Splitting properties of total enumeration degrees

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This paper (see Theorem 6 below) describes general conditions under which relative splittings and specified diamond embeddings are derivable in the local structure of the enumeration degrees. In so doing, three underlying themes are touched on — that of characterising the context for the Turing degrees provided via enumeration reducibility; secondly, general questions of definability and the role of splitting and nonsplitting; and also (emerging from the techniques) the description of new relationships between information content and degree theoretic structure.

Jockusch [1968] introduced the notion of a semirecursive set:

$A \subseteq \omega$ is *semirecursive* if there is a computable $f : \omega^2 \to \omega$ for which, for all x, y:

(i) $f(x, y) = x$ or $f(x, y) = y$,

(ii) $x \in A \vee y \in A \implies f(x, y) \in A$.

The paper begins with a result (Theorem 1) which shows that using semirecursive sets one can construct minimal pairs of e-degrees by both effective and uniform ways (see Theorem 2). Following this, new results concerning the local distribution of total e-degrees (Theorem 7) and of the degrees of semirecursive sets (Theorem 5) enable one to proceed via the natural embedding of the Turing degrees in the enumeration degrees to results concerning embeddings of the diamond lattice in the e-degrees, preserving least

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and greatest elements (see Corollaries 8, 9 and Theorem 6). A particularly striking application of these techniques is a relatively simple derivation of (a strong generalisation of) the Ahmad Diamond Theorem (see Theorem 6).

There are a number of known constraints on further progress in this direction. One of these is the result of Ahmad and Lachlan [1998] showing the existence of a nonsplitting Δ_2^0 e-degree $> \mathbf{0}_e$ (see also Theorem 10 below), with a corresponding non-diamond theorem for the e-degrees (Arslanov, Cooper, Kalimullin and Li [ta]). There are others limiting the local distribution of total e-degrees (e.g. Cooper and Copestake [1988], and more recently Arslanov, Cooper, Kalimullin and Li [ta]). But there remain a number of open questions, two of which (Questions 12, 13) are briefly discussed at the end of the paper.

For further background concerning enumeration reducibility and its degree structure, the reader is referred to Cooper [1990], Sorbi [1997] or Odifreddi [1999], chapter XIV.

1 Semirecursive sets

The results of this section will imply that e-degrees of semirecursive sets are situated "closely enough" to $\mathbf{0}$, the e-degree of the c. e. sets. On the other side, it follows from Theorem 5 that the class of e-degree of semirecursive sets generates under $\cup$ at least all the total e-degrees. Therefore, this class is an automorphism base in the structure of all e-degrees since it is known that any e-degree is the greatest lower bound of two total e-degrees (see Sorbi [1998]).

Theorem 1 *Let B be a semirecursive set. Then there is a computable function h such that for all i and j,*

$$\Phi_i^B \bigcap \Phi_j^{\overline{B}} \subseteq W_{h(i,j)} \subseteq \Phi_i^B \bigcup \Phi_j^{\overline{B}},$$

where $\{\Phi_e\}_{e\in\omega}$ and $\{W_e\}_{e\in\omega}$ are effective enumerations of all e-operators and c. e. sets, accordingly.

Proof. Let f be a computable function (the *semirecursive function* for B) such that

for all x, y, $f(x,y) \in \{x,y\}$, and

$$\{x, y\} \cap B \neq \emptyset \to f(x, y) \in B.$$

Given i and j enumerate a c. e. set $C(i, j)$ in the following manner: for any $x \in \omega$, $x \in C$ iff

a) there are finite sets F and G such that $x \in \Phi_i^F$ and $x \in \Phi_j^G$, and

b) $f(x, y) = x$ for all $x \in F$ and all $y \in G$.

This strategy, obviously, provides an effective enumeration for C. Therefore, there is a computable function $h(i, j)$ such that $C = W_{h(i,j)}$ for all $i, j \in \omega$. Now let $x \in \Phi_i^B \bigcap \Phi_j^{\overline{B}}$. Then $x \in \Phi_i^F$ for some finite set $F \subset B$ and $x \in \Phi_j^G$ for some finite set $G \subset \overline{B}$. Therefore, by the construction, $x \in W_{h(i,j)}$. Now let $x \in W_{h(i,j)}$. Then $x \in \Phi_i^F$ and $x \in \Phi_j^G$ for some finite sets F and G such that $f(x, y) = x$ for all $x \in F$ and all $y \in G$. Suppose, in order to get a contradiction, that $x \notin \Phi_i^B \bigcup \Phi_j^{\overline{B}}$. This means that $F \not\subseteq B$, and therefore there is an $a \in F$ such that $a \in \overline{B}$. Similarly, there is a $b \in G$ such that $b \in B$. But then one has $f(a, b) = b$ which contradicts clause b) of the construction. $\square$

As a corollary one obtains the following

Theorem 2 *For any semirecursive set B*
1) (Effectiveness) $\deg_e(B) \cap \deg_e(\overline{B}) = \mathbf{0}$, and, moreover, given i and j such that $\Phi_i^B = \Phi_j^{\overline{B}}$ one can effectively find an index of a c. e. set C such that $C = \Phi_i^B = \Phi_j^{\overline{B}}$;
2) (Uniformity) For any set $X \subseteq \omega$, the following holds:

$$\deg_e(X) = \deg_e(X \oplus B) \bigcap \deg_e(X \oplus \overline{B}).$$

Proof. 1) If $\Phi_i^B = \Phi_j^{\overline{B}}$ for some i and j, then $\Phi_i^B \bigcap \Phi_j^{\overline{B}} = \Phi_i^B \bigcup \Phi_j^{\overline{B}}$. Now the proof immediately follows from Theorem 1.
2) Immediately follows from Theorem 1 relativised to X with obvious changes.
$\square$

Recall that if $A \in \mathbf{a}$ then $\mathbf{a}' = \deg_e(J(A) \oplus \overline{J(A)})$, where $J(A) = \{e : e \in \Phi_e(A)\}$. It is easy to see that $A \equiv_e J(A)$, $\overline{A} \leq_e \overline{J(A)}$ and $\overline{K} \leq_e \overline{J(A)}$. Therefore $A \oplus \overline{A} \oplus \overline{K} \leq_e J(A) \oplus \overline{J(A)}$. The following theorem shows that if A is semirecursive with a non-c.e. complement then $J(A) \oplus \overline{J(A)}$ has the least possible degree (namely, the degree of $A \oplus \overline{A} \oplus \overline{K}$).

Theorem 3 *Let A be a semirecursive set. Then either $\overline{A}$ is c. e. or $\overline{J(A)} \leq_e \overline{A} \oplus \overline{K}$.*

Proof. Since A is semirecursive then by Jockusch [1968] A is a lower cut for some computable linear ordering $<_L$ of ω. Let $\max_L(F)$ denotes the maximum element of a finite set F under $<_L$.

