ON LACHLAN'S MAJOR SUB-DEGREE PROBLEM *

S. Barry Cooper†
Department of Pure Mathematics
University of Leeds
Leeds LS2 9JT
West Yorkshire, U.K.
pmt6sbc@leeds.ac.uk

Angsheng Li‡
State Key Laboratory of Computer Science
Institute of Software
Chinese Academy of Sciences
P.O. Box, 8718, Beijing, 100190
P. R. of CHINA
angsheng@ios.ac.cn

Abstract

The *Major Sub-degree Problem* of A. H. Lachlan (first posed in 1967) has become a long-standing open question concerning the structure of the computably enumerable (c.e.) degrees. Its solution has important implications for Turing definability and for the ongoing programme of fully characterising the theory of the c.e. Turing degrees. A c.e. degree $\mathbf{a}$ is a *major subdegree* of a c.e. degree $\mathbf{b} > \mathbf{a}$ if for any c.e. degree $\mathbf{x}$, $\mathbf{0}' = \mathbf{b} \vee \mathbf{x}$ if and only if $\mathbf{0}' = \mathbf{a} \vee \mathbf{x}$. In this paper, we show that every c.e. degree $\mathbf{b} \neq \mathbf{0}$ or $\mathbf{0}'$ has a major sub-degree, answering Lachlan's question affirmatively.

1 Introduction

The lack of natural local definitions of many of the most important degree-theoretic relations is a central problem of computability theory. There is growing evidence

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that the structural theorems currently provable provide inadequate ingredients for such missing definitions, and that the technical basis for the next level of structure theory needed must transcend that currently available (essentially that derived from the work of Sacks, Yates, Lachlan and their contemporaries in the sixties to mid-seventies, and subsequently consolidated by Harrington and others). Over the years, the difficulties encountered by those who worked on Lachlan's major subdegree problem seemed to both confirm the need to prioritise the development of new technical resources, while providing a promising context within which such a research programme might be progressed. The proof below does contain a number of new and potentially useful features. A number of these are due to other researchers who have spent (often considerable amounts of) time on this problem, and others arose out of the need to fill significant gaps in earlier partial solutions to the problem.

We say that a set $A \subseteq \omega$ is *computably enumerable* (c.e.), if there is an algorithm to enumerate the elements of it. For $A, B \subseteq \omega$, we say that A is *Turing reducible to* (or *computable in*) B, if there is an algorithm to decide for every $x \in \omega$, whether or not $x \in A$, when given answers to all questions of the form "Is $y \in B$?". We use $A \leq_{\mathrm{T}} B$ to denote that A is Turing reducible to B, and we write $A \equiv_{\mathrm{T}} B$ if $A \leq_{\mathrm{T}} B$ and $B \leq_{\mathrm{T}} A$. A (Turing) *degree* is an equivalence class of A under $\equiv_{\mathrm{T}}$ for some set $A \subseteq \omega$. We say that a degree $\mathbf{a}$ is *computably enumerable* (c.e.) if it contains a c.e. set. Post (1944) observed that there is a greatest c.e. degree $\mathbf{0}'$, and asked whether or not there is a c.e. degree other than $\mathbf{0}$ (the least degree) and $\mathbf{0}'$. Muchnik [1956], and independently Friedberg [1957], answered Post's question affirmatively. Furthermore, Sacks [1963], [1964] showed that:

1.1 THEOREM. (Sacks Splitting Theorem, 1963) For any c.e. degree $\mathbf{a} \neq \mathbf{0}$, there exist c.e. degrees $\mathbf{a_0}, \mathbf{a_1}$ such that $\mathbf{a_0}, \mathbf{a_1} < \mathbf{a}$ and $\mathbf{a} = \mathbf{a_0} \vee \mathbf{a_1}$.

1.2 THEOREM. (Sacks Density Theorem, 1964) For any c.e. degrees $\mathbf{b} < \mathbf{a}$, there is a c.e. degree $\mathbf{c}$ such that $\mathbf{b} < \mathbf{c} < \mathbf{a}$.

Consequently, Shoenfield [1965] raised the following:

1.3 CONJECTURE. Given finite upper semi-lattices $P \subseteq Q$ in the language $L(\leq, \vee, 0, 0')$, any embedding of P into $\boldsymbol{\mathcal{E}}$ (the set of all c.e. degrees), in which degree theoretic joins and $\mathbf{0}$, $\mathbf{0}'$ are consistent with Q, can be extended to an embedding of Q into $\boldsymbol{\mathcal{E}}$.

