RECURSIVE FUNCTION THEORY: NEWSLETTER

No. 3. October, 1972.

The newsletter is an informal means of circulating information amongst recursion theorists and others interested in recursive function theory. Results and other announcements should be kept fairly brief and sent to:

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Next Deadline: December 15th, 1972.

(a) HIERARCHIES OF RECURSIVE FUNCTIONS.

29. Characterization of an extended Grzegorczyk-hierarchy by means of infinite terms.

Infinite terms of finite type are defined in Tait 1965. A term of typelevel n is called predicative, if all its subterms have typelevels $\leqslant n$. We consider finite notations for certain infinite terms; a sequence $\langle t_i \rangle_{i<\omega}$ is coded by (among others) a pr-rec index of a function assigning to each i a notation for t_i, and a standard-notation for an ordinal $\alpha < \varepsilon_0$ greater than the depth (defined as usual) of each t_i-notation. Let $\mathcal{T}_\alpha$ be the class of all functions definable by a predicative term notation with depth $< \omega(\alpha + 1)$. Theorem: $\mathcal{T}_\alpha = \mathcal{E}_\alpha$ for $\omega \leqslant \alpha < \varepsilon_0$. ($\mathcal{E}_\alpha$ is the αth class of the extended Grzegorczyk-hierarchy as defined in Robbin, 1965).

To each pr-rec functional (in the sense of Gödel) we assign in a primitive recursive way an ordinal $< \varepsilon_0$ as its measure of complexity. This ordinal is obtained as the depth of a canonically constructed predicative term notation representing the given functional. Let $\mathfrak{K}_\alpha$ be the class of all functions definable as pr-rec functionals with an assigned measure of complexity $< \omega(\alpha + 1)$. Theorem: $\mathfrak{K}_\alpha = \mathcal{E}_\alpha$ for $\omega \leqslant \alpha < \varepsilon_0$. -- Let $\tilde{\mathfrak{F}}_n$ be the class of all functions definable as pr-rec functionals in such a way that all functionals occurring in the definition have typelevels $\leqslant n$. Corollary (implicit in Howard, 1970, §5): $\tilde{\mathfrak{F}}_n = \bigcup_{\alpha<\omega_n} \mathcal{E}_\alpha$ for $n \geqslant 1$ ($\omega_0 = 1$, $\omega_{n+1} = \omega^{\omega_n}$).

By a slight extension of the method used, one obtains the following generalization. Theorem: Let F be a pr-rec functional of typelevel n, whose definition uses only functionals of typelevels $\leqslant n + m$. Then there is a definition of F that uses functionals of typelevels $\leqslant n$ only, but contains α-recursions, $\alpha < \omega_{m+1}$, as well as primitive recursions.

References: 1. Howard, W.A., Assignment of ordinals to terms for pr-rec functionals of finite type. Intuitionism and Proof Theory, Amsterdam, 1965, 443-458.

2. Robbin, J.W., Subrecursive hierarchies. Thesis, Princeton, 1965.

3. Tait, W.W., Infinitely long terms of transfinite type. Formal Systems and recursive functions, Amsterdam, 1965, 176-185.

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(b) RECURSIVELY ENUMERABLE SETS.

30. M. Lerman has shown [J.S.L. vol. 36 (1971), 193-215] that if a is any r.e. degree with $a' = 0''$ and B is any r.e. nonrecursive set, then B has a major subset of degree a . The converse to Lerman's result holds in the sense that whenever A is a major subset of B then $a' = 0''$, where a is the degree of A . (The proof of this statement will appear in my review of Lerman's paper in Math. Reviews.) Bob Soare has pointed out that the following stronger result can be proved by much the same argument: If A, B are r.e., $A \subseteq B$, $B - A$ is infinite, and

$a' < 0''$ (where a is the degree of A), then there is an infinite sequence of pairwise disjoint, uniformly recursive subsets of B, all intersecting $B - A$. Thus in particular $B - A$ is not superimmune in the sense of D.A. Martin [Z. Math. Logic vol. 12 (1966), footnote on p. 306].

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31. Post's problem and his hypersimple set.

