The Non-Isolating Degrees are Upwards Dense in the Computably Enumerable Degrees

Joint work with Guohua Wu and Matthew Salts

S. Barry Cooper - TAMC 2008, Xidian University, Xi'an, China, April 28, 2008

Computably enumerable sets

A set of A is Computably enumerable (c.e.) iff its members can be effectively listed in some order - Examples:

- The set of theorems of an axiomatisable theory is c.e.
- The set of inputs on which a given Turing machine halts is c.e. - as is the graph of a partial computable function
- A c.e. set can be simulated, without making mistakes, but can not necessarily be computed
- The c.e. sets $\mathcal{E}$ are naturally occurring

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The importance of seeking data non-monotonically

"… if a machine is expected to be infallible, it cannot also be intelligent. There are several theorems which say almost exactly that. "

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- By Arslanov's Completeness Criterion - Every complete extension of Peano arithmetic of c.e. degree is of degree 0′
- But - by the Low Basis Theorem - some are even low

The Ershov difference hierarchy

- Idea: Iterate boolean operations on c.e. sets to get a hierarchy for the Δ_2 sets
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- A is 2-c.e. or d.c.e. iff A = B - C for some c.e. sets B, C

- Dynamically: A has a sequence of finite approximations A^s such that $|\{s: A^{s+1}(x) \neq A^s(x)\}| \leq 2$