In particular, Shoenfield listed two consequences of the conjecture:

C1. For any c.e. degrees $\mathbf{a}, \mathbf{b}$, if $\mathbf{a}, \mathbf{b}$ are incomparable, the greatest lower bound of $\mathbf{a}, \mathbf{b}$ does not exist.

C2. For any c.e. degrees $\mathbf{0} < \mathbf{b} < \mathbf{c}$, there is a c.e. degree $\mathbf{a} < \mathbf{c}$ such that the least upper bound of $\mathbf{a}, \mathbf{b}$, denoted by $\mathbf{a} \vee \mathbf{b}$, equals $\mathbf{c}$.

C1 was refuted by the *minimal pair theorem* of Lachlan [1966] and (independently) Yates [1966].

1.4 THEOREM. (Minimal Pair Theorem) There exist c.e. degrees $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$, where $\mathbf{a} \wedge \mathbf{b}$ is the greatest lower bound of $\mathbf{a}, \mathbf{b}$.

However, the search for a characterisation of exactly *which* $P \subseteq Q$ Shoenfield's conjecture holds for gave rise to an extensive body of results relating to what is often referred to as the algebraic structure of the c.e. degrees, and eventually led to Slaman and Soare's [1995] full solution to this extensions of embeddings problem.

In looking for decision procedures for fragments of the elementary theory of c.e. sets the notion of major subset plays an important role, and Lachlan suggested a corresponding notion for the c.e. degrees:

1.5 DEFINITION. (Lachlan, 1967) Given c.e. degrees $\mathbf{a} < \mathbf{b}$, we say that $\mathbf{a}$ is a *major sub-degree of* $\mathbf{b}$, if for every c.e. degree $\mathbf{x}$, we have that $\mathbf{b} \vee \mathbf{x} = \mathbf{0}'$ holds if and only if $\mathbf{a} \vee \mathbf{x} = \mathbf{0}'$ holds.

Given that every incomputable c.e. set B has a major subset (Lachlan [1968]), it is natural to ask:

1.6 MAJOR SUB-DEGREE PROBLEM. (Lachlan, 1967, private communication, see Jockusch and Shore [1983] or Ambos-Spies, Lachlan and Soare [1993]) Does every c.e. degree $\mathbf{b} \neq \mathbf{0}$ or $\mathbf{0}'$ have a major sub-degree?

Attempts to solve the problem unexpectedly led to an appreciation of the importance of the notion of *continuity* in the c.e. degrees, where we regard a structural property P as continuous if whenever it holds at some point it also holds in a neighborhood of that point, and a number of results have been proved. The first progress with the major sub-degree problem is related to consequence C2 (above) of Shoenfield's conjecture. We say that a c.e. degree $\mathbf{a} \neq \mathbf{0}$ is *noncuppable*, if for any c.e. degree $\mathbf{x}$, $\mathbf{a} \vee \mathbf{x} = \mathbf{0}'$ if and only if $\mathbf{x} = \mathbf{0}'$, and is *cuppable* otherwise. Then:

1.7 THEOREM. (Cooper [1974a] and Yates) There exists a noncuppable c.e. degree.

More extensions of theorem 1.7 were obtained by Harrington [1976] (see Miller [1981]). By definition, if $\mathbf{b}$ is noncuppable, then any nonzero c.e. degree $\mathbf{a} < \mathbf{b}$ is a major sub-degree of $\mathbf{b}$. So major sub-degrees exist, since every noncuppable c.e. degree has one.

Many partial results relating to the major sub-degree problem are describable within a useful hierarchy of the class of all c.e. degrees, i.e., the high/low hierarchy, which we now describe. We say that a c.e. degree $\mathbf{a}$ is high$_n$ (low$_n$) if $\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$ ($\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$), for all $n > 0$, where $\mathbf{a}^{(0)} = \mathbf{a}$, $\mathbf{a}^{(n+1)} = (\mathbf{a}^{(n)})'$, and $\mathbf{x}'$ is the Turing *jump* of $\mathbf{x}$. Let $\mathbf{H}_n$ ($\mathbf{L}_n$) be the set of all high$_n$ (low$_n$) c.e. degrees. If $n = 1$, then we also call an element of $\mathbf{H}_1$ ($\mathbf{L}_1$) *high* (*low*).

