) Design: The system has been constructed by the designer, by describing local elementary interactions between components (e.g., artificial creatures and elements of the environment) in a language $L_1$.

2) Observation: The observer is fully aware of the design, but describes global behaviors and properties of the running system, over a period of time, using a language $L_2$.

3) Surprise: The language of design $L_1$ and the language of observation $L_2$ are distinct, and the causal link between the elementary interactions programmed in $L_1$ and the behaviors observed in $L_2$ is non-obvious to the observer - who therefore experiences surprise. In other words, there is a cognitive dissonance between the observer’s mental image of the system’s design stated in $L_1$ and his contemporaneous observation of the system’s behavior stated in $L_2$.


Descriptions and Emergent Structure

- **Notice** - It is often possible to get descriptions of emergent properties in terms of the elementary actions.

- E.g., this is what Turing did for the role of Fibonacci numbers in relation to the sunflower etc.

- In mathematics, it is well-known that complicated descriptions may take us beyond what is computable.

- A potential source of surprise in emergence...

Intuition - entities exist because of, and according to, mathematical laws. In the words of Leibniz [1714] -

“... there can be found no fact that is true or existent, or any true proposition, without there being a sufficient reason for its being so and not otherwise, although we cannot know these reasons in most cases.”
... and definability
the key concept

- So natural phenomena not only generate descriptions, but arise and derive form from them...

- ... so connecting with a useful abstraction - the concept of mathematical definability...

- ... formalising desirability in a mathematical structure

- Giving precision to our experience of emergence as a potentially non-algorithmic determinant of events

I believe the following aspects of evolution to be true, without knowing how to turn them into (respectable) research topics.

- Important steps in evolution are robust. Multicellularity evolved at least ten times. There are several independent origins of eusociality. There were a number of lineages leading from primates to humans. If our ancestors had not evolved language, somebody else would have.

Supervenience ‘represents the idea that mentality is at bottom physically based, and that there is no free-floating mentality unanchored in the physical nature of objects and events in which it is manifested’

-A set of properties A supervenes upon another set B just in case no two things can differ with respect to A-properties without also differing with respect to their B-properties.”

Supervenience ‘represents the idea that mentality is at bottom physically based, and that there is no free-floating mentality unanchored in the physical nature of objects and events in which it is manifested’

The role of a clarified notion of emergence in pinning down the nature of supervenience - and so, of intelligence

- Physicalism and dualism reconciled...
Descartes revisited ...

- Achieve a non-reductive physicalism, delivering -
- Mind-body supervenience
- The physical irreducibility of the mental - including consciousness, qualia
- And the causal efficaciousness of the mental
- With definability removing conflict between ‘vertical’ determination and ‘horizontal’ causation

Emergence and Mathematical Intuition

“At first Poincaré attacked [a problem] vainly for a fortnight, attempting to prove there could not be any such function ...” [quoting Poincaré]:

‘Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it ... I did not verify the idea ... I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience sake, I verified the result at my leisure.”

from Jacques Hadamard [1945], “The Psychology of Invention in the Mathematical Field”, Princeton Univ. Press

Intelligent Thoughts as Emergent Phenomena

- Need to bridge the gap between ‘emergent’ higher mental functionality and ... what algorithmic ‘design’?

Difficult - Rodney Brooks [Nature, 2001]: “neither Al nor Alife has produced artifacts that could be confused with a living organism for more than an instant.”
Intelligent Thoughts as Emergent Phenomena

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So does emergence explain what we observe ... is that all there is?

Connectionist Models of Computation?

- These have come a long way since Turing's [1948] discussion of ‘unorganised machines’, and McCulloch and Pitts [1943] early paper on neural nets

- But for Steven Pinker “... neural networks alone cannot do the job”.

There is a reasonable chance that connectionist models will lead to the development of new somewhat-general-purpose self-programming, massively parallel analog computers, and a new theory of analog parallel computation: they may possibly even challenge the strong construal of Church’s Thesis as the claim that the class of well-defined computations is exhausted by those of Turing machines.