Suppose $\overline{A}$ is not c. e. Note that if $e \notin \Phi_e(A)$ then $V_e \subset \overline{A}$, where V_e is the following c. e. set:

$$\{x : (\exists \text{ finite } F)[\max_L(F) <_L x \;\&\; e \in \Phi_e(F)]\}.$$

Hence, if $e \notin J(A)$ then $\overline{A} - V_e \neq \emptyset$. This implies that for each x

$$e \in \overline{J(A)} \text{ iff } (\exists x \in \overline{A})(\forall \text{ finite } F)[\max_L(F) <_L x \rightarrow e \notin \Phi_e(F)],$$

which provides $\overline{J(A)} \leq_e \overline{A} \oplus \overline{K}$. □

Theorem 4 *If A is a semirecursive set such that $A, \overline{A}$ are not c. e., then e-degree of A is quasiminimal.*

Proof. Let $B \oplus \overline{B} \leq_e A$ for some set B. Let $B = \Phi^A$ and $\overline{B} = \Psi^A$. Let A is a lower cut for a computable linear ordering $\leq_L$ of ω. We may assume (see the previous theorem) that if $\langle x, F\rangle \in \Theta$ then $|F| = 1$, where $\Theta = \Phi$ or Ψ. Let $S = \{n : \exists m \leq_L n(\Phi^n \cap \Psi^n \neq \emptyset)\}$.

Clearly, $S \subseteq \overline{A}$.

Case 1. Let there is an integer $a \in \overline{A}$ such that $a \leq_L x$ for all $x \in S$. Then for all y, $y \in B \iff \exists n <_L a(\langle y, n\rangle \in \Phi)$. Therefore, B is c. e.

Case 2. Otherwise. Then, obviously, $\overline{A} = \{a : \exists x \in S(x <_L a)\}$. This means that $\overline{A}$ is c. e. □

Jockusch [1968] proved that for any noncomputable set A there is a semirecursive set $B \equiv_T A$ such that B and $\overline{B}$ are non c. e. Below in Theorem 5 we generalise this result. This generalization will be used in the proof of Theorem 6.

Theorem 5 *Let $C <_T A$. Then there is a semirecursive set $B \equiv_T A$ such that B and $\overline{B}$ are not c. e. in C.*

Proof. We consider two cases.

Case 1. $\deg_T(A)$ is not c. e. in C. Let $\Lambda = \{\,0, 1\,\}$ with the usual order $0 <_\Lambda 1$, and $T = \Lambda^{<\omega}$. Define $\delta \in \Lambda^\omega$ by the relation $\delta(x) = 1 \iff x \in A$, and let

$B = \{\sigma \in T : \exists \tau \subset \delta(\sigma <_L \tau)\}$. (Here $<_L$ is a standard ordering defined on the strings of the tree T.) Obviously, $A \equiv_T B$ and B is semirecursive.

Case 2. $\deg_T(A)$ is c. e. in C. Without loss of generality we can assume that A is itself c. e. in C. Let $\{A_s\}_{s\in\omega}$ be a C-computable sequence of finite sets such that $A_s \subseteq A_{s+1}$ and $A = \bigcup_s A_s$. Now let $\Lambda = \{\,-1, 0, 1\,\}$ with the ordering $-1 <_\Lambda 0 <_\Lambda 1$, and $T = \Lambda^{<\omega}$. We again define the desired set B as $B = \{\sigma \in T : \exists \tau \subset \delta(\sigma <_L \tau)\}$ for some $\delta \in \Lambda^\omega$. We shall define a C-computable approximation to δ, $\{\delta_s\}_{s\in\omega}$, where $\delta_s \in \Lambda^{<\omega}$, $|\delta_s| = s$, and for all x $\delta(x) = \lim_s \delta_s(x)$, meeting for all $e \in \omega$ requirements:

$$\mathcal{N}_{2e} : \overline{A} = \Phi_e^{B\oplus C\oplus\overline{C}} \to \overline{A} \text{ c. e in } C,$$

$$\mathcal{N}_{2e+1} : \overline{A} = \Phi_e^{\overline{B}\oplus C\oplus\overline{C}} \to \overline{A} \text{ c. e in } C,$$

where $\{\Phi_e\}_{e\in\omega}$ is an effective enumeration of all computable enumeration operators. We will also meet the condition $\delta \equiv_T A$.

These requirements together ensure that $B \equiv_T A$ and B and $\overline{B}$ are not c. e. in C, as required. Indeed, since $B \equiv_T \delta$ and $\delta \equiv_T A$, we have $A \equiv_T B$. Further, if, for instance, B is c. e. in C, then $\overline{A} \leq_e A \oplus \overline{A} \equiv_e B \oplus \overline{B} \leq_e \overline{B} \oplus C \oplus \overline{C}$ which implies to the contrary that $\overline{A}$ is c.e. in C.

Without loss of generality we may assume that for all e-operators Φ_e and any finite set F the following holds for any $s \in \omega$

$$\langle x, F\rangle \in \Phi_{e,s} \longrightarrow (\forall \sigma \in F)(|\sigma| < s).$$

Define for $\Phi_{e,s}$ the C-computable use-function $u(G; e, x, s)$ for a finite set G as the finite set $F \subseteq G$ with the least canonical index such that $\langle x, F \oplus H_1 \oplus H_2\rangle \in \Phi_{e,s}$ for some finite sets $H_1 \subseteq C$ and $H_2 \subseteq \overline{C}$. If there is no such F then $u(G; e, x, s) = \emptyset$.

The construction (which is computable in C).

Stage $s = 0$. Let $\delta_0 = \lambda$ (the empty string), and $B_0 = \emptyset$.

Stage $s+1$. We assume by induction that δ_s and B_s already are defined. Let

$l(2e, s) = \max\{k \leq s : \overline{A}_s\lceil k = \Phi_{e,s}^{B_s\oplus C\oplus\overline{C}}\lceil k\}$,
$l(2e+1, s) = \max\{k \leq s : \overline{A}_s\lceil k = \Phi_{e,s}^{\overline{B}_s\oplus C\oplus\overline{C}}\lceil k\}$,
$R(2e, s) = \bigcup\{F : \exists t \leq s \exists x \leq l(e, t)(F = u(B_t, e, x, t))\}$,

$R(2e+1,s) = \bigcup\{F : \exists t \leq s \exists x \leq l(e,t)(F = u(\overline{B}_t, e, x, t))\}$.

Case 1. $\forall k < s(k \in A_{s+1} \to k \in A_s)$. Then define $\delta_{s+1} = \delta_s \frown n$, where

$$n = \begin{cases} 0, & \text{if } s \notin A_{s+1}, \\ 1, & \text{if } s \in A_{s+1}. \end{cases}$$

Case 2. $\exists k < s(k \in A_{s+1} - A_s)$. Let k_0 be the least such k. Now let m be the least integer (if any) such that $m = 2e+i, i < 2$, and $\{\sigma \in T : |\sigma| \geq k_0\} \cap R(m,s) \neq \emptyset$. (If there is no such m then let $m = 0$.) Now define

$$\delta_{s+1}(k_0) = \begin{cases} 1, & \text{if } i = 0, \\ -1, & \text{if } i = 1. \end{cases}$$

$$\delta_{s+1}(k) = \delta_s(k),$$

if $k < k_0$,

$$\delta_{s+1}(k) = \begin{cases} 0, & \text{if } k \notin A_{s+1}, \\ 1, & \text{if } k \in A_{s+1}, \end{cases}$$

if $k_0 < k < s+1$.

