RECURSIVE FUNCTION THEORY: NEWSLETTER

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The newsletter is an informal means of circulating information amongst recursion theorists and others interested in recursive function theory. Results and other announcements should be kept fairly brief and sent to

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(a) ITEMS RELATING TO A. MEYER'S PROBLEM (NEWSLETTER #4, ITEM 60)

64. Solution of A.R. Meyer's Problem

Theorem. There is a creative set C which is not approximable to within any $\varepsilon < 1$.

Proof. It is sufficient to put $C = \{x \in N; (\exists\, y \in D)(y! \leqslant x+1 < (y+1)!)\}$, where D is an arbitrary creative set of naturals.

This theorem can strengthened e.g. in this way:

Theorem. Let α, β be recursive real numbers, $0 \leqslant \alpha \leqslant \beta \leqslant 1$. Then there is a creative set C such that

(i) for every recursive set $A \subseteq C$, $B \subseteq N - C$

$$(1) \quad \liminf_{n \to \infty} \frac{1}{n} \left| \{0, 1, \ldots, n-1\} - (A \cup B) \right| \geqslant \alpha ,$$

$$(2) \quad \limsup_{n \to \infty} \frac{1}{n} \left| \{0, 1, \ldots, n-1\} - (A \cup B) \right| \geqslant \beta ;$$

(ii) there are recursive sets A, B such that in (1) and (2) equality holds.

In the proof a convenient simple set, the set C from the first theorem, a creative set $C_1 \subseteq \{n^2; n \in N\}$ and convenient recursive sets are used. (Received March 13, 1973.)

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65. Recursive Approximations to the Halting Problem

Definition: If A is any set, r any number, we say:

"dens(A) < r a.e." ("the density of A is less than r almost everywhere") if for all but finitely many n , $\frac{|A \cap \{0, 1, \ldots, n-1\}|}{n} < r$.

"dens(A) < r i.o." ("the density of A is less than r infinitely often") if for infinitely many n , $\frac{|A \cap \{0,1,\ldots,n-1\}|}{n} < r$.

Analogous definitions may be used for $>$, $\leqslant$ or $\geqslant$ as well.

Noting that every creative set is the halting set for some Gödel numbering, we may easily show:

Proposition 1: There is a Gödel numbering ϕ and recursive sets A and B , $A \subseteq K_\phi$ and $B \subseteq \overline{K}_\phi$, such that $(\forall \varepsilon > 0)[\text{dens}(A \cup B) > 1 - \varepsilon \text{ a.e.}]$.

That is, some Gödel numberings have their halting sets very closely approximable by recursive sets. On the other hand, the existence of a very sparse simple set allows us to conclude:

Proposition 2: There is a Gödel numbering ϕ such that: $(\forall A, B \text{ recursive}, A \subseteq K_\phi \text{ and } B \subseteq \overline{K}_\phi)(\forall \varepsilon > 0)[\text{dens}(A \cup B) < \varepsilon \text{ a.e.}]$.

That is, some Gödel numberings are very non-approximable. This Proposition answers a question posed by Albert Meyer in the last Newsletter, No. 60.

We may also obtain Gödel numberings with arbitrary degrees of approximability, as well as degrees of approximability which are not well-determined. Restriction to "optimal Gödel numberings" (Schnorr) changes the statements of the results slightly:

Definition: A Gödel numbering ϕ is called "optimal" if for any Gödel numbering $\alpha = \{\alpha_0, \alpha_1, \ldots\}$, there is a 1-1 total recursive function f and a constant c such that $(\forall i)[\alpha_i = \phi_{f(i)}$ and $f(i) \leqslant ci]$.

This condition on Gödel numberings is an attempt to obtain sharper results by ruling our artificial Gödel numberings containing many extra indices for certain functions. It insures that all functions have indices not much larger than their indices in any other Gödel numbering. We obtain

Proposition 3: For any $\varepsilon > 0$, there exists an optimal Gödel numbering ϕ and recursive sets A and B, $A \subseteq K$ and $B \subseteq \overline{K}_\phi$, such that $\text{dens}(A \cup B) > 1 - \varepsilon$ a.e.