A standard enumeration of the recursively enumerable (r.e.) sets is a 1:1 recursive function f with range $\{\langle x, y\rangle : x \in W_y\}$, where $\{W_y\}_{y \in N}$ is an acceptable numbering of the r.e. sets in the sense of Rogers, "Theory of recursive functions and effective computability," p. 41. In his quest for an incomplete r.e. set Post (Bull. Amer. Math. Soc. 50 (1944), 284-316) constructed a hypersimple set H_f relative to a fixed but unspecified standard enumeration f. Although it was later shown that hypersimplicity does not guarantee incompleteness, the ironic possibility remained that Post's own particular hypersimple set might be incomplete. This seemed plausible in view of the fact that Post's hypersimple set construction is almost a priority argument (although requirements are never injured), and there is a great deal of "negative restraint" which keeps elements out of the set. We settle the question by proving that the H_f may be either complete or incomplete depending upon <u>which</u> standard enumeration f is used. In contrast, D.A. Martin has shown that Post's <u>simple</u> set S is complete for <u>any</u> standard enumeration.

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32. <u>Automorphisms of the lattice of recursively enumerable sets. I: Maximal sets.</u>

Let $\mathcal{E}$ be the lattice of recursively enumerable (r.e.) sets under inclusion (as explained in Rogers, "Theory of recursive functions and effective computability," Chapter 12). Let $A \subseteq^* B$ denote that $A - B$ is finite. For any r.e. set A let $\mathcal{C}(A) = \{W\colon W \text{ r.e. and } W \subseteq^* A \text{ or } W \,{}^*\!\supseteq \overline{A}\}$. Note that $\mathcal{C}(A)$ is a lattice and is the class of r.e. sets with respect to which $\overline{A}$ is cohesive. <u>Theorem</u>. If A and B are any non recursive r.e. sets, there is a permutation p of N such that $p(A) = B$ and p induces an isomorphism from $\mathcal{C}(A)$ to $\mathcal{C}(B)$. A co-infinite r.e. set A is <u>maximal</u> just if $\mathcal{C}(A) = \mathcal{E}$. Call r.e. sets A and B <u>inequivalent</u> if their symmetric difference, $(A - B) \cup (B - A)$, is infinite. <u>Corollary</u> 1. If for some k, $1 \leqslant k < \omega$, $\{A_i\colon 1 \leqslant i \leqslant k\}$ and $\{B_i\colon 1 \leqslant i \leqslant k\}$ are two finite classes of sets with pairwise disjoint complements, each containing exactly k inequivalent maximal sets, then there is an automorphism ϕ of the lattice $\mathcal{E}$ such that $\phi(A_i) = B_i$ for all i, $1 \leq i \leq k$. For $k = 1$, the problem was proposed to the author by D.A. Martin, and negatively answers the question raised by A.H. Lachlan of whether there are any nontrivial elementary classes of maximal sets ("On the lattice of recursively enumerable sets," Trans. Amer. Math. Soc. 130 (1968), 36). Corollary 2. Turing degree is not invariant under automorphisms of $\mathcal{E}$. (Corollary 2 also has a more direct proof.)

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33. Automorphisms of the lattice of recursively enumerable sets. II: Complete sets.

Post's program (as opposed to Post's problem) was to find a simple property P of complements of recursively enumerable (r.e.) sets which guarantees that a set satisfying P has degree strictly between $\underset{\sim}{0}$ and $\underset{\sim}{0}'$. Although Post's problem was solved by Friedberg and Muchnik, his program remains open as has recently been pointed out by G.E. Sacks in "Degrees of unsolvability," and by C.E.M Yates in his review (J. Symbolic Logic, March 1971) of Harley Rogers' book, "Theory of recursive functions and effective computability." We give a partial answer to the question by proving that no lattice-invariant property guarantees incompleteness. Theorem. Given any nonrecursive r.e. set A there is an automorphism Φ of the lattice of r.e. sets $\mathcal{E}$ such that $\Phi(A)$ has degree $\underset{\sim}{0}'$. Corollary (Yates). There is a complete maximal set.

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(c) DEGREES OF UNSOLVABILITY.

34. If P is a property of infinite sets which is hereditary under inclusion and possessed by at least one arithmetical set, then the degrees of sets which have the property P are closed upwards. This implies upward closure for the degrees of cohesive sets, quasi-cohesive sets, etc. and is proved using a result from my paper "Ramsey's theorem and recursion theory."

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35. Theorem. There exists a non-recursive r.e. degree $\underset{\sim}{d}$ such that any two r.e. sets of degree $\underset{\sim}{d}$ are weak truth table equivalent. The degree that I constructed to show that there exists a non-recursive r.e. degree such that every r.e. set of that degree is mitotic satisfies the theorem.