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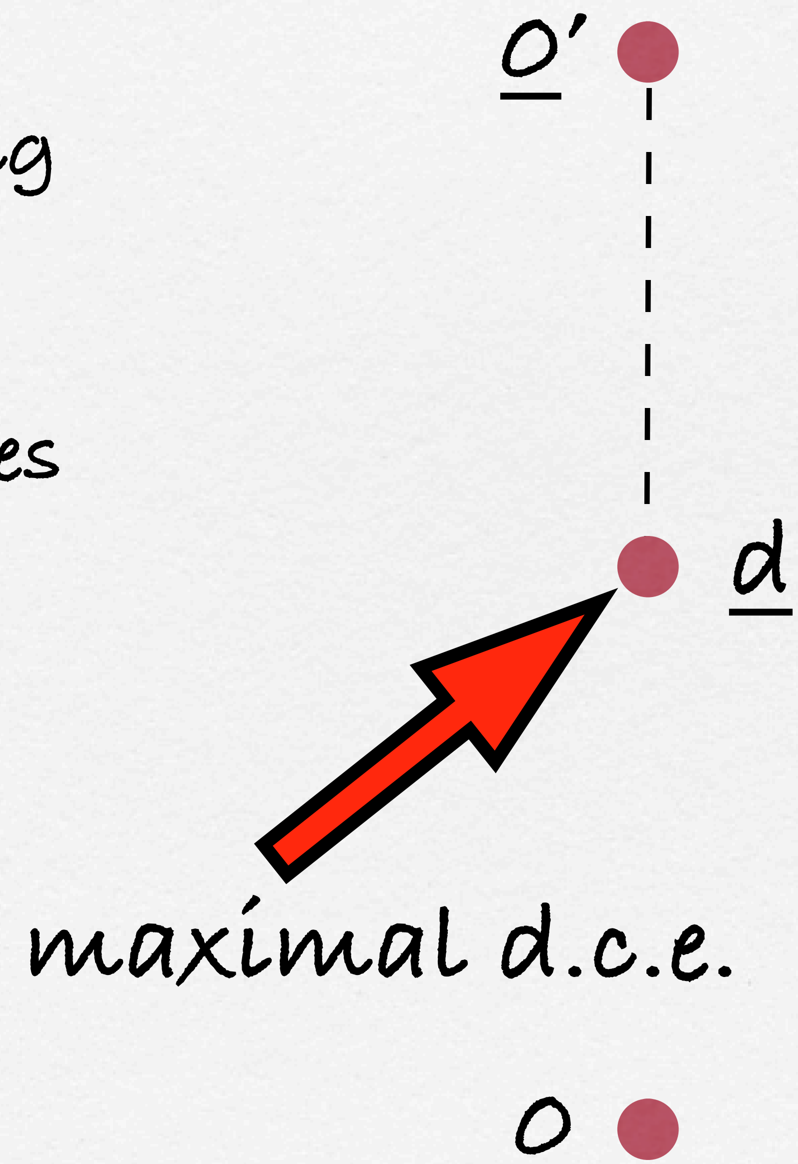
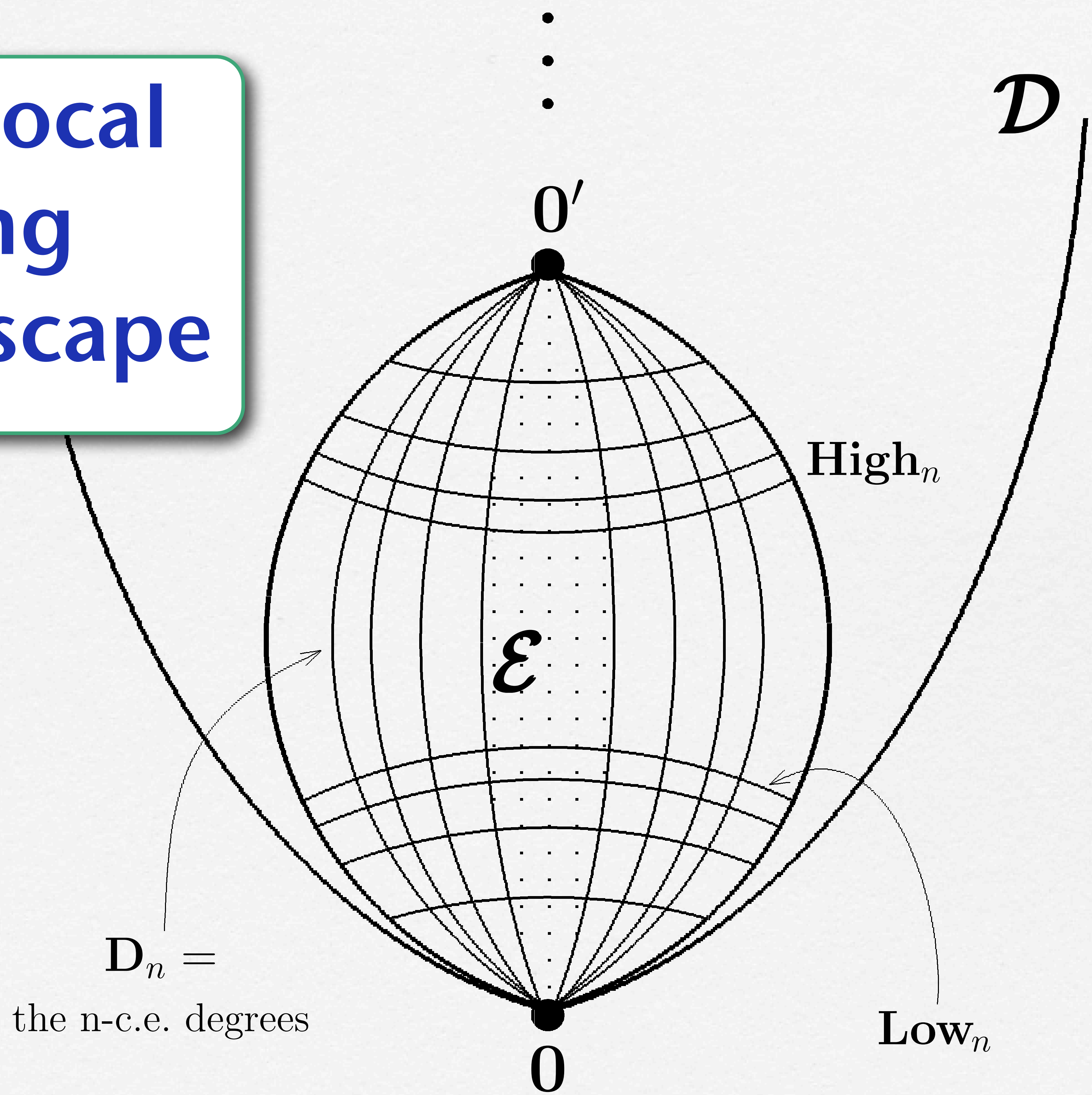
- C., Harrington, Lachlan, Lempp, Soare, 1991: There exist maximal incomplete d.c.e. degrees