Proposition 4: For any $\varepsilon > 0$, there exists an optimal Gödel numbering ϕ such that $(\forall A, B$ recursive, $A \subseteq K_\phi$ and $B \subseteq \overline{K}_\phi)$ $[\text{dens}(A \cup B) < \varepsilon$ a.e.$]$.

In fact, the apparent weakness of Propositions 3 and 4 is inherent; it is impossible to obtain optimal Gödel numberings for which the halting set is arbitrarily approximable or non-approximable. Specifically,

Proposition 5: All optimal Gödel numberings ϕ have the following two properties.:

(a) There exists $\varepsilon > 0$ and recursive sets A and B, with $A \subseteq K_\phi$, $B \subseteq \overline{K}_\phi$, and $\text{dens}(A \cup B) > \varepsilon$ a.e.

(b) There exists $\varepsilon > 0$ such that for all recursive sets A and B, with $A \subseteq K_\phi$ and $B \subseteq \overline{K}_\phi$, $\text{dens}(A \cup B) < 1 - \varepsilon$ a.e.

Open Question: Is it true that for any rational r strictly between 0 and 1, there is an optimal Gödel numbering ϕ such that:

$(\exists A, B$ recursive, $A \subseteq K_\phi$ and $B \subseteq \overline{K}_\phi)(\forall \varepsilon > 0)[\text{dens}(A \cup B) > r - \varepsilon$ a.e.$]$, and

$(\exists A, B$ recursive, $A \subseteq K_\phi$ and $B \subseteq \overline{K}_\phi)(\forall \varepsilon > 0)[\text{dens}(A \cup B) < r + \varepsilon$ a.e.$]$?

That is, can we obtain optimal Gödel numberings with arbitrary degrees of approximability? (Received March 14, 1973.)

Reference:

Schnorr, C.P. Optimal Enumerations and Optimal Gödel Numberings. Mathematisches Seminar, Universität Frankfurt, Aug. 1972.

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66. Solution to Meyer's Problem

There is a creative set C such that, if $1 > \varepsilon > 0$, C is not approximable to within ε in the sense of [1]. In fact there is a creative set C with this much "worse" property: (1) For all recursive sets $A \subset C$, $B \subset \overline{C}$,

$$\limsup_{n \to \infty} \frac{1}{n} \cdot |\{0, 1, \ldots, n-1\} - (A \cup \overline{C})| = 1$$

and

$$\limsup_{n \to \infty} \frac{1}{n} \cdot |\{0, 1, \ldots, n-1\} - (C \cup B)| = 1 .$$

Indeed, (2) every isomorphism degree that contains a productive set contains a set $\overline{C}$ fulfilling (1).

These results follow from the proofs in §3 of [2], which allow us to conclude that if P is productive then there is a strictly increasing recursive function h such that for every recursive set R,

$$\limsup_{n \to \infty} \frac{1}{n} \cdot \left|\{h(0),h(1),\ldots,h(n-1)\} - ((P \cap R) \cup (\overline{P} \cap \overline{R}))\right| = 1 .$$

Let P be any productive set and h the function guaranteed for it. Let g be a recursive permutation such that (i) g is increasing on Range h and (ii) g maps $\overline{\text{Range } h}$ into the powers of 2. Since to be a recursive set is to be the g image of a recursive set, it follows that for any recursive R

$$\limsup_{n \to \infty} \frac{1}{n} \cdot \left|\{gh(0),gh(1),\ldots,gh(n-1)\} - ((g(P) \cap R) \cup (\overline{g(P)} \cap \overline{R}))\right| = 1 .$$

And since (ii) assures that

$$\lim_{n \to \infty} \frac{1}{n} \cdot \left|\{0, 1, \ldots, n-1\} \cap g(\text{Range } h)\right| = 1 ,$$

it follows via (i) that

$$\limsup_{n \to \infty} \frac{1}{n} \cdot \left|\{0, 1, \ldots, n-1\} - ((g(P) \cap R) \cup (\overline{g(P)} \cap \overline{R}))\right| = 1 .$$

Now taking in turn the cases $\overline{R} = A \subset \overline{g(P)}$ and $R = B \subset g(P)$ we can verify the claim (2), using $g(P)$ as $\overline{C}$. And when $\overline{P}$ is creative, $\overline{g(P)}$ is a creative set C fulfilling (1). (Received February 21, 1973.)