In both cases let $B_{s+1} = \{\sigma \in T : \sigma <_L \delta_{s+1}\}$. Therefore, in Case 2 we have

$$i = 0 \to \delta_s <_L \delta_{s+1} \to B_s \subseteq B_{s+1},$$

$$i = 1 \to \delta_{s+1} <_L \delta_s \to B_{s+1} \subseteq B_s \to \overline{B}_s \subseteq \overline{B}_{s+1}.$$

This means that the requirement $\mathcal{N}_m$ is not injured at stage $s+1$. This completes the construction. Let $\delta = \lim_s \delta_s$ and $B = \lim_s B_s = \{\sigma \in T : \exists \tau \subset \delta(\sigma <_L \tau)\}$.

Is is clear that B is semirecursive. The reducibility $A \leq_T \delta$ follows from $x \in A \iff \delta(x) \neq 0$ for any x. The reducibility $\delta \leq_T A$ follows from the fact that if $A_s\lceil(x+1) = A\lceil(x+1)$ and $s > x$ then $\delta_s(x) = \delta(x)$.

Now we show that all requirements $\mathcal{N}_m$ are satisfied for all $m \geq 0$.

Suppose, in order to get the contradiction, that there is a (least) $m = 2e+i$ such that either the requirement $\mathcal{N}_m$ is not satisfied (which means that $\overline{A} = \Phi^{B \oplus C \oplus \overline{C}}$, if $i = 0$, and $\overline{A} = \Phi^{\overline{B} \oplus C \oplus \overline{C}}$, if $i = 1$), or the set $R(m) = \bigcup\{R(m,s) : s \in \omega\}$ is infinite. Then one necessarily has $\limsup_s l(m,s) = \infty$. Obviously, we have also that the set $R = \bigcup\{R(m') : m' < m\} = \bigcup_{m' < m} \bigcup_{s \in \omega} R(m', s)$

is finite. Fix a stage s' such that for every $k \leq \max\{n : \exists\sigma \in R(n = |\sigma|)\}$ we have $A(k) = A_{s'}(k)$. Then for all $x \in \omega$, $x \in \overline{A} \leftrightarrow \exists s > s'(x \in \overline{A}_s \ \&\ x < l(m,s))$. Therefore, $\overline{A}$ is c. e. in C since the construction above is C-computable. Since A is also c. e. in C this implies that $A \leq_T C$ which is a contradiction. □

2 Splitting properties and the diamond embedding

This section begins with the following strong generalisation of Ahmad's Diamond Theorem (see Ahmad [1991]):

Theorem 6 *1) Let* $\mathbf{a}$ *be a given e-degree and* $\mathbf{b}$ *be total degree such that* $\mathbf{a} < \mathbf{b}$*. Then the diamond lattice is embeddable in the e-degrees with* $\mathbf{b}$ *as the top element and* $\mathbf{a}$ *as the least element, given a total degree* $\mathbf{c}$ *such that* $\mathbf{a} \leq \mathbf{c} < \mathbf{b}$*.*

2) Let $\mathbf{a} > \mathbf{0}$ *be a total* Δ^0_2 *degree. Then the diamond lattice is embeddable in the low e-degrees with* $\mathbf{a}$ *as the top element and* $\mathbf{0}$ *as the least element.*

Proof. 1) Let $A \in \mathbf{a}$, $C \oplus \overline{C} \in \mathbf{c}$ and $B \oplus \overline{B} \in \mathbf{b}$. Then $C <_T B$. By Theorem 5 there exists a semirecursive set $X \equiv_T B$ such that X and $\overline{X}$ are not c. e. in C which means that $X \not\leq_e C \oplus \overline{C}$ and $\overline{X} \not\leq_e C \oplus \overline{C}$. It follows that $X \not\leq_e A$ and $\overline{X} \not\leq_e A$. Now let $\mathbf{u} = \deg_e(A \oplus X)$ and $\mathbf{v} = \deg_e(A \oplus \overline{X})$. It follows from Theorem 2 that $\mathbf{u} \cap \mathbf{v} = \mathbf{a}$, also obvious that $\mathbf{u} \cup \mathbf{v} = \mathbf{b}$.

2) Let $A \subset \omega$ be a set such that $A \oplus \overline{A} \in \mathbf{a}$. Let B be a semirecursive set such that $A \equiv_T B$ and B and $\overline{B}$ are not c. e. Since $B \oplus \overline{B} \equiv_e A \oplus \overline{A}$, then $\deg_e(B) \cup \deg_e(\overline{B}) = \mathbf{a}$. The part $\deg_e(B) \cap \deg_e(\overline{B}) = \mathbf{0}_e$ follows from Theorem 2. Since B and $\overline{B}$ are not c. e. then $\deg_e(B) \not\leq \deg_e(\overline{B})$ and $\deg_e(\overline{B}) \not\leq \deg_e(B)$. Now apply Theorem 3. □

(By Ahmad and Lachlan [1998], it is not possible in part 2) of the previous theorem to replace 'total Δ^0_2' with 'Δ^0_2'.)

In view of Theorem 6 it is important to investigate for which degrees $\mathbf{a} < \mathbf{b}$, with $\mathbf{b}$ total, there is a total degree $\mathbf{c}$ such that $\mathbf{a} \leq \mathbf{c} < \mathbf{b}$. Below in Theorem 10 we prove that in general the answer to this problem is negative: There is a Δ^0_2 degree $\mathbf{a}$ and a total degree $\mathbf{b} > \mathbf{a}$ such that there is no total degree $\mathbf{c}$ strongly between $\mathbf{a}$ and $\mathbf{b}$) (see Corollary 11). However, from Theorem 7 it follows that the answer to this question is "yes" at least for $\mathbf{b} = \mathbf{0}'$.

Theorem 7 *Let* $\mathbf{a} < \mathbf{0}'_e$ *be a* Δ^0_2*-e-degree. Then there is a total* Δ^0_2*-e-degree* $\mathbf{b}$ *such that* $\mathbf{a} \leq \mathbf{b} < \mathbf{0}'_e$.

Note that Theorem 7 allows one to easily obtain without using the results from previous sections the following corollary, which was first proved in Arslanov, Sorbi [1999] by a direct construction.

Corollary 8 $\overline{K}$ *is splittable in* Δ^0_2*-e-degrees above any* Δ^0_2*-e-degree.*

Proof. Let A be a Δ^0_2-set such that $A <_e \overline{K}$. By the previous theorem there is a set B such that $A <_e B$ and $B <_T K$. Now a splitting of $\mathbf{0}'$ above $\deg_T(B)$ in Turing degrees (which is possible by a relativised version of Sacks' Splitting Theorem — see Soare [1987]) obviously provides a splitting of $\mathbf{0}'_e$ above $\deg_e(A)$ in the e-degrees. $\square$

Theorem 7 shows now that one can prove essentially more:

Corollary 9 *For any given* Δ^0_2*-e-degree* $\mathbf{a} < \mathbf{0}'$ *the diamond lattice is embeddable into the* Δ^0_2 *e-degrees preserving* $\mathbf{0}'_e$ *and* $\mathbf{a}$.

