RECURSIVE FUNCTION THEORY: NEWSLETTER

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The newsletter is an informal means of circulating information amongst recursion theorists and others interested in recursive function theory. Results and other announcements should be kept fairly brief and sent to:

Barry Cooper/Dick Epstein,
Group in Logic and the Methodology of Science
Department of Mathematics
University of California,
Berkeley,
Cal. 94720
U.S.A.

Next Deadline: March 15th, 1973.

(a) α - Recursion Theory

46. Maximal Sets in α - Recursion Theory

Friedberg's maximal set theorem (cf. Rogers pp. 235-6) reads: there is an r.e. set of natural numbers whose complement is infinite but cannot be split by an α - r.e. set into two infinite parts. Such a set is called a maximal r.e. set. We ask: for which admissible ordinals α do maximal α - r.e. sets exist? We obtain a variety of existence and non-existence results, by a variety of methods. Our most quotable result reads: if α is an uncountable admissible ordinal then maximal α - r.e. sets do not exist. (This result answers a question of Sacks, Trans. A.M.S. 124 (1966) 1-23.) Here a maximal α - r.e. set is defined to be an α - r.e. set whose complement is unbounded in α but cannot be split by an α - r.e. set into two parts each unbounded in α .

[M. Lerman,
Department of Mathematics,
Yale University,
New Haven, Conn. 06520,U.S.A.,
and S. G. Simpson,
Department of Mathematics,
University of California,
Berkeley, Calif. 94720, U.S.A.]

47. The Friedberg Completeness Theorem in α - Recursion Theory

Let α be an admissible ordinal. Denote by $\&_\alpha$ the set of all equations in the Kleene-Kripke equation calculus for α . If $E \subseteq \&_\alpha$, denote by S(E) (resp. R(E)) the set of all equations in

$\&_\alpha$ deducible from E (resp. via α - finite deduction trees). If $X \subseteq \alpha$ write

$$\Delta_X = \{g(\underset{\sim}{\gamma}) = \underset{\sim}{0} \mid \gamma \in X\} \cup \{g(\underset{\sim}{\gamma}) = \underset{\sim}{1} \mid \gamma \in \alpha - X\}$$

and define

$$X^{c\alpha} = \{\varepsilon < \alpha \mid f(\underset{\sim}{\varepsilon}) = \underset{\sim}{0} \in S(E \cup \Delta_X) \text{ where } \varepsilon \text{ is the Gödel number of a finite set of equations } E \subseteq \&_\alpha\}$$

and

$$X^{w\alpha} = \{\varepsilon < \alpha \mid f(\underset{\sim}{\varepsilon}) = \underset{\sim}{0} \cup R(E \in \Delta_X) \text{ where } \varepsilon \text{ is as above.}$$

These are called respectively the α - *calculability jump* of X and the *weak* α - *jump* of X. Each in its own way naturally generalizes the notion of Turing jump to α - recursion theory.

Define *regular* and *hyperregular* as in Sacks, Trans. A.M.S 124 (1966) 1-23.

Proposition 1. $X \subseteq \alpha$ is regular and hyperregular if and only if $X^{w\alpha} = X^{c\alpha}$.

Proposition 2. If $X, Y \subseteq \alpha$ are regular and hyperregular then $X \leqslant_\alpha Y$ implies $X' \leqslant_\alpha Y'$.

Thus, when X is regular and hyperregular (and only then), we are justified in writing $X' = X^{w\alpha} = X^{c\alpha}$ and in calling it the *jump* of X. Furthermore, the jump operator is well defined on regular, hyperregular α - degrees.

Friedberg's completeness theorem (cf. Rogers pp. 265-6) can be generalized to α - recursion theory as follows:

Theorem. Let $Y \subseteq \alpha$ be regular. Then there is a regular, hyperregular $X \subseteq \alpha$ such that

$$X' \equiv_{\alpha} X \oplus 0' \equiv_{\alpha} Y \oplus 0' .$$

Corollary 1. An α - degree $\geqslant 0'$ is regular if and only if it is the jump of a regular, hyperregular α - degree.

Corollary 2. Let α be a countable admissible ordinal. Then there is an α - degree $\underset{\sim}{d}$ such that every α - degree $\geqslant \underset{\sim}{d}$ is the jump of a regular, hyperregular α - degree. (Cf. the following abstract, Cones of Regular α - degrees.)