References:

[1] A.R. Meyer, An Open Problem on Creative Sets, Recursive Function Theory: Newsletter No. 4, January, 1973, 15-16, #60.

[2] G.F. Rose and J.S. Ullian, Approximation of Functions on the Integers, Pacific J. Math. 13(1963), 693-701.

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(b) RETRACEABLE SETS

67. Boolean-r.e. Sections of t-retraceable Sets.

If A is a set of natural numbers, A infinite, we let p_A denote the function enumerating A in order of magnitude. A retraceable (infinite) set A is called t-retraceable if, whenever ϕ_e is a partial recursive function, there exists a number $n(e)$ such that $(\forall m)[(m \geqslant n(e) \ \& \ \phi_e(p_A(m)) \text{ is defined}) \implies \phi_e(p_A(m)) < p_A(m+1)]$. A variety of results about t-retraceable sets (and theorems applying such sets) have been developed, e.g., in [1], [2], [3], [4], and [5]. In particular, Degtev has proved and applied the following two theorems in [1], [2], respectively:

Theorem 1 ([1]). If A is a t-retraceable set with r.e. complement and if B is a co-infinite r.e. set extending the complement of A, then the complement of B admits a general recursive retracing function.

Theorem 2 ([2]). If A is as in Theorem 1 and B is an r.e. superset of A such that $B - A$ is infinite, then $B - A$ admits a general recursive retracing function.

Inasmuch as Degtev has succeeeded in using these results (especially Theorem 2) in an interesting way to obtain information about the many-one semilattice, it seems to us worthwhile to draw attention to an extremely simple proof of the following general fact which includes them both:

Theorem 3. Let A be a t-retraceable set, and let B be an infinite set of natural nubmers such that (i) B belongs to the boolean algebra generated by the r.e. sets and (ii) $B \cap A$ is infinite. Then $B \cap A$ is retraceable; moreover, if A admits a general recursive retracing function then so does $B \cap A$.

Proof. Let B be written in the form $\bigcup_{i=1}^{n} (W_{e_i} - W_{f_i})$, for some integer n and suitable r.e. sets W_{e_i}, W_{f_i}, $1 \leqslant i \leqslant n$. Let it be understood that for each i between 1 and n the r.e. set W_{e_i} (W_{f_i}) is the domain of the one-place partial recursive function ϕ_{e_i} (ϕ_{f_i}). Let h be a retracing function for A. We can of course assume, without loss of generality, that $(\forall x)[h(x)$ defined $\Longrightarrow h(x) \leq x]$, that range $(h) \subseteq$ domain (h), and that h has a unique fixed point (namely, $p_A(0)$). Under this assumption, we denote by $\hat{h}(x)$ the set $\{x, h(x), h(h(x)), \ldots\}$, for each $x \in$ domain of h. On the understanding that $(\forall e)[\phi_e \simeq U((\mu y)T(e,x,y)]$, T = the Kleene Enumeration Predicate, we define partial recursive functions $\hat{\phi}_{e_i}$ and $\hat{\phi}_{f_i}$ for each i between 1 and n as follows: $\hat{\phi}_{e_i} \simeq (\mu y)T(e_i,x,y)$; $\hat{\phi}_{f_i} \simeq (\mu y)T(f_i, x, y)$. Applying the definition of t-retraceability, we let $n_0 = \max\{n(e_i) + n(f_i) | 1 \leq i \leq n\}$. There is no loss of generality in assuming $p_A(n_0) \in B$, which we shall accordingly assume. We modify h to form a new partial recursive function h_0 , as follows:

If $x = p_A(n_0)$, we set $h_0(x) = x$ if no element of $\hat{h}(p_A(n_0)) - \{p_A(n_0)\}$ belongs to B ; otherwise we set $h_0(x) = \max\{y | y \in B \cap (\hat{h}(p_A(n_0)) - \{p_A(n_0)\})\}$;

If $x \in \hat{h}(p_A(n_0)) - \{p_A(n_0)\}$ and $(\exists y)[y \in \hat{h}(x) - \{x\}\ \&\ y \in B]$, we set $h_0(x) = \max\{y | y \in B \cap (\hat{h}(x) - \{x\})\}$;

If $x \in \hat{h}(p_A(n_0) - \{p_A(n_0)\}$ and $\neg (\exists y)[y \in \hat{h}(x) - \{x\}\ \&\ y \in B]$ and $x \in B$, we set $h_0(x) = x$;

If $x \in \hat{h}(p_A(n_o) - \{p_A(n_o)\}$ and $\neg(\exists y)[y \in \hat{h}(x) \cap B]$, we set $h_o(x) = p_A(n_o)$;

If $x \notin \hat{h}(p_A(n_0)$ and $p_A(n_0) \notin \hat{h}(x)$, but $x \in$ domain (h) , we set $h_0(x) = p_A(n_0)$; and, finally,

If $p_A(n_0) \in \hat{h}(x) - \{x\}$, we set $h_0(x) = h(x)$.

Now, we notice the following facts about h_0 :

(i) h_0 is partial recursive and has the same domain as h ;

(ii) the range of h_0 is included in the domain of h and hence in the domain of h_0 ;

(iii) $(\forall x)[x \in$ domain of $h_0 \Longrightarrow (\exists y)[h_0^{y+1}(x) = h_0^y(x)]$, so that $\hat{h}_0(x)$ is a computable finite set for all $x \in$ domain (h_0) ;

(iv) $h_0 \big|_{\hat{h}(p_A(n_0))}$ retraces $B \cap \hat{h}(p_A(n_0))$; and

(v) $(x \in$ domain $(h_0)\ \&\ x \notin \hat{h}_0(p_A(n_0))) \Longrightarrow p_A(n_0) \in \hat{h}_0(x)$.

Finally, we shall construct from h_0 a partial recursive function h_1 , with domain $(h_1) =$ domain (h_0) , such that h_1 retraces

$B \cap A$. For $x \in$ domain (h_0), we set:

$$h_1(x) = h_0(x) \ , \quad \text{if} \quad x \in \hat{h}_0(p_A(n_0)) \ ;$$

$$\begin{aligned} h_1(x) = \max\{y \mid y \in \hat{h}_0(x) - \{x\} \ \& \ (\forall z)[1 \leq z \leq n \Longrightarrow \\ (\exists s)[s < x \wedge T(e_z, y, s)]] \ \& \ (\forall z)[1 \leq z \leq n \Longrightarrow \\ \neg(\exists s)[s < x \wedge T(f_z, y, s)]]\} \ , \quad \text{if} \quad p_A(n_0) \in h_0(x) - \{x\} \ . \end{aligned}$$

Because of our choice of n_0, h_1 is defined for each $x \in$ domain (h_0); and, in view of the choice of n_0 and the various properties of h_0 (using also our assumption that $p_A(n_0) \in B$), we see without difficulty that h_1 retraces $A \cap B$. That completes the proof.

Corollary. If A is a t-retraceable set and B is an infinite boolean-r.e. section of A, then B is t-retraceable also; moreover, B admits a general recursive retracing function if A does.

Proof. By Theorem 3, since obviously any retraceable subset of a t-retraceable set is t-retraceable.

References:

[1] A.N. Degtev, On retraceable sets with hypersimple complements, Algebra and Logic (Russian), 10(3), 1971, 235-246.

[2] __________, On many-one degrees of simple sets, Algebra and Logic (Russian), 11(2), 1972, 130-139.

[3] Erik Ellentuck, Universal Cosimple isols, Pacific J. Math., 42(3), 1972, 629-638.

[4] Judith Gersting, A rate of growth criterion for universality of regressive isols, Pacific J. Math., 31(3), 1969, 669-677.