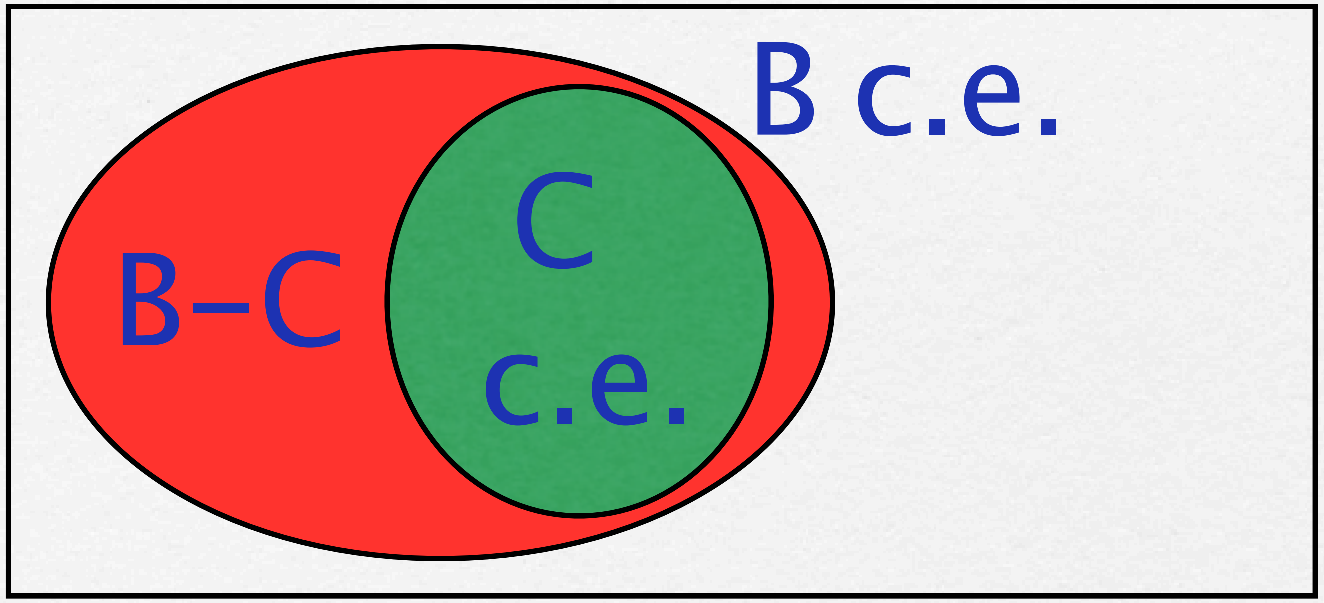
The local Turing landscape



Distribution of n.c.e. degrees in c.e. degrees?

Alistair Lachlan

- Notice - $A = B - C$ is c.e. in C – So either A is c.e. or C is incomputable



- Now - exchange C for $E = \{(x,s) : x \in B^s - C\}$ - E still c.e.

- And - A still c.e. in E and $E \leq_T A$, in fact $\leq_m A$ - so:

Every incomputable d-c.e. A has an incomputable c.e. $E \leq_T A$

Distribution of n.c.e. degrees in c.e. degrees?