[S. G. Simpson,
Department of Mathematics,
University of California,
Berkeley, California 94720, U.S.A.]

48. Cones of Regular α - Degrees

Let α be an admissible ordinal. Write $|\alpha|$ for the cardinality of α (= the least ordinal equipollent with α). Define the notion of α - degree as in G. Sacks, Post's Problem, Admissible Ordinals, and Regularity; Trans. A.M.S. 124 (1966) 1-23. (It is a satisfactory generalization of the notion of Turing degree to α - recursion theory.)

A set $X \subseteq \alpha$ is α - regular if $X \cap \gamma \in L_{\alpha}$ for all $\gamma < \alpha$. An α - degree is regular if it contains an α - regular set. Much of the intricacy of α - recursion theory stems from the fact that, in general, not every α - degree is regular.

Theorem 1. Assume $V = L$. Then for every admissible ordinal α there is an α - degree $\underset{\sim}{d}$ such that either (a) every α - degree $\geqslant \underset{\sim}{d}$

is regular, or (b) every α - degree $\geqslant \underset{\sim}{d}$ is non-regular. The assumption $V = L$ can be dropped provided $|\alpha|$ is a regular cardinal.

In the proof of Theorem 1 we obtain necessary and sufficient conditions on an admissible ordinal α for conclusion (a) to hold. Namely:

Theorem 2. If $|\alpha|$ is regular then (a) holds iff every $X \subseteq \alpha$ of cardinality less than $|\alpha|$ is a member of L_α. In particular, (a) holds for countable α.

Theorem 3. Assume $V = L$. Then (a) holds iff there is a function $f : |\alpha| \xrightarrow{\text{onto}} \alpha$ such that $f \restriction \gamma \in L_\alpha$ for all $\gamma < |\alpha|$.

The following Lemmas for Theorem 3 may be of independent interest to students of α - recursion theory. In the lemmas, suppose that every α - degree is less than or equal to a regular α - degree. Then:

Lemma 1. $\forall \beta < \alpha(\mathrm{cf}(\beta) = \mathrm{cf}(\alpha) \ \vee \ \mathrm{cf}(\beta) = \mathrm{cf}^{L\alpha}(\beta))$.

Lemma 2. (G.C.H.) $\mathrm{cf}(\alpha) = \mathrm{cf}(|\alpha|)$.

[S. G. Simpson,
Department of Mathematics,
University of California,
Berkeley, California 94720, U.S.A.]

(b) Turing Degrees

49. There is a minimal degree $\underline{m}$ such that

$$\underline{m} \cup \underline{0}' \neq \underline{m}'.$$

This answers a question of S. Simpson. It should be noted that for hyper degrees this is not true.

[L. Sasso,
Department of Mathematics,
University of California,
Irvine, California, U.S.A.]

(c) Recursively Enumerable Degrees

50. Let L be the first-order language with $\leqslant$, 0, LUB, and GLB as non-logical symbols, where LUB(x,y,z) means $x = y \cup z$. Theorem 1: The decision problem for the set of existential sentences of L true in the set of RE degrees is solvable. In fact, such a sentence is true in this set iff it is true in some (finite) poset with 0. This last fact becomes false if $0'$ is added to the language.

A subset A of a poset B is a sublattice if for $x,y \in A$, $x \cup y$ and $x \cap y$ exist in B and are in A. Theorem 1 follows from Theorem 2: Every finite lattice is isomorphic to a sublattice of the set of RE degrees. Methods of proof come from Lachlan's paper in the London volume. (Springer Lecture Notes).

[J. R. Shoenfield,
Department of Mathematics,
Duke University,
Durham, North Carolina 27706, U.S.A.]

(d) Recursively Enumerable Sets

51. Definition 1. A set X of integers is said to be monotone if for every total recursive function f there exist an integer k such that

either i) $\forall n,m\ (n\epsilon X,\ m\epsilon X,\ n,m > k \rightarrow f(n) = f(m))$ or ii) $\forall n,m(n\epsilon X,\ m\epsilon X,\ n,m > k$ and $n < m \rightarrow f(n) \leqslant f(m))$. A set is said to be strictly monotone if in the definition of monotone we require $f(n) < f(m)$ in place of $f(n) \leqslant f(m)$.