Jockusch and Shore [1983] gave the first instance of a cuppable c.e. degree which has a major sub-degree:

1.8 THEOREM. (Jockusch and Shore, 1983) There exist c.e. degrees $\mathbf{l} < \mathbf{h} < \mathbf{0}'$, such that $\mathbf{l}$ is low, $\mathbf{h}$ is high, and such that $\mathbf{h}$ is cuppable and $\mathbf{l}$ is a major sub-degree of $\mathbf{h}$.

General continuity results were first achieved in relation to the dual of the major sub-degree problem.

1.9 THEOREM. (Harrington and Soare, 1989, Continuity of Capping) For any c.e. degrees $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$, if $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$, then there exists a c.e. degree $\mathbf{c} > \mathbf{a}$ such that $\mathbf{c} \wedge \mathbf{b} = \mathbf{0}$.

Theorem 1.9 was extended by Seetapun [1991], giving a full solution to the dual of the major sub-degree problem (see Giorgi [2001]).

1.10 THEOREM. (Seetapun Continuity Theorem, 1991) For any c.e. degree $\mathbf{b} \neq \mathbf{0}, \mathbf{0}'$, there exists a c.e. degree $\mathbf{a} > \mathbf{b}$ such that for any c.e. degree $\mathbf{x}$, $\mathbf{b} \wedge \mathbf{x} = \mathbf{0}$ if and only if $\mathbf{a} \wedge \mathbf{x} = \mathbf{0}$.

A consequence of Seetapun's theorem is the continuity of Lachlan nonbounding degrees. A c.e. degree $\mathbf{a} \neq \mathbf{0}$ is called a *(Lachlan) nonbounding degree*, if it bounds no minimal pairs.

1.11 THEOREM. (i) (Lachlan [1979] Nonbounding Theorem) There exists a nonzero c.e. degree which bounds no minimal pairs.

(ii) (Cooper [1974b] Minimal Pair Theorem) Every high c.e. degree bounds a minimal pair.

(iii) (Seetapun, 1991) There is no maximal Lachlan nonbounding degree.

Also, a partial result relating to the major sub-degree problem was provided by Ambos-Spies, Lachlan and Soare [1993]:

1.12 THEOREM. (Continuity of Cupping to $\mathbf{0}'$) For any c.e. degrees $\mathbf{a}, \mathbf{b}$, if $\mathbf{0} < \mathbf{a} < \mathbf{0}'$ and $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'$, then there is a c.e. degree $\mathbf{c} < \mathbf{a}$ such that $\mathbf{c} \vee \mathbf{b} = \mathbf{0}'$.

The proof for this result provides a base for Seetapun's result in Theorem 1.18 below, and a naive module for solving the major sub-degree problem in the present paper, Theorem 1.19.

In contrast to the situation regarding Lachlan nonbounding degrees, there are a number of results concerning splitting and nonsplitting of c.e. degrees. First of all, extending the Sacks splitting theorem, Robinson [1971] showed:

1.13 THEOREM. (Robinson Low Splitting Theorem) For any c.e. degrees $\mathbf{l} < \mathbf{a}$, if $\mathbf{l}$ is low, then there exist c.e. degrees $\mathbf{b}, \mathbf{c}$ such that $\mathbf{l} \leq \mathbf{b} < \mathbf{a}$, $\mathbf{l} \leq \mathbf{c} < \mathbf{a}$ and such that $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$.

On the other hand, Lachlan [1975] showed that:

1.14 THEOREM. (Lachlan Nonsplitting Theorem) There exist c.e. degrees $\mathbf{a} < \mathbf{b}$ such that for any c.e. degrees $\mathbf{x}, \mathbf{y}$, if $\mathbf{a} \leq \mathbf{x} < \mathbf{b}$, $\mathbf{a} \leq \mathbf{y} < \mathbf{b}$, then $\mathbf{x} \vee \mathbf{y} \neq \mathbf{b}$.

This was extended by Harrington [1980]:

1.15 THEOREM. (Harrington Nonsplitting Theorem) There exists an incomplete c.e. degree $\mathbf{a}$ above which $\mathbf{0}'$ is not *splittable*, that is there are no c.e. degrees $\mathbf{x}$, $\mathbf{y}$ such that $\mathbf{a} < \mathbf{x} < \mathbf{0}'$, $\mathbf{a} < \mathbf{y} < \mathbf{0}'$, and $\mathbf{x} \vee \mathbf{y} = \mathbf{0}'$.