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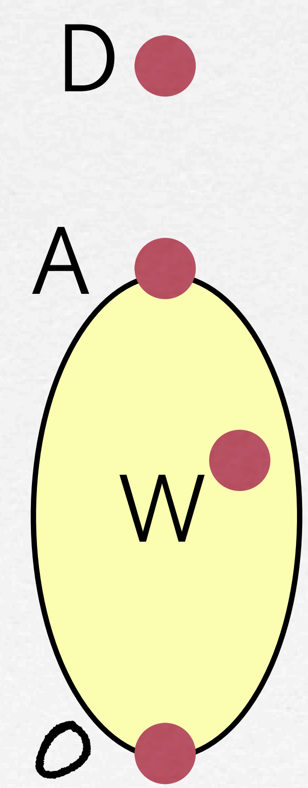
- C. 1988, C.-Yi, 1993: A d.c.e. degree $> \mathbf{0}$ is <u>isolated</u> iff it has a largest c.e. degree below it

- And … there do exist isolated d.c.e. degrees

$$\mathcal{N}_i : \quad W_i = \Phi_i^{D,A} \Rightarrow W_i \leqslant_T A,$$

$$\mathcal{R}_i : \quad D \neq \Theta_i^A,$$



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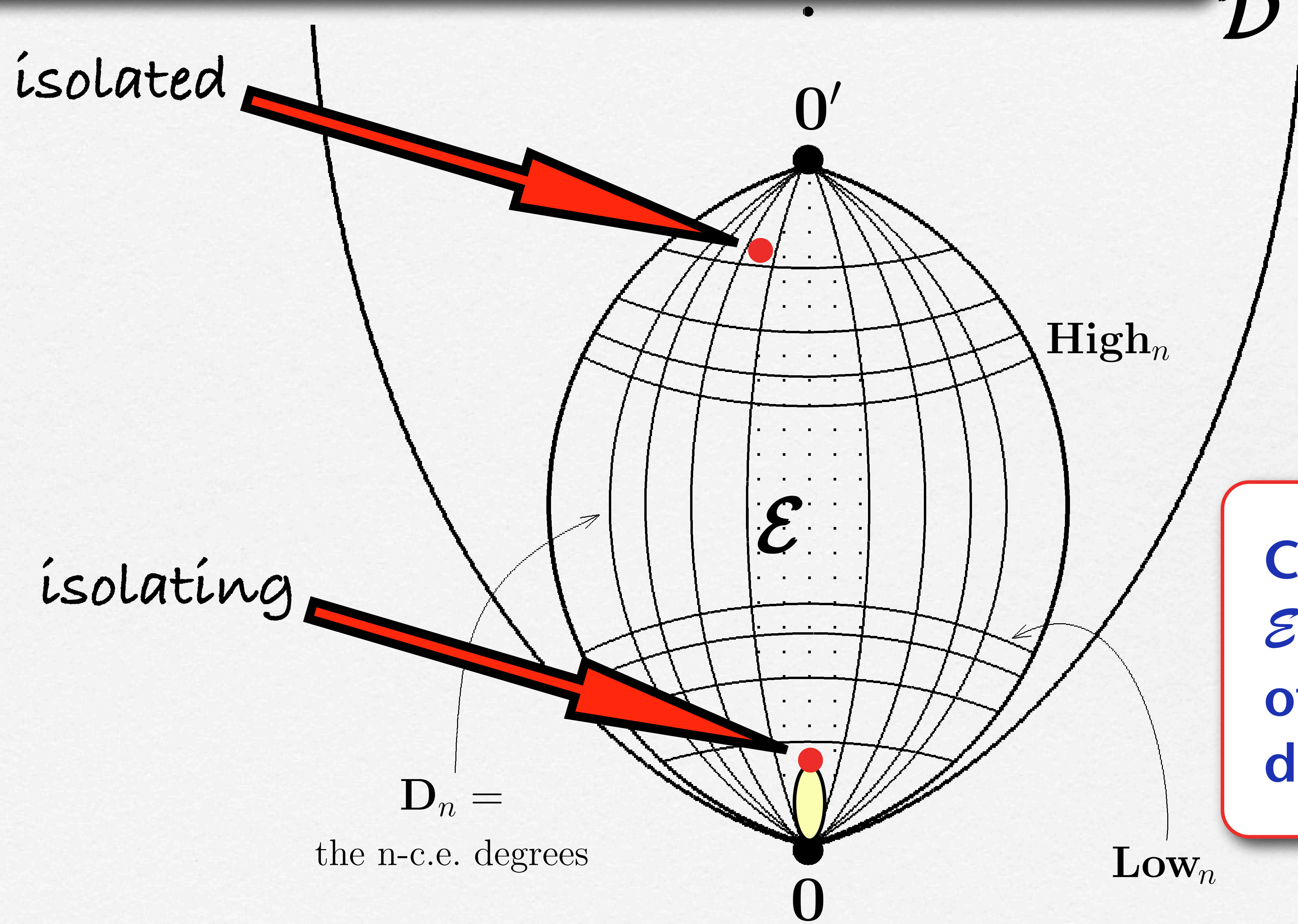
And ... there do exist properly d.c.e. degrees which are not isolated - and, in fact, ones which can be a minimal upper bound for an ascending sequence of c.e. degrees