[5] T.G. McLaughlin, Thinning the branches of a semicomputable tree, to appear.

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(c) DEGREES OF UNSOLVABILITY

68. The Halting Problem Relativized to Complements.

Let $H^A = \{e \mid \text{domain } \{e\} \cap A \neq \emptyset\}$. A.L. Selman has called the classification of H^A the "relativized halting problem," and has conjectured that there exist sets A such that H^A is Turing-incomparable to $H^{\bar{A}}$.

Theorem: There exists a set A such that H^A is Turing incomparable to $H^{\bar{A}}$ in every Turing degree $\underset{\sim}{a}$ satisfying one of the following:

(i) $\underset{\sim}{a}$ is r.e., with $\underset{\sim}{a}' > \underset{\sim}{0}'$;

(ii) $\underset{\sim}{a} \geqslant \underset{\sim}{0}''$;

(iii) $\underset{\sim}{a} \geqslant \underset{\sim}{0}'$ and $\underset{\sim}{a}$ is r.e. in $\underset{\sim}{0}'$.

This contrasts with the fact that H^A is comparable to $H^{\bar{A}}$ for almost all sets A , and for all sets in almost all degrees $\underset{\sim}{a}$.

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(d) ADMISSIBLE ORDINALS AND RECURSION THEORY

69. Hyperhypersimple α-recursively enumerable sets.

Let α be an admissible ordinal. For the notation see [Lerman, Maximal α-r.e. sets, preprint]. Following Lachlan, we call an α-r.e. set H hyperhypersimple if H is not α-recursive and the lattice of α-r.e. supersets of H is a boolean algebra. We try to characterize the existence of hyperhypersimple α-r.e. sets in terms of projecta and cofinalities of α .

Theorem: If H is α-r.e. but not α-recursive and $[\alpha) - H$ has ordertype $< ts2p(\alpha)$, then H is hyperhypersimple.

Corollary: There exist $\aleph_\omega^L$ hyperhypersimple sets.

Theorem: If H is α-r.e. but not α-recursive, $s3cf(\alpha) > \omega$, and $[\alpha) - H$ has ordertype $\geqslant ts2p(\alpha)$, then H is not hyperhypersimple.

Corollary: There are no hyperhypersimple $\aleph_1^L$-r.e. sets.

Theorem: All hyperhypersimple $\aleph_\omega^L$-r.e. sets H are such that $[\aleph_\omega^L) - H$ has ordertype $< \aleph_\omega^L = ts2p(\aleph_\omega^L)$.

Corollary: All hyperhypersimple $\aleph_\omega^L$-r.e. sets have complete $\aleph_\omega^L$-r.e. degree.

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(e) DESCRIPTIVE SET THEORY

70. Long Projective Well-Orderings

Mansfield has shown that any $\mathbf{\Sigma}^1_2$ well-ordering of reals has length $< \aleph_2$. Martin has shown that any $\mathbf{\Delta}^1_2$ pre-well-ordering of the continuum has length $< \aleph_2$, and, assuming the existence of a measurable cardinal, any $\mathbf{\Delta}^1_3$ pre-well-ordering of 2^ω has length $< \aleph_3$. The following shows that without further assumptions these results are best possible.

<u>Theorem</u>. There are models of ZFC in which: $\overline{\overline{2^\omega}}$ is as large as desired, there is a $\mathbf{\Pi}^1_2$ well-ordering of a set of reals which is as long as desired, and there is a $\mathbf{\Delta}^1_4$ well-ordering of 2^ω. (A similar result can be obtained for ZFC + there exists a measurable cardinal. In that case it is possible to have long $\mathbf{\Pi}^1_3$ well-orderings, and a $\mathbf{\Delta}^1_5$ well ordering of 2^ω.)

The question of the consistency of a long $\mathbf{\Delta}'_3$ well ordering of 2^ω still seems to be open.

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(f) OPEN PROBLEMS

71. From John Myhill

Let T be the free-variable theory with symbols + , · , successor, zero. Axioms are Peano's (with induction as a <u>rule</u>) and the recursive definitions of + and · . Prove that T is undecidable.