Definition 2. A set X of integers is said to one to one if for every total recursive function f there exists an integer k such that either i) $\forall n,m\ (n\epsilon X,\ m\epsilon X,\ n,m > k \rightarrow f(n) = f(m))$ or ii) $\forall n,m\ (n\epsilon X,\ m\epsilon X,\ n,m > k$ and $n \neq m \rightarrow f(n) \neq f(m))$.

Definition 3. An r.e. set is comonotone(c.m.), co one-one (c.1-1) or co-strictly monotone (c.s.m.) if it has infinite complement and the complement is respectively monotone, one-one, or strictly monotone.

The study of these notions was suggested by Owings who showed in his thesis that maximal sets were c.s.m. and as c.s.m. easily implies c.1-1 and c.m. we have that maximal sets are c.1-1. and c.m. However, there exist cohesive non co.-r.e. sets that are not monotone, one to one or strictly monotone.

Co-monotone and co-one to one sets are r-maximal and dense simple and hence using Robinson's characterization of dense simple to note that Lachlan's proof of an r-maximal set with no hyperhyper simple superset actually provides us with an r-maximal set with no dense simple superset we have the existence of r-maximal sets with no co-monotone or co-one to one supersets. We have shown that there exist r-maximal and dense simple sets that are not comonotone and comonotone sets that are not maximal so the notion of comonotone lies strictly between the join of r-maximal with dense simple and maximal.

[Dilip B. Madan,

Department of Economics,

University of Maryland,
College Park, Maryland 20742, U.S.A.]

52. There exists an infinite r.e. class of r.e. sets with no proper infinite r.e. subclass. More specifically, there exists a total recursive function f such that $\{W_{f(n)} | n \varepsilon \omega\}$ is an infinite collection of r.e. sets and for every total recursive function g, if the collection $\{W_{g(n)} | n \varepsilon \omega\}$ is infinite and contained in the collection $\{W_{f(n)} | n \varepsilon \omega\}$ then it is equal to the latter. This answers a question of Putnam and Pour-El.[1]

[Dilip B. Madan,
Department of Economics,
University of Maryland,
College Park, Maryland 20742, U.S.A.]

53. Automorphisms of the lattice of recursively enumerable sets III: recursive and nonrecursive sets

Let $\mathcal{E}^*$ denote the lattice $\mathcal{E}$ of r.e. sets $\{W_x\}_{x \in N}$ under inclusion modulo the ideal $\mathcal{F}$ of finite sets. An r.e. set A is <u>low</u> if its jump $K^A = \{x : x \in W_x^A\}$ has degree $\underset{\sim}{0}'$, and <u>weakly low</u> if $I^A = \{x : W_x \cap \overline{A} \neq \phi\}$ has degree $\underset{\sim}{0}'$. (It is obvious that $I^A \leqslant_T K^A$ and easy to show that there exists a <u>weakly low</u> set in every r.e. Turing degree.) Recursive sets are low as are most incomplete nonrecursive r.e. sets which are constructed by a finite-injury priority argument, such as in the Friedberg-Muchnik theorem.

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[1] Marian Boykan Pour-El and Hilary Putnam "Recursively enumerable classes and their application to recursive sequences of formal theories" Archiv für Mathematische, Logik und Grunlagenforschung 8/34 pp.104-121.

For each r.e. set A, Lachlan considers the principal filter $\mathcal{L}^*(A) = \{W_x : W_x \supset A\}$ modulo $\mathcal{F}$ and has asked whether $\mathcal{L}^*(A) \cong \mathcal{L}^*(B)$ if A and B are both low and simple. Theorem. If A is weakly low and coinfinite then $\mathcal{L}^*(A) \cong \mathcal{E}^*$. Corollary 1: (Lachlan). If A is low and coinfinite and H is any hh-simple set (e.g., a maximal set) then for some r.e. $B \supseteq A$, $\mathcal{L}^*(B) \cong \mathcal{L}^*(H)$. Corollary 2: In every Turing degree there exists an r.e. set A such that $\mathcal{L}^*(A) \cong \mathcal{E}^*$. Corollary 3: $\mathcal{L}^*(A) \cong \mathcal{L}^*(B)$ does not imply that there is an automorphism of $\mathcal{E}^*$ mapping A to B. (We are grateful to Carl Jockusch for pointing out that our original hypothesis "low" could be weakened to "weakly low.")