The c.e. degree $\mathbf{a} < \mathbf{0}'$ in Theorem 1.15 is called a *Harrington nonsplitting base*. Harrington showed (see Shore and Slaman [1990]):

1.16 THEOREM. (Harrington Splitting Theorem) For any c.e. degrees $\mathbf{a} < \mathbf{l}$, if $\mathbf{l}$ is low$_2$, then there exist c.e. degrees $\mathbf{x}, \mathbf{y}$ such that $\mathbf{a} \leq \mathbf{x} < \mathbf{l}$, $\mathbf{a} \leq \mathbf{y} < \mathbf{l}$ and $\mathbf{x} \vee \mathbf{y} = \mathbf{l}$.

Cooper and Li [2002] have shown:

1.17 THEOREM. (Cooper–Li Nonsplitting Theorem) There exists a low$_2$ c.e. degree above which $\mathbf{0}'$ is not splittable.

By using the techniques in the proofs of Theorems 1.12 and 1.16, Seetapun proved:

1.18 THEOREM. (Seetapun, 1991) Every nonzero low$_2$ c.e. degree has a major sub-degree.

Seetapun (1992, unpublished) also described a module to prove the following partial solution to the Major Sub-degree Problem:

For any c.e. degree $\mathbf{b} \neq \mathbf{0}, \mathbf{0}'$, there exists a c.e. degree $\mathbf{a} \not\geq \mathbf{b}$ such that for any c.e. degree $\mathbf{x}$, if $\mathbf{0}' = \mathbf{b} \vee \mathbf{x}$, then $\mathbf{0}' = \mathbf{a} \vee \mathbf{x}$.

Seetapun's notes do not constitute a full proof, there being no construction or verification. But we now know from our main theorem here, Theorem 1.19, that the partial result Seetapun tried to prove is correct.

Building on this earlier work, especially on the module developed for the proof of the continuity of cupping to $\mathbf{0}'$ by Ambos-Spies, Lachlan and Soare [1993], we show that:

1.19 THEOREM. (The Major Sub-degree Theorem) Every c.e. degree $\mathbf{b} \neq \mathbf{0}, \mathbf{0}'$ has a major sub-degree.

In fact, the construction given in the proof of the theorem is a uniform one, which hence provides a computable function f such that for every e, if $W_e \not\leq_{\mathrm{T}} \emptyset$, and $K \not\leq_{\mathrm{T}} W_e$, then $W_{f(e)} <_{\mathrm{T}} W_e$, and $\deg_{\mathrm{T}}(W_{f(e)})$ is a major sub-degree of $\deg_{\mathrm{T}}(W_e)$.

An immediate consequence of this main theorem is a continuity result for Harrington nonsplitting bases.

1.20 COROLLARY. There is no minimal Harrington nonsplitting base.

Proof. Suppose that $\mathbf{b}$ is a Harrington nonsplitting base. By the choice of $\mathbf{b}$ and the Sacks Splitting Theorem, $\mathbf{b} \neq \mathbf{0}, \mathbf{0}'$. By Theorem 1.19, let $\mathbf{a}$ be a major sub-degree of $\mathbf{b}$. Then, for any c.e. degrees $\mathbf{x_0}, \mathbf{x_1}$, if $\mathbf{x_0} \vee \mathbf{x_1} = \mathbf{0}'$ but $\mathbf{x_i} \vee \mathbf{a} \neq \mathbf{0}'$ for both $i = 0$ and 1, then $\mathbf{b} \vee \mathbf{x_i} \neq \mathbf{0}'$ holds for both $i = 0$ and 1. Let $\mathbf{y_i}$ be a c.e. degree such that

$\mathbf{b} \vee \mathbf{x_i} \leq \mathbf{y_i} < \mathbf{0}'$ for $i = 0, 1$. Since $\mathbf{x_0} \vee \mathbf{x_1} = \mathbf{0}'$, we have that $\mathbf{y_0} \vee \mathbf{y_1} = \mathbf{0}'$, but $\mathbf{b} \leq \mathbf{y_0}, \mathbf{y_1} < \mathbf{0}'$, $\mathbf{b}$ is not a Harrington nonsplitting base, contradicting the choice of $\mathbf{b}$. Therefore for any c.e. degrees $\mathbf{x_0}, \mathbf{x_1}$, if $\mathbf{x_0} \vee \mathbf{x_1} = \mathbf{0}'$, then $\mathbf{x_i} \vee \mathbf{a} = \mathbf{0}'$ for some $i \leq 1$, so $\mathbf{a}$ is a Harrington nonsplitting base. So $\mathbf{b}$ is not a minimal Harrington nonsplitting base, and the corollary follows. □

There are also consequences for lattice embeddings into $\boldsymbol{\mathcal{E}}$. For instance, the embeddability of the non-distributive lattice N_5, first proved by Lachlan [1972], is an immediate corollary.

