RECURSIVE FUNCTION THEORY: NEWSLETTER

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The Newsletter is an informal means of circulating information amongst recursion theorists and others interested in recursive function theory. Results and other announcements should be kept fairly brief and sent to:

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Next deadline: September 15th, 1972.

(a) Partial degrees.

WEAK TURING DEGREES

16. Weak Turing degrees are degrees of partial functions induced by recursive operators (cf. Rogers': "Theory of Recursive Functions...") or by compact functionals (cf. Davis': "Computability and Unsolvability"). Weak Turing degrees are a finer partition of partial functions than enumeration degrees (cf. Rogers) but a grosser partition than Turing degrees studied previously by us. (Cf. AMS-Notices: 666-39-June, 1968; 71T-E74-August, 1971; 691-02-10-January, 1972). The distinction is illustrated as follows: Let C_α be the characteristic function of a set α and S_α be the function taking value o on α and undefined elsewhere. C_α and $S_\alpha \oplus S_{\bar{\alpha}}$ are always in the same enumeration degree but only in the same Weak Turing degree if α is recursive. For sets α, β $S_{\alpha\cup\beta}$ and $S_\alpha \oplus S_\beta$ are always in the same Weak Turing degree but can be in different Turing degrees. The Weak Turing degrees are an upper semi-lattice (under the naturally induced order) and the total degrees (degrees containing a total function) are isomorphic to the degrees of unsolvability in the sense of the literature. There are no minimal degrees but there are quasi-minimal degrees (non-total degrees below which the only total degree is $\underset{\sim}{0}$), in fact any S_α with α not r.e. has quasi-minimal degree.

Spectors' well-known result that any countable ideal is the intersection of two principal ideals holds for Weak Turing degrees, hence the degree structure is not a lattice. Any countable ideal has a quasi-minimal upper bound (ar upper bound with all total degrees below it being in the ideal). Finally, there is an upper-semi-lattice isomorphic embedding of the total degrees properly into degrees which contain some S_α . Most of the proofs are direct modifications of the corresponding proofs for either enumeration or Turing degrees.

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(b) Degrees of unsolvability.

17.

LEMMA. Let A be any non-recursive Δ_2^0 set of numbers. Let $\{\sigma_s | s \geqslant 0\}$ be a sequence of strings each σ_s being of length s , such that $A(x) = \lim_s \sigma_s(x)$ for each $x \geqslant 0$. Then there is no recursive function f which dominates C_A where

$$C_A(x) = \mu s(\sigma_s[x] = A[x]) .$$

Proof: If f dominates C_A , to compute A(x) look for the least n for which $\sigma_s(x) = \sigma_{f(n)}(x)$ for each s between n and f(n) . (This can be applied in extending results on r.e. degrees to the degrees recursive in $\underset{\sim}{0}'$).

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18. THEOREM. $\underline{0} < a \leqslant \underline{0}'$ and $\underline{0} < \underline{b} \leqslant \underline{0}'$, then there is a degree $\underline{c}$ with $\underline{a} \cup \underline{c} = 0'$ and $b \nleqslant c$.

The proof makes use of Cooper's lemma [17].

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19. The degrees of the members of the Boolean algebra generated by the set of r.e. sets and their complements extend the r.e. degrees (also proved by A.H. Lachlan), contain no minimal degrees, and contain no pair complementary below $\underset{\sim}{0}'$.

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(c) Recursively enumerable sets.

20. Following A.H. Lachlan's recent proof of the existence of r.e. degrees $\underset{\sim}{a}, \underset{\sim}{b}$, with $\underset{\sim}{a} < \underset{\sim}{b}$, and the property that for no incomparable pair of r.e. degrees $\underset{\sim}{c}_0, \underset{\sim}{c}_1$ can one have $\underset{\sim}{c}_0 | \underset{\sim}{c}_1$, $\underset{\sim}{a} < \underset{\sim}{c}_0$, $\underset{\sim}{a} < \underset{\sim}{c}_1$ and $\underset{\sim}{b} = \underset{\sim}{c}_0 \cup \underset{\sim}{c}_1$, it is

easy to show that the usual method of splitting is bound to fail locally in the r.e. degrees. Using Yates' index-set results, one can show that given an r.e. set D , the index set of the set of r.e. sets recursive in some pair of r.e. sets forming a disjoint splitting of D is at most Σ_4 . And if D is chosen to be Ladner's complete non-mitotic r.e. set, there must be some $A <_T D$ with no disjoint splitting of D in which A is recursive, since otherwise the index set would be $\underset{\sim}{C}(< \underset{\sim}{O}') \in \Pi_4 - \Sigma_4$.