[Robert I. Soare,
Department of Mathematics,
University of Illinois at Chicago Circle,
Chicago, Illinois 60680, U.S.A.]

(e) Index Sets

54. Theorem 1. Every countable partial ordering can be embedded in the 1-degrees of index sets in $\underset{\sim}{0}'$.

Theorem 2. Every 1-degree of an index set in Δ_2^0 has an immediate successor in the partial ordering of 1-degrees of index sets.

Theorem 3. For θC in Δ_2^0, C non-trivial, the following are equivalent:

(a) $\phi \notin C$,

(b) $(\exists A)(\phi \notin A \ \& \ \theta A \cong \theta C)$,

(c) $(\forall A)\ (\theta A \cong \theta C \rightarrow \phi \notin A)$,

(d) $K \times \theta C \cong \theta C$,

(e) $(\exists S)(\theta C \cong \{\langle x,y\rangle \mid \phi_x(y)$ is defined & $\in S\})$.

[Louise Hay,
Department of Mathematics,
Rutgers College,
New Brunswick, New Jersey 08903, U.S.A.]

(f) Recursive Arithmetic and Consistency, etc.

55. The Metatheory of the Elementary Equation Calculus.

We develop a quantifier-free, logic-free, induction-free, formal system of natural number arithmetic called 'The Elementary Equation Calculus' (E.E.C.). This is a subsystem of Professor Goodstein's 'Primitive Recursive Arithmetic' [1] containing as terms only Kalmar's Elementary Functions, as axioms only the recursive definitions of these functions, and as rules of inference only Goodstein's deduction schema T, Sb_1, Sb_2 and four special cases of U .

It is shown that induction is a derived rule of E.E.C. and that a 'bounded predicate logic' can be introduced into it in the manner of Goodstein [1] . The system is developed to the point at which the prime factorization theorem and the binomial theorem can be proved within it.

An 'arithmetization of syntax', based on that used by Pozsgay in [2] , is introduced and used to prove versions of Gödel's first incompleteness theorem and Löb's theorem (see [3]), from which several forms of Gödel's second incompleteness theorem are deduced. Finally, following Rose [4] and Rose and Cleave [5], the various equations which had been shown to be unprovable in E.E.C. are derived in an extended system E.E.C*. which consists of E.E.C. plus a new function $e_4(x,y)$ whose defining equations are

$$e_4(0,y) = y$$

$$e_4(Sx,y) = f_3(Se_4(x,y),\ Se_4(x,y))$$

where f_3 is as in [6]. This provides a formal proof of the consistency of E.E.C.

References:

[1] R.L. Goodstein, Recursive Number Theory, North-Holland Publishing Company, 1957.

[2] L.J. Pozsgay, Gödel's second theorem for elementary arithmetic, Zeitschr.f.math.Logik, Bd. 14, s. 67-80 (1968).

[3] M.H. Löb, Solution of a problem of Leon Henkin, Journal of Symbolic Logic 20 (1955), 115-118.

[4] H.E. Rose, On the consistency and undecidability of recursive arithmetic, Zeitschr.f.math.Logik, Bd. 7, s. 124-135 (1961).

[5] J.P. Cleave and H.E. Rose, $\mathcal{E}^n$-arithmetic. Article in 'Sets, Models and Recursion Theory' (ed. J.N. Crossley), North-Holland Publishing Company (1967).

[6] A. Grzegorczyk, Some classes of recursive functions, Rozprawy Matematyczne IV, Warsaw, 1953.

[Alan Bundy,

Metamathematics Unit,

Hope Park Square,

Meadow Lane, Edinburgh EH8 9NN, Scotland.]