We now prove our main result, Theorem 1.19. The paper is organised as follows. In section 2, we formulate the conditions of Theorem 1.19 into requirements and describe the basic modules for satisfying the requirements; in section 3, we describe the strategies to satisfy typical combinations of the requirements; in section 4, we introduce some rules and principles of the construction and the proofs in the paper; in section 5, we describe the general strategies to satisfy the requirements; in section 6, we build the priority tree of strategies; in section 7, we describe a full construction to build the objects we need; and finally, in section 8, we verify that the construction satisfies all of the requirements.

Our notation and terminology are standard and generally follow Cooper [2004] or (with regard to notation) Soare [1987]. During the course of a construction, notations such as A, Φ are used to denote the current approximations to these objects, and if we want to specify the values immediately at the end of stage s, then we denote them by A_s, $\Phi[s]$ etc. For a *partial computable* (p.c., or for simplicity, we also call Turing) functional, Φ say, the use function is denoted by the corresponding lower case letter φ. The value of the use function of a converging computation is the greatest number which is actually used in the computation. For a p.c. functional which is not built by us, if a computation is not defined on some argument, then we define the value of its use function on that argument to be -1. For a p.c. functional which is built by us, if a computation is not defined on some argument, then we regard the value of its use function on this argument to be ω. During the course of a construction, whenever we define a parameter, p say, as *fresh*, we mean that p is defined to be the least natural number which is greater than any number mentioned so far, in particular, if p is defined afresh at stage s, then $p > s$. In the description of the proofs throughout the paper, when we call $p[s]$ *unbounded*, we mean that the parameter p will be unbounded during the course of the construction. In the description of the strategies and construction, we say that a node α *is below* (or *above)* β, if $\beta \subset \alpha$ (or $\alpha \subset \beta$), and that α *is to the left* (or *to the right) of* β, if $\alpha <_{\mathrm{L}} \beta$ (or $\beta <_{\mathrm{L}} \alpha$).

2 Requirements and Basic Modules

In this section, we describe the requirements to prove Theorem 1.19 and the basic modules to satisfy the requirements.

2.1 The Requirements

To prove Theorem 1.19, given a c.e. set B, we construct c.e. sets A, D to satisfy the following requirements:

$$\begin{aligned} \mathcal{T}: &\quad A \leq_{\mathrm{T}} B \\ \mathcal{R}_e: &\quad D = \Phi_e(B, X_e) \rightarrow \ B \leq_{\mathrm{T}} X_e \oplus A \\ \mathcal{S}_e: &\quad B = \Theta_e(A) \rightarrow [B \leq_{\mathrm{T}} \emptyset \text{ or } K \leq_{\mathrm{T}} B] \end{aligned}$$

where $e \in \omega$, $\{(\Phi_e, X_e, \Theta_e) \mid e \in \omega\}$ is a standard enumeration of all triples (Φ, X, Θ) with both Φ and Θ Turing functionals, and with X a c.e. set. K is a fixed creative set.

Suppose that $B \not\leq_{\mathrm{T}} \emptyset$, and that $K \not\leq_{\mathrm{T}} B$. By the $\mathcal{T}$-requirement and the $\mathcal{S}$-requirements, $A <_{\mathrm{T}} B$. Given a c.e. set X, if $K \leq_{\mathrm{T}} B \oplus X$, then there is an e such that $X = X_e$ and such that $D = \Phi_e(B, X_e)$. By $\mathcal{R}_e$, $B \leq_{\mathrm{T}} X_e \oplus A = X \oplus A$. By the assumption of $K \leq_{\mathrm{T}} B \oplus X$, $K \leq_{\mathrm{T}} X \oplus A$. Therefore for any c.e. set X, if $K \leq_{\mathrm{T}} B \oplus X$, then $K \leq_{\mathrm{T}} X \oplus A$. So the requirements are sufficient to prove the theorem.