Ishmukhametov-Wu isolation -



Guohua Wu

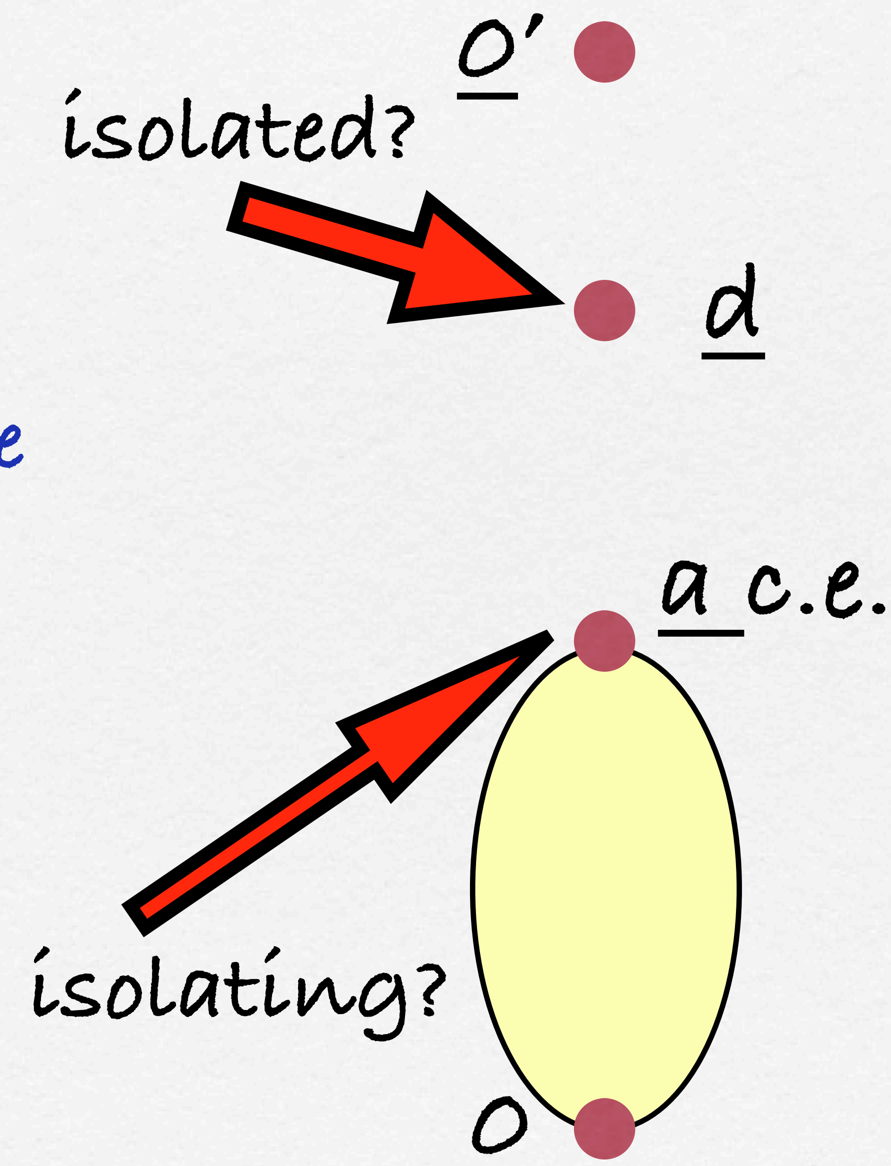
Constrains $\mathcal{E}$-definability of the high d.c.e. sets

