Total Degrees and Nonsplitting Properties of $\Sigma^0_2$
Enumeration Degrees

Joint work with Marat Arslanov, Iskander Kalimullin
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Two different models of relative computability

Compute using an oracle - i.e., call deterministically on values of a total function

Compute using emergent or enumerated information - i.e., call non-deterministically on values of a partial function
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- Compute using an oracle - i.e., call deterministically on values of a total function
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Enumeration reducibility

Let's consider a computation process:

- \( b_0, b_1, \ldots \in B \)
- \( a_0, a_1, \ldots \in A \)
- \( \Psi \subseteq B \times A \)
- \( \Psi \) is a relation between elements of \( B \) and \( A \)

The process is defined as:

\[
\exists \text{ a finite } D \subseteq A [\langle n, D \rangle \in \Psi].
\]
Enumeration reducibility

\[ b_0, b_1, \ldots \in B \text{ in some order} \]

\[ \exists a, a_0, a_1, a_2, \ldots \in A \text{ enumerated in any order} \]

\[ n \in \Psi_i^A \iff \text{defn} \left( \exists \text{ a finite } D \subseteq A \right) [\langle n, D \rangle \in \Psi]. \]
The link with NT-reducibility

If $g$ is total, we have $f \leq_{NT} g \iff f \leq_T g$.

**THEOREM**

Let $f, g$ be partial functions. Then

$$f \leq_{NT} g \iff \text{Graph}(f) \leq_e \text{Graph}(g).$$
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f \leq_{NT} g \iff \text{Graph}(f) \leq_e \text{Graph}(g).
\]
Define $A \equiv_e B \iff \text{defn } A \leq_e B \& B \leq_e A$.

The **enumeration degree** — or **e-degree**, written $\text{deg}_e(A)$ — of $A$ is

$$\text{deg}_e(A) = \text{defn } \{X \mid X \equiv_e A\}.$$  

We define $\text{deg}_e(A) \leq \text{deg}_e(B) \iff \text{defn } A \leq_e B$.

We write $\mathcal{D}_e = \text{defn }$ the set of all e-degrees with the ordering $\leq$.

The **partial degree** of a partial function $f$ is

$$\text{deg}(f) = \text{defn } \{g \mid \text{Graph}(f) \equiv_e \text{Graph}(g)\} = \{g \mid f \equiv_{NT} g\}.$$  

We write $\mathcal{P}$ = the set of all partial degrees, with ordering $\leq$ defined by $\text{deg}(f) \leq \text{deg}(g) \iff \text{defn } \text{Graph}(f) \leq_e \text{Graph}(g) \iff f \leq_{NT} g$.

We say that an e-degree $a_e$ is **total** if there is a total function $f$ with $\text{Graph}(f) \in a_e$.

We write $\text{TOT} =$ the set of total e-degrees.
Define $A \equiv_e B \iff \text{defn } A \leq_e B \land B \leq_e A$.

The enumeration degree — or e-degree, written $\text{deg}_e(A)$ — of $A$ is

$$\text{deg}_e(A) = \text{defn } \{ X \mid X \equiv_e A \}.$$  

We define $\text{deg}_e(A) \leq \text{deg}_e(B) \iff \text{defn } A \leq B$.

We write $\mathcal{D}_e = \text{defn}$ the set of all e-degrees with the ordering $\leq$.

The partial degree of a partial function $f$ is

$$\text{deg}(f) = \text{defn } \{ g \mid \text{Graph}(f) \equiv_e \text{Graph}(g) \}$$

$$= \{ g \mid f \equiv_{NT} g \}$$

We write $\mathcal{P} = \text{the set of all partial degrees, with ordering } \leq \text{ defined by}$

$$\text{deg}(f) \leq \text{deg}(g) \iff \text{defn } \text{Graph}(f) \leq_e \text{Graph}(g) \iff f \leq_{NT} g.$$

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$\text{Graph}(f) \in a_e$.

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The Natural Embedding

\[ \text{TOT} \cong \mathcal{D} \]

\[ \mathcal{P} \cong \mathcal{D}_e \]
Extending the Jump Operator

Let \( K_A = \{ x \mid x \in \Psi^A_x \} \).