56. A system-internal 'consistency'-deduction in elementary number theory by primitive recursive computations

I refer to my abstract in JSL 36, 376-377 (erratum : for Ou read $\leqslant$ Ou). We change the last 7 postulates l.c.1.: $\bigvee \to 0 = 0$, $a = b \leftrightarrow {}'a = {}'b$, ${}'a = 0 \to \bigwedge$, $0 = {}'a \to \bigwedge$, $+\,abc \leftrightarrow (b = 0 \wedge a = c) \vee \vee \bigvee x \bigvee y \bigvee z(b = {}'y \wedge a = x \wedge c = {}'z \wedge +\,xyz)$, $\cdot abc \leftrightarrow (b = 0 \wedge c = 0) \vee \vee \bigvee x \bigvee y \bigvee z(b = {}'y \wedge a = x \wedge +\,zac \wedge \cdot xyz)$.

The Secretaries of the recent Congrès de logique d'Orléans (JSL 36, 809) have distributed a canonical system - 148 productive rules on 5 pages - which simplifies the calculus l.c.2.; the above modification yields 149 (or 145) rules.

An inspection of the propositional forms which are required

(i) for an arithmetisation of my new calculus,

(ii) for the arithmetisation of 27 eliminating procedures on directly advancing derivations (simplest case: the 'derivation'-term $(d)_6$ eliminates an application of the imperative translation ${}^{e}A \Rightarrow {}^{e}A$ of each postulate $A \rightarrow A$ on the level $'e$, where ${}^{e}A$ notifies the $\Rightarrow$-translation of A on the level e),

(iii) for 3 arithmetised 'underivability'-deductions as being uniform in the formal level-variable e , that eliminate the imperative translations of $\bigwedge \rightarrow A$, $'a = 0 \rightarrow \bigwedge$, $0 = {}'a \rightarrow \bigwedge$,

has shown that we can deduce the consistency-critical 'consistency'--proposition $\vdash R_D(a) \wedge (a)_{3,1} = \ulcorner \bigvee \urcorner \rightarrow (a)_{3,2} \neq \ulcorner \bigwedge \urcorner$ in terms of several (complicated) primitive recursive functions $r(a) = b$, whereas the refuting deduction $\vdash \bigvee \rightarrow \bigwedge$ uses predicates of the form $\bigvee y(r(a,y) = 0)$.

[Eduard Wette,
5608 Radevormwald,
Blumenstrasse 14, West Germany]

(g) Complexity of Recursive Functions

57. On the greatest natural number of definitional or information complexity $\leq n$

The growth of this number $a(n)$ as a function of n serves to measure a number of very general phenomena.

For example, consider the time $t(n)$ it takes that program with not more than n bits to halt that takes the longest time to halt. This grows with n approximately in the same way as $a(n)$. More exactly, for all n, $a(n) \leqslant t(n+c)$ and $t(n) \leqslant a(n+c')$.

Consider those programs that halt and whose output is the set $S(n)$ of all binary strings of complexity not greater than n. Any program that halts and whose output includes $S(n)$, must either have more than $a(n-c)$ bits, or must take a time to halt exceeding $a(n-c)$. Both extremes are possible: few bits of program and very long running time, or vice versa. Thus those programs with about n bits which halt and whose output set is precisely $S(n)$ are among the programs of length about n that take most time to halt.

Or consider a program that outputs the r.e. but not recursive set of all programs that halt. The time it takes this program to output all programs of length not greater than n that halt, grows with n approximately like $a(n)$.

Or consider the set P having a binary string iff the string's information or definitional complexity is less than its length. P is "simple", that is, P is r.e. and its complement with respect to the set of all binary strings is infinite and contains no infinite r.e. subset. In fact, P is closely related to Post's original construction of a simple set. The time that it takes a program that outputs P to output all P's elements of length not greater than n, grows with n approximately like $a(n)$.

Each of these results can be interpreted as the precise measure of a limitation of formal systems. For example, a formal system can be sufficiently powerful for it to be possible to prove within it that each program that halts in fact does so. Suppose that it is only possible to prove that a program halts if this is true. Then the maximum length of the proofs needed to establish that each program of length not greater than n that halts in fact does so, grows with n in approximately the same manner as $a(n)$.

[G. J. Chaitin,
Mario Bravo 249,
Buenos Aires, Argentina]

58. A necessary and sufficient condition for an infinite binary string to be recursive

Loveland and Meyer* have provided a necessary and sufficient condition for an infinite binary string to be recursive, in terms of the relative information or definitional complexity of its initial segments of length n , given n . In their notation: x is an infinite binary string for which there exists a constant $c > 0$ such that $K(x^n/n) \leqslant c$ for all n , iff x is recursive. Based on this result and other considerations we provide a necessary and sufficient condition using the absolute complexity of the initial segments, instead of the conditional complexity. An infinite binary string x is recursive iff there exists a constant c such that for all n the complexity $K(x^n)$ of its initial segment of length n is bounded by $c+\log_2 n$.