We assume that the use function for a given Turing functional, Φ or Θ, say, is increasing in arguments and will dominate the identity function at the stages at which it is defined. We also assume that for any $s_1 < s_2$, if $\Phi(B, X; x)[s_1]$ and $\Phi(B, X; x)[s_2]$ are both defined, then $\varphi(B, X; x)[s_1] \leq \varphi(B, X; x)[s_2]$ even if there exists a stage $s_3 \in (s_1, s_2)$ at which $\Phi(B, X; x)$ is undefined. The same is true for Θ.

2.2 The $\mathcal{T}$-Strategy

The $\mathcal{T}$-strategy is the usual *permitting method*. We ensure that for any x, s, if x is enumerated into $A_{s+1} - A_s$, then there is an element $y \leq x$ such that $y \in B_{s+1} - B_s$. This ensures that $A \leq_{\mathrm{T}} B$.

We will see that all $\mathcal{R}$-strategies will build Turing functionals Γ with use functions γ. Every element of A is a γ-marker $\gamma_\tau(y)$ for some $\mathcal{R}$-strategy τ, and some y. During the course of the construction, for a γ-marker $x = \gamma_\tau(y)$ for some τ and some y, we initially define the *permitting marker* $m(x)$ of x (at the stage at which x is defined as a γ-marker) to be strictly less than x, and we may update the *permitting marker* $m(x)$ *of* x by redefining it to be a number less than the previous value. We ensure that:

(1) If x is a γ-marker of some Turing functional Γ, then $m(x)$ is defined.

(2) At all stages, $m(x) < x$.

(3) If x is enumerated into A at stage s, then there is an element $b \leq m(x)$ which is enumerated into B at the same stage s.

Clearly, the three properties above ensure that A is computable in B.

2.3 An $\mathcal{R}$-Module

Given an $\mathcal{R}$-requirement, $\mathcal{R}$ say (we drop the index), we define the length of agreement function $l = l(D, \Phi(B, X))$ as usual. We say that s is $\mathcal{R}$-*expansionary*, if

$l[s] > l[v]$ for all $v < s$. If there are only finitely many $\mathcal{R}$-expansionary stages, then $\mathcal{R}$ is satisfied.

Suppose that there are infinitely many $\mathcal{R}$-expansionary stages. In this case, we will build a Turing functional Γ to show that $\Gamma(X, A) = B$ unless $\Phi(B, X)$ is partial. Γ will be built via cycles $k \geq 0$. For all k, cycle $k+1$ begins when $\Gamma(k)$ has already been defined.

Cycle $k \geq 0$

1. Define an *agitator* $d(k)$ afresh.

   [*Remark.* Notice that $d(k) > k$.]

2. (Defining Γ) Wait for a stage v, say, at which $\Phi(B, X; d(k)) \downarrow = 0 = D(d(k))$. Then define $\Gamma(X, A; k) \downarrow = B(k)$ with $\gamma(k)$ fresh.

3. Wait for a stage $s > v$ at which one of the following occurs:

   Case 3a. $B_v \upharpoonright (\varphi(d(k))[v] + 1) \neq B_s \upharpoonright (\varphi(d(k))[v] + 1)$, then enumerate $\gamma(k)$ into A, and cancel $d(k')$ for all $k' > k$. We say that $d(k)$ *receives honestification at stage s via B-change*, and go back to step 2.

   Case 3b. $X_v \upharpoonright (\varphi(d(k))[v] + 1) \neq X_s \upharpoonright (\varphi(d(k))[v] + 1)$, then set $\Gamma(X, A; k)$ to be undefined, and cancel $d(k')$ for all $k' > k$. We say that $d(k)$ *receives honestification at stage s* via *X-change*, and go back to step 2.

In addition to the above steps, whenever X changes both below $\gamma(k)$ and strictly above $\varphi(d(k))$ before step 3 occurs, then the strategy enumerates new axioms with the same value and the same use functions automatically.

We note that the $\mathcal{R}$-module will never enumerate any agitator $d(k)$ into D. We say that the agitator $d(k)$ is *honest* if either $\gamma(k)$ is undefined, or $B \upharpoonright (\varphi(d(k)) + 1)$ has not changed since $\gamma(k)$ was last created, and *dishonest*, otherwise. The use rules of Φ will ensure that if $d(k)$ is honest, then both B and X have not changed below $\varphi(d(k))$ since the γ-marker $\gamma(k)$ was last created.