A general result concerning non-mitotic r.e. sets can be obtained using the correspondingly stronger Yates index set results.

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DEFICIENCY SETS

21. Recall that the deficiency set of a 1 - 1 , recursive function f is $\{m: (\exists n > m)[f(n) < f(m)]\}$ and that an r.e. set β is a deficiency set for an r.e. set α if β is the deficiency set of some 1 - 1 , recursive function with range α . Let Δ_α be the set of deficiency sets of α for any r.e. α and observe that $\Delta_\alpha \cup \{\alpha\}$ is always contained in a single Turing degree. For truth table and finer degree structures the following possibilities exist for

non-recursive, r.e. sets α, β :

(i) Δ_α is contained in a single many-one degree,

(ii) Δ_α is split by a truth table degree,

(iii) $\beta|_{tt}\alpha$ and $\Delta\alpha \cap \Delta_\beta \neq \phi$.

In proving (i) a "self-deficient" α (i.e., $\alpha \in \Delta_\alpha$) is constructed.

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(d) Decision problems.

22. THEOREM. For $n = 4$ there is no algorithm for deciding of any given n-complex whether it is a combinatorial n-manifold. This follows from a result which can be extracted from the Markov paper ("Insolubility of the problem of homeomorphy", Proc. Intern. Congress of Mathematicians, 1958 (C.U.P.) 300-306)

LEMMA 1. For $n = 3$ there is no algorithm for deciding of any combinatorial manifold whether it is a combinatorial n-sphere,

and also

LEMMA 2. Let M be an n-complex. $\{a, b\}M$ is an $(n + 1)$-manifold iff M is a combinatorial n-sphere.

($\{a, b\}M$ is the double suspension of M from the vertex pair {a, b}) .

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23. WS1S is the weak monadic second order theory of successor. Büchi has shown that WS1S is recursive, and in fact primitive recursive. Let $t(0, n) = n$; $t(k + 1, n) = 2^{t(k,n)}$.

THEOREM. There is an $\in > 0$ such that any procedure (Turing machine) which decides the truth of sentences in the language of WS1S requires time and space exceeding $t(\in \cdot \log_2 n, n)$ on some sentences of length n for all sufficiently large integers n .

COROLLARY. *WS1S is not elementary-recursive in the sense of Kalmar.* The proof follows by exhibiting formulas of length n in the language of WS1S which define the predicate $\lambda y[y \geqslant t(\in \cdot \log_2 n \ \ n)]$, and then using these formulas to obtain formulas of roughly length n which describe Turing machine computations of length $t(\in \cdot \log_2 n, n)$.

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(e) Complexity of recursive functions.

24. PRIORITY ARGUMENTS IN COMPLEXITY THEORY

Abstract (machine-independent) complexity theory is basically a theory which establishes limits on what may be said about the complexity of specific recursive functions. (For example, the speed-up theorem [1] shows that it is not always possible to discover close upper and lower bounds for the complexity of a function.) As such, its basic proof method is diagonalization.

Recursion theorists have developed extensive diagonalization machinery for theorems about degrees of unsolvability. Thus far, we have not applied any except the simplest of these methods to complexity theory. We present here two results which arise very naturally out of complexity theory and whose proofs seem to require use of priority constructions.

The problems we deal with involve finding pairs of recursive sets which are complex, but for "different reasons"--that is, they don't "help" each other's computation.

To handle questions of this type, we use a generalization of Blum's axioms for complexity of partial recursive functions, to the case of relative algorithms (as represented by Turing machines with oracles). Specifically, the axioms we use are:

(1) $(\forall i, x, A)\ \phi_i^{(A)}(x)\downarrow \Leftrightarrow \Phi_i^{(A)}(x)\downarrow$

(2) $(\exists\psi$, a relative algorithm$)(\forall i, x, y, A)$

$$\psi^{(A)}(i, x, y) = \begin{cases} 1 & \text{if } \Phi_i^{(A)}(x) = y , \\ 0 & \text{otherwise.} \end{cases}$$

We call the functions $\Phi_i^{(\phi)}$ "running times."

