RECURSIVE FUNCTION THEORY: NEWSLETTER

No. 7. October, 1973

The Newsletter is an informal means of circulating information amongst recursion theorists and others interested in recursive function theory. Results and other announcements should be kept fairly brief, should be submitted in the original and in the form in which they are to appear, and should be sent to:

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Next deadline: January 15, 1974

(a) OBJECTS OF HIGHER TYPE

95. 1-Envelopes of Type 2 Functionals.

In order to state our results we need some definitions.

Let M be an admissible set. M is recursively inaccessible if for every $x \in M$ there is an admissible set y such that $x \in y \in M$. M is recursively Mahlo if, for every $x \in M$ and every $A \subseteq M$ which is $\Delta(M)$, there is y such that $x \in y \in M$ and $\langle y, \in, y \cap A \rangle$ is an admissible structure. These two definitions are generalized to admissible structures $\mathfrak{M} = \langle M, \in, R \rangle$ in place of admissible sets M, in the obvious way.

A Spector class (over the structure of arithmetic) is a class of relations on the natural numbers which contains all recursive relations, is closed under recursive substitution and positive propositional connectives and existential and universal number quantification, is parametrized (i.e., contains a complete relation), and has the prewellordering property. If C is a Spector class, then a companion to C is an admissible structure $\mathfrak{M}_C = \langle M_C, \in, R \rangle$ such that C is the class of all numerical relations which are $\Sigma(\mathfrak{M})$. It is known that every Spector class has a companion and the companion is unique up to a suitable equivalence relation on admissible structures. (See Chapter 9 of Y. N. Moschovakis, Elementary Induction on Abstract Structures, North Holland, to appear, wherein are treated Spector classes over arbitrary acceptable structures.)

Let F be a variable ranging over type 2 functionals. The 1-*envelope* of F is the class of all numerical relations which are semirecursive in F in the sense of Kleene. Let C be any class of numerical relations.

Theorem 1. The following are equivalent.

(i) There exists F such that C is the 1-envelope of the pair $(F, {}^2E)$ where ${}^2E$ is Kleene's functional for numerical quantification.

(ii) C is a Spector class and the structure $\mathfrak{M}_C$ is not recursively Mahlo.

Theorem 2. The following are pairwise equivalent.

(i) There exists F such that C is the 1-envelope of the super-jump of F.

(ii) There exists F such that C is the 1-envelope of the pair (F, E_1) where E_1 is Tugue's functional.

(iii) C is a Spector class, the set M_C is recursively inaccessible, and the structure $\mathfrak{M}_C$ is not recursively Mahlo.

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(b) ADEQUATE FRIEDBERG THEORIES

96. A Splitting Theorem for Adequate Friedberg Theories.

Let Θ be a Friedberg theory in the sense of Moschovakis (Y. N. Moschovakis, Axioms for computation theories - first draft, in R. O. Gandy and C. E. M. Yates, eds., Logic Colloquium '69, North-Holland, 1971, pp. 199-255). Let $\leqslant$ be a Θ-computable prewellorder, a refinement of the canonical prewellorder $\leqslant_{\Theta}$, and let $\rho_{\leqslant}$ be the unique orderpreserving map from the domain of Θ onto some (unique) ordinal $|\leqslant|$.

Definitions:

(i) The projectum $(\leqslant)$ is the least ordinal β for which there is an injective Θ-computable mapping whose range is a subset of $\{x: \rho_{\leqslant}(x) < \beta\}$.

(ii) The r.e.-projectum $(\leqslant)$ is the least ordinal β for which there is a Θ-semicomputable non-Θ-computable set W such that $(\forall x)(x \in W \Rightarrow \rho_{\leqslant}(x) < \beta)$.

(iii) Θ is an adequate Friedberg theory if there is a Θ-computable prewellorder $\leqslant$, a refinement of $\leqslant_{\Theta}$, for which projectum $(\leqslant)$ = r.e.-projectum $(\leqslant)$ = a limit ordinal (> 0).

(iv) A set B is regular if $B \cap K$ is a Θ-finite set whenever K is Θ-finite.

One can easily define the notions "Θ-computable in" (denoted $\leq$) and "weakly Θ-computable in" (denoted $\leq_w$) analogous to "α-recursive in" and "weakly α-recursive in" for α-recursion theory.

Theorem. Assume Θ is an adequate Friedberg theory. Suppose C is a regular Θ-semicomputable set and D is a regular Θ-semicomputable non-Θ-computable set. Then there are Θ-semicomputable sets A and B such that $C = A \cup B$, $A \cap B = \emptyset$, $A \leq C$, $B \leq C$, $D \not\leq_w A$, and $D \not\leq_w B$.