QUESTIONS . . .

- Are the isolated degrees dense in $\mathcal{E}$? - "YES" - Laforte '95, Ding-Qian '96

- Are the non-isolated degrees dense in $\mathcal{E}$? - "YES" - Arslanov, Lempp, Shore 1996

- Does every c.e. degree (other than $\underline{0}$ and $\underline{0}'$) isolate some d.c.e. degree ?



Non-isolating degrees exist, but not everywhere

- Salts, 1999 - There exists an $\mathcal{E}$-interval consisting entirely of isolating c.e. degrees
- Cenzer, Laforte, Wu - Isolating intervals are dense in $\mathcal{E}$

Arslanov, Lempp, Shore, 1996 - But - the non-isolating c.e. degrees are downwards dense in $\mathcal{E}$

A closer look at what Arslanov-Lempp-Shore did

- Remember: Lachlan produced a c.e. $A \leq_T$ any d.c.e. D, with D c.e. in A - so any properly d.c.e. D is CEA a c.e. A
- Consequence: If a c.e. a isolates d, then d is CEA a
- So A-L-S built a c.e. A, and for each W^A a witness B to A not isolating W^A - where making $B \leq_T W^A$ involved a lot of restraint on A
- ... which works with permitting A below a c.e. C, but is incompatible with making A above such a C ...

The non-isolating degrees are upwards dense in $\mathcal{E}$

- BUT: We need a new construction of a non-isolating degree $\underline{a}$, in which for each d.c.e. D its Lachlan set plays a key role in witnessing its failure to be isolated by our c.e. A

- ... resulting in a $0'''$ priority argument ...

Theorem: For any c.e. $\underline{u} < \underline{0}'$ there exists a non-isolating degree $\underline{a}$ with $\underline{u} \leq \underline{a} < \underline{0}'$

Some intractable open problems -

- Are the c.e. degrees definable in the d.c.e. - or n-c.e. - degrees?

- Characterise the relationship between the d.c.e. and the CEA degrees - or more generally, between the n-c.e. and the n-CEA hierarchies

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Thank you!

$l(i,s) = \max\{x : (\forall y < x)[\Phi_i^{D,A}(y)[s]\downarrow = W_{i,s}(y)]\}$, and
$m(i,s) = \max\{0, l(i,t) : t < s\}$.

The module for $\mathcal{R}$ below one higher priority $\mathcal{N}_i$ runs as follows:

1. Select an unused witness x for $\mathcal{R}$.

2. If $l(i,s) > y$ and $\Gamma_i(y)[s]\uparrow$, define $\Gamma_i^A(y) = W_i(y)$ with $\gamma_i(y) = \varphi_i(y)$ and restrain A up to $\varphi_i(y)$.

3. Wait for $\Theta^A(x)\downarrow = 0$.

4. Enumerate x into D, restrain A up to $\theta(x)$.

5. Wait for $l(i)$ to return (outcome for $\mathcal{N}_i$ is 1).

6. If there exists y such that $\Gamma_i(y)\downarrow$ and y entered W_i after $\Gamma_i(y)$ was defined last time, remove x from D (outcome for $\mathcal{N}_i$ is 2). The reader should note that if 6) holds, $\mathcal{N}_i$ is satisfied; and if 6) does not occur, $\mathcal{R}$ is satisfied and Γ_i is not injured. In the latter case, the outcome for $\mathcal{N}_i$ is 0.