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36. Notation as in item 17 of Newsletter No. 2. Let A_i for $i < e$ be any Δ_2^0 sets of numbers and $\mathcal{C}_{A_i}$ the computation function obtained from some recursive approximation to A_1.

Lemma. If $\phi(n) \geq \min\{\mathcal{C}_{A_i}(n) | i < e\}$ for all n then $A_i \leq_T \phi$ for some $i < e$.

In particular, if the A_i's are all non-recursive then such a function ϕ could not be recursive. Using this lemma we can improve item 18 of Newsletter No. 2.

Theorem 1. If $\{\underset{\sim}{a}_i\}_{i \in N} \leq_T \underset{\sim}{0}'$ and $\{\underset{\sim}{b}_i\}_{i \in N} \leq_T \underset{\sim}{0}'$ are such that $\underset{\sim}{a}_i > \underset{\sim}{0}$ and $\underset{\sim}{b}_i > \underset{\sim}{0}$ for all i, then there is $\underset{\sim}{c}$ with $\underset{\sim}{a}_i \cup \underset{\sim}{c} = \underset{\sim}{0}'$ and $\underset{\sim}{b}_i \nleq \underset{\sim}{c}$ for all i. The lemma is required in replacing $\underset{\sim}{a}$ by $\{\underset{\sim}{a}_i\}_{i \in N}$. Replacing $\underset{\sim}{b}$ by $\{\underset{\sim}{b}_i\}_{i \in N}$ is trivial. However, the latter improvement does lead in a useful direction as indicated below.

Theorem 2. If $\underset{\sim}{a} > \underset{\sim}{0}$ is r.e. and $\{\underset{\sim}{b}_i\}_{i \in N} \leq_T \underset{\sim}{0}'$ then there is $\underset{\sim}{c}$ with $\underset{\sim}{a} \cup \underset{\sim}{c} = \underset{\sim}{0}'$ and $\underset{\sim}{b}_i \leq \underset{\sim}{c}$ just if $\underset{\sim}{b}_i = \underset{\sim}{0}$ for each i.

Corollary. If $\underset{\sim}{a} > \underset{\sim}{0}$ is r.e. and $\underset{\sim}{a}'' = \underset{\sim}{0}''$ then there is $\underset{\sim}{c}$ with $\underset{\sim}{a} \cup \underset{\sim}{c} = \underset{\sim}{0}'$ and $\underset{\sim}{a} \cap \underset{\sim}{c} = \underset{\sim}{0}$.

The corollary depends on the observation that if $\underset{\sim}{a} \leq \underset{\sim}{0}'$ and $\underset{\sim}{a}'' = \underset{\sim}{0}''$ then the degrees $\leq \underset{\sim}{a}$ are uniformly $\leq \underset{\sim}{0}'$. That is, for some sequence $\{\underset{\sim}{b}_i\}_{i \in N} \leq_T \underset{\sim}{0}'$, $\{\underset{\sim}{d} | \underset{\sim}{d} \leq \underset{\sim}{a}\} = \{\underset{\sim}{b}_i | i \geq 0\}$. In the theorem $\underset{\sim}{a}$ may be replaced by any simultaneously r.e. sequence $\{\underset{\sim}{a}_i\}_{i\in N}$ of $> \underset{\sim}{0}$ degrees; for the corollary the double jumps must be uniformly $\leq \underset{\sim}{0}''$.

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(d) PARTIAL DEGREES.

37. The results announced for truth table degrees in item 21 of the previous (July, 1972) newsletter also hold for weak truth table degrees. In the tt-cases of (ii) and (iii) α (and hence β) may have arbitrary $\neq \underset{\sim}{0}$ r.e. degree while in the wtt-cases it can be ensured that α (and hence β) are recursive in an arbitrary $\neq \underset{\sim}{0}$ r.e. degree. In both cases the unaltered constructions yield complete sets. In the tt-case of (iii) α and β are actually enumerated in the same order however in the wtt-case this is known to be impossible. In contrast to (iii) there are recursively isomorphic r.e. sets α and β with $\Delta_\alpha \cap \Delta_\beta = \phi$. Finally, in the context of self-deficient sets, all infinite, co-infinite recursive sets are self-deficient; every r.e. degree contains a self-deficient set; and there are non-recursive deficiency sets which are not self-deficient.

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(e) HIGHER TYPE RECURSION THEORY.