Degrees are defined in the usual manner using "$\leq$". Most of the usual corollaries of the splitting theorem (see J. R. Shoenfield, Degrees of Unsolvability, North-Holland, 1971, pp 66-67) follow easily. The proof of the theorem uses a blocking technique which is similar to the one used by R. Shore in proving the analogous theorem for α-recursion theory. Note that there are adequate Friedberg theories with no Θ-computable wellorder.

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(c) DEGREES OF UNSOLVABILITY

97. Complements for r. e. Degrees in $\mathbb{D}(\leq \mathbf{0}')$.

Theorem: $0 < \mathbf{a}$ r. e. $\rightarrow \exists \mathbf{m}$ minimal such that $\mathbf{m} \cup \mathbf{a} = \mathbf{0}'$.

Corollary: $\mathbf{a}$ r. e. $\rightarrow \mathbf{a}$ has a complement in $\mathbb{D}(\leq \mathbf{0}')$.

The method of proof is a combination of the techniques we established in our new proof of "$\mathbf{0} < a$ r. e. $\rightarrow \exists \mathbf{m}$ minimal such that $\mathbf{m} < \mathbf{a}$" (RFT Newsletter #6, item 81) with those of R. W. Robinson's proof of "$\mathbf{0} < \mathbf{a}$ r. e. $\rightarrow \exists \mathbf{b} < \mathbf{0}'$, $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$" (RFT Newsletter #17, item 18).

R. W. Robinson has announced the corollary in the case $\mathbf{a}'' = \mathbf{0}''$ (RFT Newsletter #3, item 36), but the methods of proof are considerably different.

The theorem is in contrast to Lachlan's result that if $\mathbf{0} < \mathbf{a}$ r. e. $< \mathbf{0}'$ then $\mathbf{a}$ has no r. e. complement in $\mathbb{D}(\leq \mathbf{0}')$.

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(d) REDUCIBILITIES; WEAK TRUTH TABLE DEGREES

98. Weak Truth Table Degrees Maximum Within r. e. Degrees.

A non-recursive r. e. Turing degree can contain a maximum weak truth table degree.

A theorem of Jockusch states that a Turing degree contains a maximum truth table degree if and only if the degree contains no hyper-immune sets. Hence, no non-recursive r. e. degree has a maximum truth table degree. The situation for weak truth table degree is quite different.

Theorem. There is a non-recursive r. e. set A such that if B is any set Turing reducible to A then $B \leq_W A$.

The Turing degree of A has as its maximum weak truth table degree the weak truth table degree of A . In contrast to this result we have:

Theorem. If A is any set of Turing degree $\underset{\sim}{0}'$ then there is another set B of Turing degree $\underset{\sim}{0}'$ such that $B \not\leq_W A$.

Hence the complete degree does not contain a maximum weak truth table degree.

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(e) COMPLEXITY OF RECURSIVE FUNCTIONS

Theorem 1: A is (effectively) speedable iff $A \times N$ is (effectively) levelable.

Theorem 2: If A and B are r.e., A is speedable and $A \leq_m B$, then B is speedable (i.e., speedability is inherited upwards in the m-reducibility ordering).

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(f) OPEN PROBLEMS

100. An open problem about some natural immune sets.

Let $\varphi_0, \varphi_1, \cdots$ and $\varphi'_0, \varphi'_1, \cdots$ be standard enumerations of the partial recursive functions. Let

$$M_\varphi = \{ i \mid \varphi_i = \varphi_j \Rightarrow j \geq i \}$$

That is, M_φ is the set of minimal indices wrt enumeration φ. M. Blum observed that M_φ is a natural example of an immune set.

Despite various recursive isomorphism theorems equating any two enumerations φ and φ', P. Young has shown that it is possible that $M_\varphi \subsetneqq M'_\varphi$, and since M_φ and $M_{\varphi'}$ are always immune, they can not be recursively isomorphic (cf [1]). I show in [1] that M_φ is always Turing equivalent to $\emptyset''$.

Open questions: (1) Are M_{φ} and $M_{\varphi'}$ always m-equivalent?

(2) Is M_{φ} always tt-equivalent to $\emptyset''$?

Reference: [1] Meyer, A. R. Program size in restricted programming languages, Information and Control, 21, 4 (November 1972), 382-394.

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(g) CORRECTIONS

101. My proof of the theorem in RFT No. 4 (Item 50) is in error (although the theorem still seems likely to be true). The amended proof will embed only certain special lattices; roughly those in which all non-trivial lower bounds are $\mathbf{0}$.

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