Proof. This follows immediately from Theorems 7 and 6. $\square$

Proof of Theorem 7. Let $A <_e \overline{K}$ be a Δ^0_2-set. To ensure that $A \leq_e X \oplus \overline{X} <_e \overline{K}$ for a set X, it is enough to construct X such that $A \leq_e X$ and $X <_T K$. One need to satisfy the requirements:

$$P : x \in A \leftrightarrow \exists y(\langle x, y\rangle \in X)$$

(which obviously gives $A \leq_e X$), and

$$S_e : K = \Phi^X_e \rightarrow \overline{K} = \Psi^A_e$$

(i. e., given a Turing partial computable functional Φ_e, we build an enumeration operator Ψ_e).

Strategy for P: Given $x \in A$ at a stage s, enumerate $\langle x, y\rangle$ into X for a big y. If later x leaves A then remove $\langle x, y\rangle$ from X. If later again x enters A then enumerate $\langle x, y'\rangle$ into X with a new big y', etc.

Strategy for S_e: For any given x, while $x \in \overline{K}$:

1) Wait for $\Phi_{e,s}^{X_s}(x) \downarrow = 0$ at a stage s with a *use* $\varphi_{e,s}(x)$. Let $F_x = \{y \leq \varphi_{e,s}(x) \mid y \in X_s\}$. By the P-strategy, for any $y \in F_x$, we have $y = \langle a, b \rangle$ for some $a \in A_s$. Define $F_{A,x} = \{a \mid \exists b \exists y(y \in F_x \ \& \ y = \langle a, b \rangle\}$.

2) Enumerate $\langle x, F_{A,x} \rangle$ into Ψ_e^A, and restrain $X \upharpoonright \varphi_{e,s}(x)$ against other strategies from now on.

In case of any A-change at $F_{A,x}$ go back to 1).

If later $x \in K$, and again $K(x) = \Phi_{e,t}^{X_t}(x)$ at a stage $t > s$, then this means that we had an $X \upharpoonright \varphi_{e,s}(x)$-change between stages s and t, which means that some $y \leq \varphi_{e,s}(x)$ is removed from F_x (since nothing from this interval can be enumerated into F_x after stage s). This means, by the P-strategy, that an element $a \in F_{A,x}$ is removed from A between stages s and t. Therefore, either $x \notin \Psi_e^A$, or later x enters A again. In the latter case, we may diagonalise $K(x)$ against $\Phi_e^X(x)$, enumerating back into X or removing from X all elements F_x removed from X (or enumerated into X) between stages s and t.

We now proceed with the formal construction and verifications.

The construction.

Throughout the construction $(\omega+1)^{<\omega}$ will be referred to as the *tree of outcomes* T. The elements of T will be called *strings* or *nodes*. If $\alpha, \beta \in S = 2^{<\omega}$, then *compat*$(\alpha, \beta)$ means that "either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$".

Each node $\sigma \in T$ will be assigned the requirement $S_{|\sigma|}$, and each node $\sigma \in T$ will be assigned a changeable parameter $r(\sigma)$ which is the restraint of the strategy σ. For each node $\sigma \in T$, each finite set F and each integer $k \in \omega$ the triple (σ, F, k) will be assigned a parameter $\alpha_k(\sigma, F) \in S$. Here $\alpha_k(\sigma, F) \in S$ is an initial segment which is needed by the requirement $S_{|\sigma|}$ for the diagonalisation $\Phi_{|\sigma|}^X(k)$ against $K(k)$ in case that $k \in \Psi_e^F - \overline{K}$ and $F \subseteq A$. Denote the value of $\alpha_k(\sigma, F)$ at point x by $\alpha_k(\sigma, F, x)$.

Stage $s = 0$. Let $X_0 = \emptyset$, and for all $\sigma \in T, k \in \omega$, and finite sets F let $r(\sigma) = 0, \delta_0 = \alpha_k(\sigma, F) = \lambda, \Psi_{|\sigma|,0} = \emptyset$.

Stage $s+1$. For any $\sigma \in T$ such that for some $\langle x, y \rangle \in X_s \lceil r(\sigma)$ we have $x \notin A_{s+1}$, we define $r(\sigma) = -1$ (remove the restraint). In this case we say that this restraint becomes *obsolete*.

Step 1 (*the definition of δ_{s+1}*). Say that the strategy $\sigma \in T$ *requires attention at stage* $s+1$ if $\exists k \in K_{s+1} \exists F \subseteq A_{s+1}(\langle x, F\rangle \in \Psi_{\sigma,s} \& \ \forall \tau \prec \sigma(compat(\alpha_k(\sigma, F), X_s\lceil r(\tau)))$ (i. e. $\alpha_k(\sigma, F)$ does not threaten to injure restraints associated with higher priority strategies).

Now we define $\delta_{s+1} \in T$ as a string of the length $s+1$ by induction. Let $\delta_{s+1}\lceil n$ is defined for $n \leq s$, and let $\sigma = \delta_{s+1}\lceil n$. Then define $\delta_{s+1}(n) = \omega$, if either $r(\sigma) > 0$, or $r(\sigma) = 0$ and σ requires attention at stage $s+1$. Otherwise, let $s_0 + 1 \leq s$ be the least stage such that $\delta_{s_0+1} \subseteq \sigma\widehat{\ }k$ for some $k \in \omega$. (If there is no such stage the let $s_0 = 0$.) Now let k be the least integer such that $k \notin K_{s+1}$, and either $\Phi^{X_s}_{n,s}(k) \uparrow \vee \Phi^{X_s}_{n,s}(k) \downarrow \neq 0$, or $use(X_t, n, k, t) \neq use(X_s, n, k, s)$, or $X_t\lceil use(X_t, n, k, t) \neq X_s\lceil use(X_s, n, k, s)$ for some $t, s_0 \leq t \leq s$. Define $\delta_{s+1}(n) = k$. Initialise all strategies $\sigma \succ \delta_{s+1}$ (namely, let $r(\sigma) = 0$ and $\alpha_k(\sigma, F) = \lambda$ for all k, F).

For every $\sigma \subseteq \delta_{s+1}$ if $r(\sigma) = -1$ (i.e., if earlier the restraint of σ became obsolete) then redefine $r(\sigma) = 0$.