38. A Hierarchy for the 1-Section of Any Type-2 Object.

Shoenfield (Trans. Amer. Math. Soc. 134 (1968), 103 - 108) constructed a hierarchy for any type-2 object F in which E is recursive by generalizing the hyperarithmetical hierarchy, using a jump operator j_F such that $A'' \leqslant_T j_F(A)$ uniformly. Now suppose F is any type-2 object and let $[e]^f$, $e < \omega$, be some standard enumeration of all unary functions primitive recursive in a unary f. Define $\mathcal{O}$, $<_o$ and f_a inductively by: (1) $1 \in \mathcal{O}$, $\neg(b <_o 1)$, $|1| = 0$, $f_1(x) = 0$. (2) If $a \in \mathcal{O}$, then $2^a \in \mathcal{O}$, $b <_o 2^a \leftrightarrow (b <_o a \vee b = a)$, $|2^a| = |a| + 1$, $f_{2^a}(x) = \langle [x]^{f_a}(o), F([x]^{f_a}) \rangle$. (3) If $a \in \mathcal{O}$, $\phi = [e]^{f_a}$, $\phi(n) \in \mathcal{O}$ and $\phi(n) <_o \phi(n+1)$ for all n, then $3^a5^e \in \mathcal{O}$, $b <_o 3^a5^e \leftrightarrow (En)\ (b <_o \phi(n))$, $|3^a5^e| = \sup_n |\phi(n)|$, and $f_{3^a5^e}(x) = f_{\phi((x)_o)}((x)_1)$. Combining methods of Shoenfield and Feferman (Trans. Amer. Math. Soc. 104 (1962), 101-122) we have:

Theorem 1. A function is recursive in F if and only if it is primitive recursive if f_a for some $a \in \mathcal{O}$.

Corollary (Grilliot). ω_1^F is admissible, for every type-2 F.

Theorem 2. Every function μ-recursive in F is primitive recursive in f_a for some $a \in \mathcal{O}$ with $|a| < \omega^3$.

Theorem 3. If $f \to \lambda x . \langle [x]^f(0), F([x]^f) \rangle$ is a jump operation, in the sense of Enderton (Trans. Amer. Math. Soc. 111(1964), 457-471), then there is a recursive function I such that if a, $b \in \mathcal{O}$ and $|a| \leqslant |b|$ then f_a is recursive in f_b with index $I(a, b)$.

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(f) DESCRIPTIVE SET THEORY.

39. A Basis Theorem for Borel Sets.

H. Friedman has conjectured that any (lightface) Borel set, which contains a non-hyperarithmetic real, contains reals of any hyperdegree $\geq$ the hyperdegree of $\mathcal{O}$. He has proven this conjecture for Borel sets with no hyperarithmetic members. A very weak approximation to the full conjecture is the following Theorem. Any (lightface) Borel set with a non-hyperarithmetic member has a member of the same hyperdegree as $\mathcal{O}$. This leads to the following slight strengthening of the Kondo-Addison Theorem:

If $\mathcal{A}$ is a nonempty Π^1_1 set of reals then there is a singleton Π^1_1 set $\{X\} \subseteq \mathcal{A}$, such that if $\mathcal{A}$ has a non-hyperarithmetic member, then $\mathcal{O} \leqslant_h X$.

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40. Degrees of Complexity of Subsets of the Baire Space.

Let ω be the set of natural numbers, let ${}^{\omega}\omega$ be $\{\alpha \mid \alpha: \omega \to \omega\}$, and let "A" and "B" range over the set of subsets of ${}^{\omega}\omega$. For any A, B : let A be <u>simpler than</u> B (or $A \leqslant B$) iff $A = f^{-1}(B)$ for some f: ${}^{\omega}\omega \to {}^{\omega}\omega$ continuous in the Baire topolgoy (i.e., recursive in some $\delta \in {}^{\omega}\omega$) ; let the <u>degree of complexity</u> of A (or dg(A)) be $\{B \mid A \leqslant B \ \& \ B \leqslant A\}$; let $-A$ be $\{\alpha \in {}^{\omega}\omega \mid \alpha \notin A\}$; let $\overline{dg}(A)$ be $dg(A) \cup dg(-A)$; and let $dg(A) \preceq dg(B)$ iff $A \leqslant B$ or $-A \leqslant B$. We work in ZF with the axiom of dependent choices; let AD and ACn be the axioms of determinateness (see Mycielski, Fund. Math. 53(1964), 205-224) and constructibility. <u>Theorem</u> 1. $AD \Rightarrow (\forall A, B)[A \leqslant B$ or $B \leqslant -A]$ & (hence) $\preceq$ is a total order.