It is intuitive to keep in mind the "time measure" (number of steps) on oracle Turing machines when considering our results.

The first theorem is a subrecursive analog to the Friedberg-Muchnik theorem. Instead of producing two sets which do not permit each other's computation, we produce two recursive sets which do not make each other's computation any easier. Another way of interpreting the result is to say it produces pairs of recursive sets which are complex, but for different reasons.

We use the notation given in our previous announcement of results (April 1972 Newsletter, (a) 3.).

THEOREM 1. There exists h, a recursive function of two variables, with the following property:

For all sufficiently large running times t_B and t_C, there exist recursive sets B and C such that:

$$\mathrm{Comp}\ B \leqslant h \circ t_B \text{ a.e.},$$

$$\mathrm{Comp}\ C \leqslant h \circ t_C \text{ a.e.},$$

$$\mathrm{Comp}^{(B)} C > t_B \text{ a.e.},$$

and

$$\mathrm{Comp}^{(C)} B > t_C \text{ a.e.}$$

The method of proof we use is to simultaneously construct the two sets B and C, using diagonalization and a finite-injury priority argument. There is a small recursive bound on the number of injuries to any condition.

We next try to fix one of the sets arbitrarily. We easily obtain the following:

<u>PROPOSITION</u>: There exists h, a recursive function of two variables, with the following property:

For any recursive set A, and any recursive function t_A with the property that $\mathrm{Comp}\ A > h \circ t_A$ i.o., there exist arbitrarily complex recursive sets B such that

$$\mathrm{Comp}^{(B)} A > t_A \text{ i.o.}$$

The proof idea is partly due to Machtey, and is similar to the initial segment constructions in [2]; there is essentially no priority involved.

Our second theorem is similar to the proposition, but involves a stronger kind of lower bound on the complexity of

A . It is a strengthening of a result announced in the April, 1972 Newsletter. Specifically:

<u>THEOREM 2</u>. There exists h , a recursive function of two variables, with the following property:

For any recursive set A , and any total running time t_A , with $t_A \geqslant \lambda x[x]$, if $\text{Comp } A > h \circ t_A$ a.e. , then there exist arbitrarily complex recursive sets B such that

$$\text{Comp}^{(B)} A > t_A \text{ a.e.}$$

The method of proof is a finite-injury priority argument with no apparent recursive bound on the number of injuries for each condition.

[1] Blum, Manuel. A Machine-Independent Theory of the Complexity of Recursive Functions, JACM, Vol. 14, No. 2, April 1967, pp. 322-336.

[2] Rogers, Hartley, Jr. Theory of Recursive Functions and Effective Computability.

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25. THEOREM. Let A and B be recursively enumerable sets. The following conditions are equivalent:

(i) $\exists W_i = A$, $W_j = B$ such that

$$\Phi_i(x) \leqslant \Phi_j(x)$$

for all but finitely many x in A.

(ii) $B \backslash A$ is recursively enumerable.

(iii) Every (0-1 valued) partial recursive function with domain A can be extended to a (0-1 valued) partial recursive function with domain B A.

The proof contains a no-injury priority argument similar to that of Theorem 7 in Blum [1].

COROLLARY 1. (Friedberg's Decomposition Theorem) Every non-recursive r.e. set can be decomposed into two disjoint non-recursive r.e. sets.

COROLLARY 2. An r.e. set A is simple if and only if some partial recursive function with domain A cannot be extended infinitely to a partial recursive function.

[1] Blum, M., "A machine independent theory of the complexity of recursive functions," Journal of the Association for Computing Machinery 14 (1967), 322-336.

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On Effectively Levellable Sets

For any set A let S_A denote the semi-characteristic function of A, i.e., the function taking 1 on A and being undefined outside. Following Blum [1] we denote by Φ_i the step counting functions of some Blum complexity measure Φ. For other notations see Blum [2] and Rogers [4].