*D.W. Loveland, A variant of the Kolmogorov concept of complexity, Report 69-4, Math. Dept., Carnegie-Mellon Univ.

[G.J. Chaitin,
Mario Bravo 249,
Buenos Aires, Argentina]

59. There are few minimal descriptions

We are concerned with the descriptive/definitional/information complexity, i.e. the complexity of something is the number of bits in the program for calculating it whose size is smallest. Thus the complexity of something is the number of bits in a minimal (complete) description. How many different programs for calculating something are of nearly optimal size, i.e. how many minimal or nearly minimal descriptions of something are there?

We give a bound $b(n)$ on the number of programs for calculating a finite binary string which are of size not greater than the complexity of the string + n . I.e. a bound $b(n)$ on the number of different descriptions of a particular string which are within n bits of a minimal description. The bound is a function of n , i.e. does not depend on the particular string nor its complexity. The particular $b(n)$ established has the property that $\log_2 b(n)$ is asymptotic to n . An application of this result is given in the announcement "A necessary and sufficient condition for an infinite binary string to be recursive."

[G. J. Chaitin,
Mario Bravo 249,
Buenos Aires, Argentina]

(h) Recursive Operators

60. Infinitary Strong Operator Recursion Theorem.

The theorem below generalizes both our strong operator recursion theorem (Item 28, R.F.T.: Newsletter No. 2, 1972) and Smullyan's Double Recursion Theorem (Theory of Formal Systems, Annals of Mathematics Studies, No. 47, Princeton Univ. Press, 1961).

Let Φ_e be the e-th enumeration operator as in Rogers (*Theory of Recursive Functions and Effective Computability*, McGraw Hill, 1967). We write $f = \underline{\mathrm{Join}}_{k \in N}(g_k) \Leftrightarrow_{df} (\forall x,y)[f(\langle x,y\rangle) = g_x(y)]$.

Theorem. $\{\theta_i \mid i \in N\}$ is a sequence of recursive operators and $(\exists$ recursive $r)(\forall i)[\theta_i = \Phi_{r(i)}] \Rightarrow$ there is a sequence $\{g_i \mid i \in N\}$ of distinct monotone increasing recursive functions such that for any $i, x, n \in N$,

$$\phi_{g_i(n)}(x) = \theta_i(\underline{\mathrm{Join}}_{k \in N}(g_k))(\langle n,x\rangle);$$

furthermore, there is a recursive function h such that $(\forall i)[\phi_{h(i)} = g_i]$ and one can effectively find an index for h from an index for r.

[John Case,
Computer Science Department,
University of Kansas,
Lawrence, Kansas 66044, U.S.A]

(i) Open Problems

60. An Open Problem on Creative Sets

by Albert R. Meyer

The set of valid sentences of first-order predicate calculus is not recursive, but the hope for general approaches to mechanical theorem proving is that a reasonable fraction of the "interesting" sentences are effectively decidable. Ignoring the qualification that sentences "interesting," we ask what fraction of the sentences can be classified effectively as valid or invalid. We formulate the question in purely recursion-theoretic terms as follows:

Definition. A set $C \subset$ is approximable to within $\in$ for $1 \geqslant \in \geqslant 0$ iff there exist recursive sets $A \subset C$, $B \subset \overline{C}$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \cdot |\{0,1,\ldots,n-1\}-(A \cup B)| \leqslant \in .$$

Problem: Is there a creative set C and an $\in > 0$ such that C is not approximable to within $\in$?

(j) Corrections

61. In item 28, R.F.T.: Newsletter, No. 2, 1972, in the proof of the theorem replace "$W_i \subseteq p_m{}^n$" and "$W_j \subseteq D_\infty$" by "$P_m{}^n \subseteq W_i$" and "$D_\infty \subseteq W_j$" respectively.

62. Item 37 of the Newsletter (No. 3, October, 1972) should have appeared under the heading "Recursively enumerable sets" (rather than "Partial degrees").