<u>Theorem</u> 2. $(\forall A, B)[A \in F_{\sigma} \cap G_{\delta} \Rightarrow A \leqslant B$ or $B \leqslant -A]$.

<u>Theorem</u> 3. $(\forall A, B)[A \in F_{\sigma}$ & B is analytic $\Rightarrow \overline{dg}(A) \preceq \overline{dg}(B)$ or $\overline{dg}(B) \preceq \overline{dg}(A)]$. <u>Theorem</u> 4. $ACn \Rightarrow (\exists A, B)[A \in F_{\sigma}$ & B is analytic & $A \nleqslant B$ & $B \nleqslant -A]$. <u>Theorem</u> 5. If all Borel subsets of ${}^{\omega}\omega$ are determinate then $\preceq$ on $\{\overline{dg}(A) \mid A$ is Borel$\}$ is a well ordering type ε_1^{Ω} (where Ω is the least uncountable ordinal and $\langle \varepsilon_{\gamma}^{\mu} \mid \gamma, \mu$ are ordinals$\rangle$ is defined recursively by: ε_{γ}^{0} is the γ^{th} epsilon number ε_{γ} ; $\varepsilon_{\gamma}^{\mu}$ is the γ^{th} ordinal δ such that $(\forall \nu < \mu)\ \varepsilon_{\delta}^{\nu} = \delta$ (note that $\varepsilon_0^{\Omega} = \Omega$)) . The proofs use an infinite-game characterization of $\leqslant$, and, for Theorem 5, Kuratowski's (α, β)-homeomorphisms and an addition operation on degrees.

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(g) CORRECTIONS.

41. A proof of the lemma mentioned in items 17 and 18 (Newsletter No. 2) can be extracted from the proof of Theorem 1.2 of Webb Miller and D.A. Martin's paper "The degrees of hyperimmune sets" [Zeit. Math. Logik, 14 (1968) 159-166].

42. In item 22 (Newsletter No. 2), replace "n = 4" by "n > 4" in the theorem, and replace "n = 3" by "n > 3" in lemma 1.

(h) MISCELLANEOUS ANNOUNCEMENTS.

43. Dr. S.S. Wainer (Dept. Mathematics, Univ. of Leeds, Leeds, England) is considering preparing a bibliography of publications on hierarchies of recursive functions, for eventual inclusion in the "Newsletter," and would be glad to receive details of such publications from authors.

44. It has been suggested that the "Newsletter" include an "open questions" section -- if anyone has interesting questions to contribute, we will try to include them in future issues.

45. THE SEMESTER OF THE FOUNDATIONS OF MATHEMATICS

[January - June, 1973]

At the beginning of this year the Stefan Banach Internation Mathematical Center in Warsaw was founded as a joint enterprise of the Academies of Socialistic Countries. Advanced training on a high level in various fields of mathematics is the aim of the Center. The Center is a part of the Institute of Mathematics of the Polish Academy of Sciences, and the Director of the latter supervises the activity of the Center. The main activity of the Center will consist of organizing Semesters in different fields of mathematics.

The scientific council of the Center, composed of representatives of the Academies, decided that the first Semester will be devoted to foundation of mathematics.

The programme of this Semester will cover: 1. the theory of models, 2. the theory of recursion, 3. the logical calculi and their applications. Seminars, lectures, and research work will constitute the basic particpants' activity. Professors Andrzej Mostowski and Helena Rasiowa will lead the seminars from the part of Polish scientists. Up to now, the participation for a least several weeks of the following foreign scientists is expected: J. Ershov, P. Hajek, R. Gandy, A. Lachlan, G. Sacks, P. Vopenka. The participants will be only by invitation. The Center will have some means to cover living expenses of some invited participants. The Scientific Council of the Center has appointed the Organizing Committee of the Semester consisting of professors S. Jablónski [USSR], G. Moisil [Rumania], H. Rasiowa [Poland], A. Grzegorczyk [Poland, the Chairman].

Those who wish to obtain some further information concerning the Semester are requested to write to Professor A. Grzegorczyk, Instytut Matematyczny, Polskiej Akademii Nauk, Warszawa 1, Ul. Sniadeckich 8, Skrytka Pocztowa 137, Poland.

[Reprinted from the first communication.]