Def. (Blum [1]) An r.e. set W is *effectively levellable* iff there exist recursive functions r and ν such that $\phi_i = S_W \wedge a_e$ total $\rightarrow$

there exist infinitely many x

$$[\Phi_i(x) > \phi_e(x) \wedge \Phi_{\nu(i,e)} \leqslant r(x)].$$

Def. An r.e. set W is *undercreative* iff there exist recursive functions δ and δ' such that

(i) $\forall j[W_{\delta(j)} = \overline{W_{\delta'(j)}}]$ and

(ii) $W_j \wedge W$ finite $\rightarrow W_{\delta(j)} \subseteq \overline{W_j - W} \wedge W_{\delta(j)} \cap (\overline{W_j \wedge W}) \neq \phi$.

Theorem. An r.e. set W is effectively levellable iff it is undercreative.

This theorem parallels a similar characterization of Blum [3].

John Gill has noticed that undercreative sets are not strongly hypersimple and that any non-strictly hypersimple set can be extended to an undercreative one.

References:

[1] Blum, Manuel: A Machine-Independent Theory of the Complexity of Recursive Functions, JACM, 14, 2 (April, 1967), pp. 322-336.

[2] Blum, Manuel: On Effective Procedures for Speeding Up Algorithms, JACM, 18, 2 (April, 1971) pp. 290-305.

[3] Blum, Manuel: Subcreative Sets and the Complexity of Algorithms (to appear).

[4] Rogers, Hartley Jr.: Theory of Recursive Functions and Effective Computability, McGraw-Hill, 1967.

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27. In (1) Blum defines a machine independent theory of complexity of recursive functions. The complexity of each partial recursive function ϕ_i is given by a partial recursive function Φ_i. The set $\{\Phi_i\}$ is completely arbitrary save for two basic restrictions, the axioms:

1. $\phi_i(u)$ is defined $\Leftrightarrow$ $\Phi_i(u)$ is defined.
2. The function

$$m(i, n, m) = \begin{cases} 1 & \text{if } \Phi_i(n) = m \\ 0 & \text{otherwise} \end{cases}$$

is total recursive.

Let M be a set of integers. The semi-characteristic function of the set M is defined as

$$S_M(u) = \begin{cases} 1 & \text{if } u \in M \\ \text{undefined otherwise} \end{cases}$$

Definition (Blum (2)). A recursively ennumerable set M is <u>levelable</u> iff there is a recursive function r such that for any integer i such that $\phi_i = S_M$, there exists an integer j such that $\phi_j = S_M$ and

$$(\forall \text{ rec } h)(\overset{\infty}{\exists} u)[\Phi_i(u) > h(u) \ \& \ \Phi_j(u) < r(u)] .$$

It can be shown that the property of being levelable is measure independent. The following is a characterization of the levelable sets in purely recursive theoretic terms:

<u>Theorem</u>. A recursively ennumerable set M is levelable iff there exists a recursive function σ such that:

(1) $(\forall i)[\phi_{\sigma(i)}$ is the characteristic function of a set $C_i]$

(2) $(\forall \text{ rec } A)[A \not\subseteq \overline{M} \rightarrow (\exists i)[A \cap C_i$ is infinite & $C_i \subseteq M]]$.

<u>Definition</u> (Blum (3)). A recursively ennumerable set M is <u>speedable</u> iff for any integer i such that $\phi_i = S_M$ and for any recursive function h there exists an integer j such that $\phi_j = S_M$ and $\Phi_i(u) > h(u, \Phi_j(u))$ for infinitely many u. Speedability is also a measure independent concept.

The following is a characterization of speedable sets:

<u>Theorem</u>. A recursively ennumerable set M is speedable iff for all recursive σ,

if $(\forall i)[w_i \cap \overline{M} = w_{\sigma(i)} \cap \overline{M}]$

then $(\exists i)[w_i \subseteq M$ and $w_{\sigma(i)}$ is infinite].

Proofs of both theorems stated on this note are not difficult.

References:

[1] Blum, M. "A Machine-Independent Theory of the Complexity of Recursive Functions," JACM, April 1967, Volume 14, Number 2.

[2] Blum, M. "On Effective Procedures for Speeding Up Algorithms," JACM, April 1971, Volume 18, Number 2.