Step 2 (*the updating of axioms for Ψ_σ*). Let $\sigma \subset \delta_{s+1}$ such that $\delta_{s+1}(|\sigma|) < \omega$ is the maximum of all possible (finite) values of $\delta_t(|\sigma|)$ for $\sigma \subseteq \delta_t, |\sigma| < t \leq s$. For each $k \in \overline{K}_{s+1}\lceil s$ such that $k < \delta_{s+1}(|\sigma|)$ and $k \notin \Psi^{A_{s+1}}_{\sigma,s}$, define the axiom $\langle x, F\rangle \in \Psi_{\sigma,s+1}$, where $F = \{x : \exists y(\langle x, y\rangle \in X_s\lceil use(X_s, |\sigma|, k, s))\}$. (It follows from the definition of δ_{s+1} that if $\delta_{s+1}(|\sigma|) < \omega$, then $\forall x < \delta_{s+1}(|\sigma|)\Phi^{X_s}_{|\sigma|,s}(k) \downarrow = 0$).

Also, if $\alpha_k(\sigma, F) = \lambda$, then $\alpha_k(\sigma, F) := X_s\lceil use(X_s, |\sigma|, k, s)$.

Step 3 (*the renewal and the restraint*).

Case 1. $\exists\sigma(\sigma \subset \delta_{s+1} \& \sigma$ requires attention $\& r(\sigma) = 0)$. Choose a node $\sigma \subset \delta_{s+1}$ of the least length such that σ requires attention and $r(\sigma) = 0$. This means that

$$\exists k \in K_{s+1} \exists F \subseteq A_{s+1}(\langle x, F\rangle \in \Psi_{\sigma,s} \& \forall \tau \prec \sigma(compat(\alpha_k(\sigma, F), X_s\lceil r(\tau)))).$$

Let $\langle k, F\rangle$ be one of such pairs with the following additional property: elements of F have least possible number of changes in A before stage s.

Define an intermediate set X^*:

$\forall z < |\alpha_k(\sigma, F)|(X^*(z) = \alpha_k(\sigma, F, z)$,

$\forall z \geq |\alpha_k(\sigma, F)|(X^*(z) = X_s(z)$.

Let $r(\sigma) = |\alpha_k(\sigma, F)|$. Say that σ *receives attention* at stage $s+1$.

Case 2. $\forall \sigma \subset \delta_{s+1}$ either σ does not requires attention or $r(\sigma) > 0$. Then define $X^* = X_s$.

Step 4 (*the definition of* X_{s+1}).

If $x \notin A_{s+1}$, then $X_{s+1}(\langle x, y\rangle) = 0$ for all y. If $x \in A_{s+1}$ and $\langle x, y\rangle < \max\{r(\sigma) : \sigma \in T\}$, then $X_{s+1}(\langle x, y\rangle) = X^*(\langle x, y\rangle)$. If $x \in A_{s+1}$ and $\langle x, y\rangle \geq \max\{r(\sigma) : \sigma \in T\}$, then $X_{s+1}(\langle x, y\rangle) = 1$.

End of the construction.

The verification.

Lemma 1 *$\delta = \liminf_s \delta_s$ exists.*

Proof. Suppose by induction that $\delta\lceil n = \sigma$ exists. It will be proved that if the set $V = \{s : \sigma \subset \delta_s \ \&\ \delta_s(n) < \omega\}$ is infinite and $\lim_{s\in V} \delta_s(n) = \infty$, then $\overline{K} \leq_e A$.

Let s_0 be a stage such that $\forall s \geq s_0(\delta_s \geq \sigma)$. Let $k > s_0$. If $k \in \overline{K}$ then for almost all s $\Phi^{X_s}_{n,s}(k)\downarrow$, and $\lim_s \mathrm{use}(X_s, n, k, s) < \infty$. Therefore, $k \in \Psi^A_\sigma$.

If $k \in K$ then $k \notin \Psi^A_\sigma$, since otherwise for almost all s such that $\sigma \subset \delta_s$ we will have $\delta_s(n) = \omega$, a contradiction. Therefore, $\forall k > s_0(k \in \overline{K} \iff k \in \Psi^A_\sigma)$.

Lemma 2 *$X = \lim_s X_s$ exists.*

Proof. Let $\alpha_k(F, \sigma)[s]$ be the value of parameter $\alpha_k(F, \sigma)$ at the end of stage s. Fix an integer $z \in \omega$. Let $z = \langle x, y\rangle$. If $x \notin A$ then obviously $\lim_s X_s(z) = 0$. Let $x \in A$ and s^* be such a stage that $\forall s \geq s^*\ x \in A_s$.

Suppose, that $\lim_s X_s(z)$ does not exist. Say that z *is removed* at stage $s+1 \geq s^*$, if $z \in X_s - X_{s+1}$. Also, say that σ *removes* z at stage $s+1$, if at this stage σ receives attention.

Let $U(\sigma, x, s) = \{F : \exists k \in K(\alpha_k(F, \sigma, x)[s] \downarrow= 0)\}$ be the set of finite sets such that x can be removed by σ. Note, that if σ removes z at stage $s+1 \geq s^*$, then $\sigma \subseteq \delta_t$ for some $t \leq s^*$. Indeed, if $\sigma \not\subseteq \delta_t$ for all $t < s^*$, then z can enter some $F \in U(\sigma, z, u+1), u > s^*$ only if $z \notin X_u$, but this means that a strategy $\tau \subseteq \delta_t, t \leq s$, restrained z. Then $\tau\widehat{\ }\omega \subseteq \sigma$ and later at a stage σ will be initialised and $u(\sigma, x, s)$ will be $= \lambda$, since otherwise $\lim_s X_s(z) = 0$. It follows that σ cannot later remove z. Therefore, there exists a node σ from the finite set $T_0 = \{\sigma : \exists t \leq s^*(\sigma \subseteq \delta_t)\}$ such that σ removes z infinitely many times.

Clearly, $\sigma \not<_L \delta$. If $\delta <_L \sigma$ then σ will be initialised infinitely many times. Therefore, for such σ there is $\sigma' \subseteq \delta$, $\sigma' \in T_0$ such that it removes z infinitely many times. It follows that there is $\sigma \subseteq \delta, \sigma \in T_0$ such that σ removes z infinitely many times. Now let $t > s^*$ such a stage that $\forall s > t(\delta_s \not<_L \delta\lceil s^*)$. Clearly, if $|\sigma| \leq s^*$ and $\sigma\widehat{\ }\omega \subseteq \delta$, then σ can not remove z at stage $\geq t$.

Consider all σ with $|\sigma| \leq s^*$ and $\exists k \in \omega(\sigma\widehat{\ }k \subseteq \delta)$. Let they are $\sigma_1 \subset \sigma_2 \subset \ldots \subset \sigma_n$. It is easy to see that if $1 \leq i \leq n$, then

$$(*) \qquad \forall s > t \exists u > s \forall F \in U(\sigma_i, z, s) \forall v \geq u(F \not\subseteq A_v).$$

(Otherwise, there is a successful diagonalisation.)

Now we will prove that for any $i, 1 \leq i \leq n$, there is $u_i > t$ such that

$$(**) \qquad \forall v \geq u_i(U(\sigma_i, z, u_i) = U(\sigma_i, z, v)).$$

For $i = n$ the claim is obvious, since after stage t we don't put new elements into $U(\sigma_n, z, v)$. Now suppose that $(**)$ is proved for all $i', i < i' \leq n$. To prove $(**)$ for i we obtain stages $v_{i'}$ from $(*)$ with $s = u_{i'}$,$i < i' \leq n$. Let $v = \max\{v_i' : i < i' \leq n\}$. Then for $i < i' \leq n$ the value of parameter $r(\sigma_{i'})$ will be 0 beginning the stage v. Therefore, after stage v there will be no new elements in $U(\sigma_i, z, s)$. This proves $(**)$ for i.