Notice that $d(k) < d(k+1)$ holds for all $k \geq 0$. The key to the satisfaction of $\mathcal{R}$ is that every agitator $d(k)$ of the $\mathcal{R}$-module is honest, as we have ensured in step 3 of the module.

During the actual construction, the actions in case 3b of step 3 will be executed automatically as a consequence of the framework, so that no explicit instructions for this case are required. That is to say, whenever $X \upharpoonright (\varphi(d(k) + 1)$ changes since $\gamma(k)$ was last created, $\Gamma(k)$ is set to be undefined, and $d(k')$ are cancelled for all $k' > k$.

The Possible Outcomes

Recall that we are presently working under the assumption that there are infinitely many $\mathcal{R}$-expansionary stages. The possible outcomes of a single cycle k are as follows:

Case 1. There are infinitely many stages at which step 3 occurs.

Let k be the least such cycle. Then $d(k)[s]$ comes to a limit, but $\varphi(d(k))[s]$ will be unbounded over the course of the construction. So $\Phi(B, X)$ is partial, and $\mathcal{R}$ is satisfied.

Case 2. Otherwise. Then $\Gamma(X, A; k) \downarrow= B(k)$.

Therefore if $\Phi(B, X)$ is total and $D = \Phi(B, X)$, then for every k we have $\Gamma(X, A; k) \downarrow= B(k)$. So $B \leq_{\mathrm{T}} X \oplus A$, and $\mathcal{R}$ is again satisfied.

We define the *possible outcomes of an $\mathcal{R}$-strategy* to be $0 <_{\mathrm{L}} 1$, denoting infinite and finite actions respectively. That is to say, outcome 1 means that there are only finitely many $\mathcal{R}$-expansionary stages, and that 0 means that there are infinitely many $\mathcal{R}$-expansionary stages.

We note that the rectification of $\Gamma(X, A; k) \downarrow\neq B(k)$ is guaranteed automatically by the honestification in case 3a of step 3 of the module, due to the fact that $k \leq d(k) \leq \varphi(d(k))$, for which the second inequality follows from the convention regarding φ prescribed in subsection 2.1. That is to say, if B changes at k, then B changes below $\varphi(d(k))$, and case 3a of step 3 has already enumerated $\gamma(k)$ into A.

The actual construction will be divided into odd and even stages. At odd stages, B and A are enumerated outside of the tree, and at even stages, the tree construction will be carried out. We will make sure that the true path of the construction will be determined by the construction in even stages.

2.4 An $\mathcal{S}$-Module

Given an $\mathcal{S}$-requirement, $\mathcal{S}$ say (we drop the index), we define the length of agreement function $l = l(B, \Theta(A))$ as usual. We say that s is *$\mathcal{S}$-expansionary*, if $l[s] > l[v]$ for all $v < s$. If there are only finitely many $\mathcal{S}$-expansionary stages, then $B \neq \Theta(A)$, and $\mathcal{S}$ is trivially satisfied. Suppose that there are infinitely many $\mathcal{S}$-expansionary stages. Then the $\mathcal{S}$-module will build a computable function f with $f = B$. f is built as follows:

1. Let y be the least x such that $f(x) \uparrow$.
2. Wait for a stage at which $\Theta(A; y) \downarrow= B(y)$. Then define $f(y) = B(y)$, and restrain $A \upharpoonright (\theta(y) + 1)$.

The key to the satisfaction of the requirement is to ensure that the A-restraint of the module is never injured. Then $\mathcal{S}$ will be satisfied via one of the following cases:

Case 1. f is built infinitely often. In this case, f is total and $f = B$.

Case 2. Otherwise. Then $l(B, \Theta(A))$ is bounded over the course of the construction, giving $B \neq \Theta(A)$ and the satisfaction of $\mathcal{S}$.

We use $-1 <_{\mathrm{L}} 2$ to denote the possible outcomes of the $\mathcal{S}$-module, where -1 means that f is built infinitely often, and 2 means that $l(B, \Theta(A))$ is bounded over the course of the construction. (Notice that we have used 0 and 1 to denote possible outcomes of $\mathcal{R}$-strategies, so we use -1 and 2 to denote the possible outcomes of an $\mathcal{S}$-strategy.)