And focussing on our elusive higher functionality, he points to a "kind of mental fecundity called recursion"...
Emergent mental images re-used

We humans can take an entire proposition and give it a role in some larger proposition. Then we can take the larger proposition and embed it in a still-larger one. Not only did the baby eat the slug, but the father saw the baby eat the slug, and I wonder whether the father saw the baby eat the slug, the father knows that I wonder whether he saw the baby eat the slug, and I can guess that the father knows that I wonder whether he saw the baby eat the slug, and so on.


Not a closed system...

“As the brain forms images of an object - such as a face, a melody, a toothache, the memory of an event - and as the images of the object affect the state of the organism, yet another level of brain structure creates a swift nonverbal account of the events that are taking place in the varied brain regions activated as a consequence of the object-organism interaction. The mapping of the object-related consequences occurs in first-order neural maps representing the protoself and object; the account of the causal relationship between object and organism can only be captured in second-order neural maps. ... one might say that the swift, second-order nonverbal account narrates a story: that of the organism caught in the act of representing its own changing state as it goes about representing something else.”

- Antonio Damasio [1999], The Feeling Of What Happens, p.170

The role of external interaction ...

... taking us beyond thinking of intelligence as something that resides purely within the autonomous brain:

“Real computational systems are not rational agents that take inputs, compute logically, and produce outputs ... It is hard to draw the line at what is intelligence and what is environmental interaction. In a sense, it does not really matter which is which, as all intelligent systems must be situated in some world or other if they are to be useful entities.”


And computing with changing data ...

“. . . if a machine is expected to be infallible, it cannot also be intelligent. There are several theorems which say almost exactly that.”

And computing with changing data ...

“. . . if a machine is expected to be infallible, it cannot also be intelligent. There are several theorems which say almost exactly that.”


Note: Can model computation relative to changing data via oracle Turing machines.

Nature in the driving seat

- Getting intelligent machines themselves via emergence . . .

“I used to think we’d do it by engineering. Now I believe we’ll evolve them. We’re likely to make thinking machines before we understand how the mind works, which is kind of backwards.”

- Daniel Hillis, Chief Technology Officer of Applied Minds, Inc. (and ex-Vice President, Research and Development at Walt Disney Imagineering), April 2001.

Nature in the driving seat

Problem - Hard to control emergence, or predict its outcomes.

Can do it experimentally - e.g., wind-tunnels, computer simulations.

Or can make probabilistic predictions - e.g., predicting outcomes of measurement at quantum level, mapping details of radioactive decay.

Nature evolved us - but took a long time ...

Could a Computer Write “Hamlet”?
Could a Computer Write “Hamlet”? 

A: Maybe not in real-time

(Back where we started, with Nature - in the form of William Shakespeare - in the driving seat)

Footnote - Definability in What Structure?

- In modelling the physical universe -
  ... causality itself is fundamental

Lee Smolin, 'The Trouble With Physics', p.241

Turing on Description versus Computation

- Turing, 1939 - The computational content of descriptions can be captured hierarchically - but in unpredictable ways
- No consistent axiomatic theory captures arithmetic (Gödel) - but we can hierarchically transcend this barrier
- But then - identifying the route to a new theorem involves using an incomputable oracle

An explanation of why written proofs do not tell us how the proof was discovered . . .
The Turing model extended

- 1939 - Turing's oracle Turing machines appear
- Provides a model of algorithmic content of structures, based on p.c. functionals over the reals

The Turing landscape, causality and emergence ...

- Can describe global relations in terms of local structure ...
- ... so capturing the emergence of large-scale formations
- Mathematically - formalise as definability over structure based on Turing functionals?
- More generally - as invariance under automorphisms

Hartley Rogers’ programme ...

- Fundamental problem: Characterise the Turing invariant relations
Hartley Rogers’ programme ...

Fundamental problem: Characterise the Turing invariant relations

- **Intuition**: These are key to pinning down how basic laws and entities emerge as mathematical constraints on causal structure.

- **Notice**: The richness of Turing structure discovered so far becomes the raw material for a multitude of non-trivially definable relations.

Thank you!