Now let $s = \max\{u_i : 1 \leq i \leq n\}$. From $(*)$ we obtain that there is a stage $u > s$ such that $\forall i \forall v \geq u \forall F \in U(\sigma_i, z, v)(F \not\subseteq A_v)$. This means that $\sigma_i, 1 \leq i \leq n$, cannot remove z after stage u, a contradiction.

Lemma 3 *$A \leq_e X$.*

Proof. We will prove that for all x, $x \in A \iff \exists y(\langle x, y\rangle \in X)$. Indeed, if $x \notin A$, then obviously $\forall y(\langle x, y\rangle \notin X)$. Let $x \in A$. Let s^* be a stage such that $\forall s > s^*(x \in A_s)$. Now obviously integers $\langle x, y\rangle > s^*$ cannot be removed from X at stages $> s^*$, since use$(X_s, n, x, s) < s$ for all s. It follows that any such $\langle x, y\rangle > s^*$ belong X.

Lemma 4 *$X <_T K$.*

Proof. Suppose $K = \Phi_e^X$. Let $\sigma \subset \delta, |\sigma| = e$. If $\sigma\widehat{\ }k \subset \delta$ for a $k \in \omega$ then $K(k) = 0$ and $\Phi_e^X(k) \neq 0$. If $\sigma\widehat{\ }\omega \subset \delta$, then for some $k \in K$ we have $\Phi_e^X(k) \downarrow= 0$. This is a contradiction. $\square$

3 Negative results

Can one extend the above techniques to obtain a simple characterisation of those contexts allowing splitting or diamond constructions? The following theorem suggests otherwise, which opens out the prospect of such characterisations leading to nontrivially definable classes of enumeration degrees, with naturally correlated information content.

Theorem 10 *There is a Δ_2^0 set B such that $\overline{B} <_e B \oplus \overline{B}$, and $\deg_e(B \oplus \overline{B})$ is not splittable over the degree of $\overline{B}$.*

Proof. One needs to construct the set B meeting for all $e \in \omega$ requirements:

$$\begin{aligned} \mathcal{P}_e : \quad & B \neq \Xi_e(\overline{B}) \\ \mathcal{N}_e : \quad & B = \Phi_e(X_e(B) \oplus Y_e(B)) \rightarrow \\ & \qquad \exists \Gamma_e \Delta_e (B = \Gamma_e(X_e(B) \oplus \overline{B}) \vee B = \Delta_e(Y_e(B) \oplus \overline{B}), \end{aligned}$$

where $X_e(B) = \operatorname{dom} \Psi_e(B)$, $Y_e(B) = \operatorname{dom} \Theta_e(B)$ and $\{\Phi_e, \Psi_e, \Theta_e\}_{e \in \omega}$ an effective listing of all triples of e-operators Φ_e and Turing operators Ψ_e, Θ_e.

Let $T = 3^{<\omega}$ be the priority tree. If $\sigma \in T$ and $|\sigma| = e$ then we assign the node σ to both requirements $\mathcal{P}_e$ and $\mathcal{N}_e$. We will satisfy $\mathcal{P}_e$-requirement via a witness x of σ-strategy for $|\sigma| = e$. We later may choose a new witness for σ if its former witness is initialised. Let x_σ denotes the active (not initialised) witness for σ (to indicate that x_σ is the active witness for σ at stage s we sometimes will write it as $x_{\sigma,s}$).

For each witness x and any $\sigma \in T$ define the weight $w(x, \sigma)$ of x at σ as $|\sigma| +$ the number of other witnesses of σ which were assigned to σ before x. For convenience we define $w(x, \sigma) = 0$ for all x and σ such that x is not a witness of σ. (Therefore, $w(x, \sigma)$ can change its value once from 0 if later x becomes a witness for σ.)

It follows from the definition that for each $z \in \omega$ and each node σ there are only finitely many witnesses x with $w(x, \sigma) \leq z$.

For convenience we denote $\operatorname{dom} \Psi_{e,s}(Z)$ and $\operatorname{dom} \Theta_{e,s}(Z)$, for all $Z \subseteq \omega$, by $X_{e,s}(Z)$ and $Y_{e,s}(Z)$.

At each stage s of the construction we will define a string $\delta_s \in T$ of length s which is an approximation of the true path.

Construction.
Stage $s = 0$. Let $\delta_0 = \lambda$ (the empty string). Let $B_0 = \emptyset$, $\Gamma_{\sigma,0} = \Delta_{\sigma,0} = \emptyset$ for all $\sigma \in T$.
Stage $s + 1$.
Step 1 (defining δ_{s+1}). Suppose that $\delta_{s+1} \restriction n = \sigma$, $n \leq s$, is already defined. Given B_t, $t \leq s$, for each $\tau \in T$ define

$$l(\tau, t) = \max\{x \leq t : B_t \restriction x \subseteq \Phi_{|\tau|,t}(X_{|\tau|,t}(B_t) \oplus Y_{|\tau|,t}(B_t))\}.$$

If $l(\sigma, s+1) \leq \max\{x_\alpha : \sigma\widehat{\ }0 \subseteq \alpha \ \vee \ \sigma\widehat{\ }1 \subseteq \alpha\}$, define $\delta_{s+1} \restriction (n+1) = \sigma\widehat{\ }2$. Otherwise, check whether there exists a (initialised or active) witness y of a node τ, $\tau \subseteq \sigma\widehat{\ }1$, which is not restrained by some node $\alpha \prec \sigma\widehat{\ }1$, such that

$$\text{(I)} \qquad w(y, \tau) > 1 + \max\{w(x, \beta) : x \in \omega \ \& \ \sigma\widehat{\ }0 \subseteq \beta \subseteq \delta_t \ \& \ t \leq s\},$$

and

$$\text{(II)} \qquad \begin{gathered} (\exists \text{ finite } F)(\exists \text{ finite } G)[y \in \Phi_{e,s}(F \oplus G) - B_s \ \& \\ F \subseteq X_{e,s}(B_s) \cap X_{e,s}(B_s \cup \{y\}) \ \& \ G \subseteq Y_{e,s}(B_s \cup \{y\})]. \end{gathered}$$

If there is no such y then define $\delta_{s+1} \restriction (n+1) = \sigma\widehat{\ }1$. Otherwise define $\delta_{s+1} \restriction (n+1) = \sigma\widehat{\ }0$. Also let y_σ denote the least such y. (y_σ will be used at Step 2.)
Step 2 (satisfying the requirements). We will say that a node $\alpha \preceq \delta_{s+1}$ requires attention at stage $s+1$ if one of the following conditions holds:

A. $\alpha \subseteq \delta_{s+1}$ and the active witness x_α of α is not defined (in particular this happens if α was initialised at a previous stage).