Then the \textbf{e-jump} of a set \( A \) is \( J_e^A = \text{defn} A \oplus \overline{K}_A \). And the \textbf{jump} of an e-degree \( a = \deg_e(A) \) is defined to be \( a' = \deg_e(A \oplus \overline{K}_A) = \deg_e(J_e^A) \).

We iterate the jump in the usual way to obtain the \( n^{\text{th}} \) jump \( a^{(n)} \) of \( a \).

\textbf{PROPOSITION}

The e-jump agrees with the natural embedding of the Turing jump — that is, for each \( A \subseteq \mathbb{N} \) we have \( \iota(\deg(A')) = \deg_e(J_e(\chi_A)) \).
Extending the Jump Operator

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Then the **e-jump** of a set $A$ is $J_e^A = \text{defn } A \oplus \overline{K}_A$. And the **jump** of an e-degree $a = \text{deg}_e(A)$ is defined to be $a' = \text{deg}_e(A \oplus \overline{K}_A) = \text{deg}_e(J_e^A)$.

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Local information content

**THEOREM**

For each $A \subseteq \mathbb{N}, n \geq 0$ we have $\deg_e(A) \leq 0_e^{(n)} \iff A \in \Sigma_{n+1}$.

The most important case is $n = 1$, telling us that:

$\mathcal{D}_e(\leq 0'_e) =$ the set of all $\Sigma_2$ e-degrees
THEOREM
For each $A \subseteq \mathbb{N}$, $n \geq 0$ we have $\deg_e(A) \leq 0^{(n)}_e \iff A \in \Sigma_{n+1}$.

The most important case is $n = 1$, telling us that:

$D_e(\leq 0'_e) =$ the set of all $\Sigma_2$ e-degrees
Some key questions

☐ Characterise the local structure $D_e(\leq 0_e')$ of the enumeration degrees.

☐ Are the Turing degrees definable - locally or globally - in the enumeration degrees.

NOTE: The natural embedding of the computably enumerable Turing degrees is a proper subclass of the total e-degrees - the $\Pi_1$ enumeration degrees.

☐ QUESTION: Are the computably enumerable Turing degrees definable in $D_e$?
A splitting of $a$ over $b$

\[ a = c \cdot u \cdot d \]

- **Ahmad and Lachlan**: There exist non-splittable $\Delta_2$ enumeration degrees $>0_e$.

**Question**: What about splittings of total or $\Pi_1$ enumeration degrees in $\mathcal{D}_e(\leq 0_e')$?
So known results ...

- Arslanov, C, Kalimullin (2003): Results on local distribution of total e-degrees and of degrees of semirecursive sets used in natural embedding to get results on embedding of diamond lattices etc. - e.g. a simple derivation of a generalisation of the Ahmad Diamond Theorem.

- M. Soskova (2007): There is a (properly) $\Sigma_2$ e-degree $a_e$ over which $0_{e'}$ is not splittable.

- Arslanov and Sorbi (1999): One can always split $0_{e'}$ over any $\Delta_2$ enumeration degree $< 0_{e'}$. 
Theorem 1: If $a < h \leq O_e'$, $a$ is low and $h$ is total and high, then there is a low total $e$-degree $b$ such that $a \leq b < h$. 

- $O_e'$
- $h$ high total
- $\exists b$ low total
- $a$ low
**Corollary:** Let $a < h \leq O_e'$, $h$ be a high total $e$-degree, $a$ be a low $e$-degree. Then there are $\Delta_2$ $e$-degrees $c, d < h$ such that $a = c \cap d$ and $h = c \cup d$.

**Proof:** Use Thm. 6 of ACK, 2003.
... and a negative one

**Theorem 2**: There is a $\Pi_1$-e-degree $a$, and a 3-c.e. e-degree $b < a$ such that $a$ is not splittable over $b$. 

$O_e'$

$a$ which is $\Pi_1$

$c$

$d$

$b$ 3-c.e. and so $\Delta_2$
Thank you!