[3] Blum, M. "On Effectively Speedable Sets," to appear.

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(f) Turing machines.

28. Strong Operator Recursion Theorem and an Application to Automata

The following theorem (proved in [1]) generalizes the parametric form of Kleene's Recursion Theorem. (Unexplained notation or terminology is from [3].)

Strong Operator Recursion Theorem. $\underset{\sim}{\Theta}$ is a recursive operator ⇒ there is a monotone increasing recursive function h such that

$$\phi_{h(n)}(z) = \underset{\sim}{\Theta}(h)(\langle n, z\rangle) \ .$$

As an application we present below a proof of (an inessential modification of) a result stated in [1].

In [1] we consider automata which we call Turing machine automata, which we think of as embedded in a construction universal cellular space [2], [4], and each of which is functionally organized into three parts: a builder-controller, a plan for the builder-controller, and a Turing machine. T.M. automata build other T.M. automata by carrying out the following sequence of operations. (1) The builder-controller starts the T.M. with input 0 on its tape, (2) if and when the T.M. halts with some output k on its tape, the builder-controller builds a new T.M. whose index is k, (3) the builder-controller uses the plan of itself to build both a copy of itself and another copy of the plan of itself, and (4) the builder-controller joins together the new copy of itself, the new copy of the plan of itself, and the new T.M. to form its offspring automaton. Clearly, a T.M. automaton which contains a self-describing T.M. (i.e., a T.M. with index $e \ni \phi_e(0) = e$) is self-replicating.

Let the index of a T.M. automaton be the index of its T.M. Let $P_m^n =_{df} \{x/(\exists e_0, e_1, \ldots)[x = e_0 \wedge (\forall i)[\phi_{e_i}(0) = e_{i+1} \wedge e_{m+n+1+i} = e_{m+i}] \wedge e_0, e_1, \ldots, e_{m+n}$ are all distinct]} . P_m^n is the set of indices of T.M. automata without sterile descendants which after a delay of m generations repeat in generations with period $= n + 1$. Let $D_\infty =_{df} \{x/(\exists e_0, e_1, e_2, \ldots)[x = e_0 \wedge (\forall i, j)[\phi_{e_i}(0) = e_{i+1} \wedge [i \neq j \Rightarrow e_i \neq e_j]]]\}$. D_∞ is the set of indices of

T.M. automata which have no sterile descendants but all of whose descendants are distinct from one another.

Theorem. $(\forall m, n)[P^n_m, D_\infty$ are effectively inseparable].

Proof. By the strong operator recursion theorem, there is a recursive 1-1 h such that

$$\phi_{h(\langle k,\langle i,j\rangle\rangle)}(z) = \begin{cases} h(\langle k+1,\langle i,j\rangle\rangle), & \text{if } h(\langle 0,\langle i,j\rangle\rangle) \text{ appears in } W_i \text{ before it appears in } W_j \vee k < m+n; \\ h(\langle m+r,\langle i,j\rangle\rangle), & \text{if } h(\langle 0,\langle i,j\rangle\rangle) \text{ appears in } W_j \text{ before it appears in } W_i \wedge k = m+n+r; \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

Let $f(i, j) = h(\langle 0, \langle i, j\rangle\rangle)$. Suppose by way of contradiction that $W_i \subseteq P^n_m$, $W_j \subseteq D_\infty$, $W_i \cap W_j = \phi$, and $f(i, j) \in W_i \cup W_j$. If $f(i, j) \in W_i$, then $f(i, j) \in D_\infty \subseteq W_j$. Therefore, $f(i, j) \notin W_i$. If $f(i, j) \in W_j$, then $f(i, j) \in P^n_m \subseteq W_i$. Therefore $f(i, j) \notin W_i \cup W_j$, a contradiction. ▮

References:

[1] Case, J., "Periodicity in generations of automata," submitted for publication).

[2] Codd, E.F., Cellular Automata, Academic Press, New York, 1968.

[3] Rogers, H., Jr., Theory of Recursive Functions and Effective Computability, McGraw-Hill, New York, 1967.

[4] von Neumann, J., Theory of Self-Reproducing Automata (edited and completed by A.W. Burks), University of Illinois Press, Urbana, Illinois, 1966.

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