B. Otherwise, and $x_\alpha \in B_s \cap \Xi_{|\alpha|,s}(\overline{B}_s)$.

C. Otherwise, and $x_\alpha \notin \Xi_{|\alpha|,s}(\overline{B}_s)$ and $x_\alpha \in B_s - \Gamma_{\sigma,s}(X_{|\sigma|,s}(B_s) \oplus \overline{B}_s)$ for some $\sigma \subseteq \delta_{s+1}$, $\sigma\widehat{\ }1 \subseteq \alpha$.

Choose the $\preceq$–least node $\alpha \preceq \delta_{s+1}$ which requires attention (e.g. δ_{s+1} requires attention via the condition **A**).

For each γ define

$$\tilde{\Gamma}_{\gamma,s+1} = \Gamma_{\gamma,s} \cup \{\langle x_\nu, \emptyset\rangle : \nu \succ \alpha \ \& \ \nu \supseteq \gamma \ \& \ x_\nu \in B_s\}$$

if $\gamma\widehat{\ }1 \subseteq \delta_{s+1}$, otherwise define $\tilde{\Gamma}_{\gamma,s+1} = \Gamma_{\gamma,s}$. Similarly, for each γ define

$$\tilde{\Delta}_{\gamma,s+1} = \Delta_{\gamma,s} \cup \{\langle x_\nu, \emptyset\rangle : \nu \succ \alpha \ \&\ \nu \supseteq \gamma \ \&\ x_\nu \in B_s\}$$

if $\gamma\widehat{\ }0 \subseteq \delta_{s+1}$, otherwise define $\tilde{\Delta}_{\gamma,s+1} = \Delta_{\gamma,s}$.

Initialize all nodes ν, $\nu \succ \alpha$, so that each x_ν now becomes undefined.

To define B_{s+1}, $\Gamma_{\gamma,s+1}$ and $\Delta_{\gamma,s+1}$ for all $\gamma \in T$ implement the following actions depending on which of the conditions **A**, **B** or **C** hold for α.

For **A.** If the string α does not contain the symbol 0 then assign a new sufficiently large witness x_α and define $\Gamma_{\gamma,s+1} = \tilde{\Gamma}_{\gamma,s+1}$, $\Delta_{\gamma,s+1} = \tilde{\Delta}_{\gamma,s+1}$ for every $\gamma \in T$. Otherwise let σ be the $\subseteq$–greatest node such that $\sigma\widehat{\ }0 \subseteq \alpha$. Then set $x_\alpha = y_\sigma$, where y_σ is defined at Step 1. Note that it follows from the definition of weights that

$$w(x_\alpha, \alpha) \leq 1 + \max\{w(x, \beta) : x \in \omega \ \&\ \sigma\widehat{\ }0 \subseteq \beta \subseteq \delta_t \ \&\ t \leq s\},$$

therefore $w(x_\alpha, \alpha) < w(x_\alpha, \tau)$, where τ is a node from (I). Let F, G and $u > y_\sigma$ satisfies following conditions:

$$y_\sigma \in \Phi_{e,s}(F \oplus G),$$

$$F \subseteq X_{e,s}(B_s \restriction u) \cap X_{e,s}((B_s \cup \{y\}) \restriction u)$$

and

$$G \subseteq Y_{e,s}((B_s \cup \{y\}) \restriction u).$$

Define $\Gamma_{\gamma,s+1} = \tilde{\Gamma}_{\gamma,s+1}$ for $\gamma \in T$, $\Delta_{\gamma,s+1} = \tilde{\Delta}_{\gamma,s+1}$ for $\gamma \neq \sigma$, and $\Delta_{\sigma,s+1} = \tilde{\Delta}_{\sigma,s+1} \cup \{\langle x_\alpha, G \oplus (\overline{B_s \cup \{x_\alpha\}} \restriction u)\rangle\}$. Restrain the segment $B_s \restriction u$ with priority α.

In both cases let $B_{s+1} = B_s \cup \{x_\alpha\}$.

For **B.** Let $B_{s+1} = B_s - \{x_\alpha\}$. Then define $\Gamma_{\gamma,s+1} = \tilde{\Gamma}_{\gamma,s+1}$ and $\Delta_{\gamma,s+1} = \tilde{\Delta}_{\gamma,s+1}$ for every $\gamma \in T$.

For **C.** Choose the $\subseteq$–least node σ which satisfies **C**. Let v be a sufficiently large number such that

$$x_\alpha \in \Phi_{|\sigma|,s+1}(X_{|\sigma|,s}(B_s \restriction v) \oplus Y_{|\sigma|,s}(B_s \restriction v)).$$

Then $\Gamma_{\gamma,s+1} = \tilde{\Gamma}_{\gamma,s+1}$ for $\gamma \neq \sigma$, $\Gamma_{\sigma,s+1} = \tilde{\Gamma}_{\sigma,s+1} \cup \{\langle x_\alpha, X_{|\sigma|,s}(B_s) \oplus (\overline{B_s} \upharpoonright v)\}\rangle$ and $\Delta_{\gamma,s+1} = \tilde{\Delta}_{\gamma,s+1}$ for $\gamma \in T$. Restrain by priority α the segment $B_s \upharpoonright v$.

Go to stage $s + 2$.
End of the construction.

Let $\delta = \liminf_s \delta_s$ and $B = \lim_s B_s$. (It follows from the construction that $\lim_s B_s$ exists, since if we assign as a witness of α a witness x of τ then $w(x, \alpha) < w(x, \tau)$ and we initialise τ at this stage. Moreover, we can effectively bound the number of possible changes in B_s. Therefore, B is ω-c.e. set.)

To prove that all requirements are satisfied we need the following technical lemma. Let $\alpha \prec \delta$ signify that $\alpha \prec \delta \upharpoonright n$ for some n.

Lemma 1 *Every $\alpha \prec \delta$ receives attention at most finitely often. (Therefore, each $\alpha \prec \delta$ creates a finite restrain and no $\alpha \prec \delta$ is initialised infinitely often.)*

Proof. Since $\prec$ is the well ordering on T, we can proceed by induction on $\prec$. Suppose that the lemma holds for each $\alpha' \prec \alpha$. Fix a stage s_0 such that $\alpha \preceq \delta_s$ for all $s \geq s_0$. Choose a stage $t_0 \geq s_0$ such that no $\alpha' \prec \alpha$ such that $\alpha' \subseteq \delta_t$ for some $t \leq s_0$, receives attention at a stage $s \geq t_0$. Now α can receive attention after stage t_0 at most three times (by conditions **A**, **B** and **C**).

Lemma 2 *Each $\mathcal{P}_e$, $e \in \omega$, is satisfied.*

Proof. Let $\alpha \subset \delta$ be the string of length e. By the previous lemma fix the least stage s_0 such that α is not initialised at any stage $\geq s_0$. Let x_α be the permanently active witness of α (which is necessarily assigned after stage s_0.) Now $x_\alpha \in B$ unless $x_\alpha \in \Xi_{e,s}(\overline{B}_s)$ and $\alpha \preceq \delta_s$ for some stage $s \geq s_0$. In this case by the construction $x_\alpha \in \Xi_e(\overline{B}) - B$.