(g) LATE ADDITIONS

72. N-ary Almost Recursive Functions

In Closure Properties of Regressive Functions, Proc. London Math. Soc. (3) 15(1965) 226-238, Dekker generalized the notion of regressiveness to functions of n variables. We carry out a similar programme for almost recursive functions and verify that theorems analogous to Dekker's hold also in this case. $(N^n, \leqslant)$ is the direct product of n copies of the non negative integers N with the usual ordering. A subset A of N^n is initial if the set of predecessors of every element of A is included in A. A function $f: N^n \to A$ is almost recursive if (i) δf is initial, (ii) f is injective, and (iii) each of the n mappings $f(x_1, \ldots, x_n) \to x_i$ has a partial recursive extension. A set X is in ARE(n) (respectively AR(n)) if X is the range of an n-ary almost recursive (respectively strictly increasing almost recursive) function. We show that ARE(n) = ARE(1) but that $AR(n) \subsetneq AR(n+1)$.

This contrast in the behaviour of almost recursive and almost recursively enumberable sets is one more example of the rather intractable nature of almost recursive enumerability. In this respect the almost recursive sets behave more respectably, having properties analogous to those of retraceable sets. We interpret this as further evidence that the gap between almost recursivity and almost recursive enumerability is "very large."

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73. Characterization of the Elementary Functions in Terms of Nesting of Primitive Recursions

Theorem: Heinermann's R_2 (= AXT's K_2) is Grzegorczyk's E_3 (= the class of the Kalmár/Csillag elementary fucntions).

The proof (that E_3 is a subclass of R_2 ; the converse is obvious) consists of two major steps:

1. Some important elementary functions are shown to be in R_2 . These include sum, difference, product, quotient, power, the representing function of the primes, binomial coefficient, logarithm, root, and the exponent of 2 in the prime power product representation of the argument (denoted as $(\text{argument})_0$).

2. To show that an arbitrary elementary function f is in R_2 , observe that $f(x_1, \ldots, x_n)$ is computable from its arguments $x_1, \ldots, x_n$ by a Shepherdson/Sturgis register machine in time less than $F(x_1, \ldots, x_n)$ where F is in R_2 . According to the machine program, numbers s and P and functions g and G in R_2 ($g(0, x) := x$, $g(t + 1, x) := H(g(t, x), t)$ where H is in R_1 ; $G(t, x) := [g(t, x)/P] + t$) are defined which have the following properties: Consider the sequence $g(0, x)$, $g(1, x)$, ... where $x = 2^{p_s} m_1^{x_1} \ldots m_n^{x_n} P$ and the m_i are the Mersenne numbers $2^{p_i}-1$.

Delete from it all members but those less than the length (= number of states) of the program. The resulting subsequence $g(t_0, x)$, $g(t_1, x)$, $g(t_2, x)$, ... is formed by the machine states in their consecution as given by the computation, with the terminal state repeated <u>ad infinitum</u>, whereas $t_0/2^{(t_0)_o}$, $t_1/2^{(t_1)_o}$, $t_2/2^{(t_2)_o}$, ... is the corresponding sequence of the contents of registers 0, 1, ..., s, coded as Mersenne number power product. We have $t_0 = 2^{p_s} m_1^{x_1} \ldots m_n^{x_n}$ and, for all z , t_{z+1}/t_z is bounded by 2^{p_s} . Also, for t not less than $t_{\text{number of machine steps}}$, $G(t, x)$ is the least t_z not less than t . Thus, for any such t ,
$G(t, x)/2^{(G(t, x))_o} = 3^{f(x_1, \ldots, x_n)}$ (since $m_0 = 3$) , if the machine is assumed to put the result into register 0 and clear all registers with positive addresses. Hence, choosing x as above and
$t = 2^{p_s F(x_1, \ldots, x_n)} m_1^{x_1} \ldots m_n^{x_n}$, we have

$$f(x_1, \ldots, x_n) = [{}^3\log[G(t, x)/2^{(G(t, x))_o}]] .$$

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