Lemma 3 *Each $\mathcal{N}_e$, $e \in \omega$, is satisfied.*

Proof. For any stage s define the c. e. set $U_s = \{x : (\exists t > s)[B_s(x) \neq B_t(x)]\}$. It is easy to see that $U_{s+1} \subseteq U_s$, $\bigcap_s U_s = \emptyset$ and $B \subseteq^* U_s$.

Let $\sigma \subset \delta$ be the string of length e. Let $\sigma\widehat{\ }i \subset \delta$, $i < 3$. Let m be the maximum of the restraints created by strategies $\alpha \prec \sigma\widehat{\ }i$. Fix the least stage

s_0 such that $\sigma\hat{\ }i$ is not initialised at any stage $\geq s_0$ and $U_{s_0} \subseteq (m, +\infty)$. If $i = 2$ then obviously $B - \Phi_e(X_e(B) \oplus Y_e(B)) \neq \emptyset$, therefore $\mathcal{N}_e$ is satisfied.

Now assume that $i < 2$ and $B = \Phi_e(X_e(B) \oplus Y_e(B))$. First consider the case $i = 0$. To prove that $B \leq_e Y_e(B) \oplus \overline{B}$ we will show that for all $x \in U_{s_0}$

$$x \in B \iff x \in \Delta_\sigma(Y_e(B) \oplus \overline{B})$$

The left-to-right implication obviously follows from the construction. To prove ($\Longleftarrow$) suppose that for some node β there exists a node β such that $x = x_{\beta,s} \in U_{s_0}$ and

$$x \in \Delta_\sigma(Y_e(B) \oplus \overline{B}) - B$$

for some $s > s_0$. Choose the $\prec$-minimal node β with this property. The condition (II) and the definition of Δ_σ imply

$$x \in \Phi_e(F \oplus G), F \subseteq X_e(B)$$

and

$$G \subseteq Y_e(B).$$

(If it happens here that uses of X_e or Y_e are injured by some $x_{\gamma,t}$, $\gamma \prec \alpha$, then $x_{\gamma,t} \in U_{s_0} \cap \Delta_\sigma(Y_e(B) \oplus \overline{B}) - B$ for some $t > s_0$. Indeed, elements of $\overline{B}$ are protected by a $\overline{B}$-use of $\Delta_e(x)$. This means that uses of X_e and Y_e can be injured only by the extraction of a witness of some $\gamma \prec \beta$ from B.)

Hence $x \in \Phi_e(X_e(B) \oplus Y_e(B))$, which is impossible.

Finally suppose $i = 1$.

Let z be the greatest integer used in the construction at stages $s' \leq s$.

By the construction a witness x of α later can be chosen as a witness for another node τ only if $\tau \prec \alpha$. Let $Node(x)$ denote the $\prec$-least node τ such that x is a witness of τ. Then $w(x) =_{\mathrm{dfn}} w(x, Node(x))$ is the *final* weight of x. Let $w = \max_{x \leq z} w(x)$.

It follows from the above mentioned property of the weights of witnesses that the set $\{x : w(x) \leq w\}$ is finite. Therefore, we can fix a stage $s_1 \geq s_0$ such that $w(y) > w$ for each $y \in U_{s_1}$. This implies that every witness $y \in U_{s_1}$ satisfies, for the node σ, the condition (I) from the construction.

Now it remains to prove that for every $x \in U_{s_1}$,

$$x \in B \iff x \in \Gamma_\sigma(X_e(B) \oplus \overline{B}).$$

The left-to-right implication immediately follows from construction. To prove the right-to left implication suppose that there exists $x \in U_{s_1}$ such that $x \in \Gamma_\sigma(X_e(B) \oplus \overline{B}) - B$. Let F and G be finite sets such that $F \subseteq X_e(B)$, $H \subseteq \overline{B}$ and $\langle x, F \oplus G\rangle \in \Gamma_\sigma$. Then similar reasons as for case $i = 0$, and the definition of Γ_σ, imply $F \subseteq X_e(B \cup \{x\})$, $G \subseteq Y_e(B \cup \{x\})$ and $\langle x, F \oplus G\rangle \in \Phi_e$ for some finite G. Hence, for the node σ, the condition (II) from the construction is satisfied. Therefore, there exists a stage $s > s_0$ such that $\sigma^\frown 0 \subseteq \delta_s$. This contradicts to the choice of s_0. $\square$

Corollary 11 *There is a total degree* $\mathbf{b}$ *and a* Δ^0_2*– degree* $\mathbf{a} < \mathbf{b}$ *such that there is no total degree* $\mathbf{c}$ *such that* $\mathbf{a} \leq \mathbf{c} < \mathbf{b}$.

Proof. Immediately follows from Theorem 10 and Theorem 6.

Nies and Sorbi [ta] give a number of other results concerning embeddings into the Σ^0_2 enumeration degrees, and show that all incomplete Σ^0_2 e-degrees are *meet reducible* (that is, non-trivially branching) in the e-degrees below $\mathbf{0}'_e$. In Arslanov, Cooper, Kalimullin and Li [ta] various negative results are proved, including the fact that there exists an enumeration degree $\mathbf{a} < \mathbf{0}'$ such that the diamond lattice is not embeddable into the Σ^0_2 e-degrees preserving $\mathbf{0}'_e$ and $\mathbf{a}$. This latter result shows that theorem 6 and corollary 9 are optimal.

In conclusion, we consider the following important open question in view of Theorem 7:

Question 12 *Are the* n*-c. e. e-degrees dense for each* $n \geq 3$*? Are the* ω*-c. e. e-degrees dense?*

An affirmative answer to the following partial question could help obtain a negative answer to Question 12:

Question 13 *Let* $\mathbf{a} < \mathbf{0}'_e$ *be an* ω*-c. e. e-degree. Is there a total* ω*-c. e. e-degree* $\mathbf{b}$ *such that* $\mathbf{a} \leq \mathbf{b} < \mathbf{0}'_e$*?*

Indeed, let $D <_T \emptyset'$ be a d-c. e. set with no ω-c. e. set C such that $D <_T C <_T \emptyset'$ (as constructed in Cooper, Harrington, Lachlan, Lempp and Soare [1991]). Then the 3-c. e. set $D \oplus \overline{D}$ has maximal e-degree below $\mathbf{0}'_e$. Indeed, we have $D \oplus \overline{D} <_e \overline{K}$. Suppose that there is an ω-c. e. set C such that $D \oplus \overline{D} <_e C <_e \overline{K}$. Then, if the answer to Question 13 is "yes", there is an ω-c. e. set X such that $C <_e X \oplus \overline{X} <_e \overline{K}$. Therefore, $D \oplus \overline{D} <_e X \oplus \overline{X} <_e \overline{K}$ and $D <_T X <_T K$, which contradicts the choice of D.

It follows from Theorem 7 that the answer to Question 13 is "yes" at least for the Δ^0_2-enumeration degrees.

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