Using the above modules, we now describe the strategies to satisfy the $\mathcal{S}$-requirements priority below one $\mathcal{R}$-strategy.

2.5 Satisfying $(\mathcal{R}, \mathcal{S})$

Our $\mathcal{S}$-strategy is a uniform version of the module in the proof of the continuity theorem of Ambos-Spies, Lachlan and Soare [1993]. To better understand the $\mathcal{S}$-strategy, we first outline the basic idea of the ALS strategy.

The ALS Strategy. Suppose that $\Phi(B, X)$ is total, that $D = \Phi(B, X)$, and that we want to satisfy the following requirements: $\mathcal{R}$, $\mathcal{S}$.

The $\mathcal{R}$-strategy is building a Turing functional Γ such that $\Gamma(X, A)$ is total, and $B = \Gamma(X, A)$. An $\mathcal{S}$-strategy will satisfy its $\mathcal{S}$-requirement, $B \neq \Theta(A)$, while its priority ranking is given to the building of Γ.

The $\mathcal{S}$-strategy assumes that the $\mathcal{R}$-strategy works as prescribed in subsection 2.3. It will work with a fixed *base point* n, say, and will build a Turing functional Δ, and a computable partial function f.

$\Delta(B)$ and f will be defined as follows: For each $k > n$, the base point of the $\mathcal{S}$-strategy, we define an *agitator* $d(k)$ of the $\mathcal{S}$-strategy to be different from and greater than the agitator $d(k)$ chosen by the $\mathcal{R}$-strategy. When the $\mathcal{R}$-strategy defines $\Gamma(k)$, we require that the length function $l(\Phi(B, X), D)$ is above not only the agitator $d(k)$ of the $\mathcal{R}$-strategy itself, but the new agitator $d(k)$ of the $\mathcal{S}$-strategy. The request to enumerate $\gamma(k)$ into A may be issued by either the $\mathcal{R}$- or the $\mathcal{S}$-strategy. If B changes below the φ-use at the agitator $d(k)$ of the $\mathcal{R}$-strategy, then the $\mathcal{R}$-strategy requests the enumeration of $\gamma(k)$ into A. If the B-change is below the φ-use of the agitator of the $\mathcal{S}$-strategy, then the $\mathcal{S}$-strategy requests the enumeration of $\gamma(k)$ into A. Having chosen agitator $d(k)$, the $\mathcal{S}$-strategy then waits for a stage v, say, at which both $\Phi(B, X; d(k)) \downarrow = 0 = D(d(k))$, and $l(\Theta(A), B) > \varphi(d(k))$ hold, then we define $\Delta(B; k) \downarrow = K(k)$ with $\delta(k) = \theta\varphi(d(k))$.

Suppose that at a stage $s > v$, $\Delta(B; k) \downarrow \neq K(k)$ occurs. We rectify $\Delta(B; k)$ as follows, performing no further actions for any $k' \neq k$ until this process of the rectification of $\Delta(B; k)$ is completed.

Case 1. If there is no $y > n$ such that $\gamma(y) \leq \delta(k)$, then we notice that $\Theta(A)[v] \upharpoonright (\varphi(d(k))[v] + 1)$ has been preserved since stage v, that no permission has occurred for γ-markers less than or equal to $\theta\varphi(d(k))[v]$, and that it will be preserved forever, unless $\gamma(b)$ is enumerated into A for some $b \leq n$, the base point of the $\mathcal{S}$-strategy. Therefore we can define $f = B$ on the initial segment $\varphi(d(k))[v]$, in which case, we consider the procedure of the rectification of $\Delta(B; k) \neq K(k)$ finished. In fact, in this case, we are allowed to set Δ to be totally undefined.

Case 2. If there is a $y > n$ such that $\gamma(y) \leq \delta(k)$, then let m be the greatest such y, enumerate $d(m)$ into D, and wait for the next $\mathcal{R}$-expansionary stage:

Speeding up the enumeration of B, and X: Wait for the next stage $t > s$ at which $l(\Phi(B, X), D) > d(m)$, then we consider the following cases:

Case 1. There is an element $x \leq \varphi(d(m))[v]$ which has been enumerated into B since stage s. Let b be the least such x. There are two subcases:

Subcase 1a. If $m \leq k$, and $\gamma(m-1) < b \leq \varphi(d(m))[v]$, and b is above the φ-use at the agitator $d(m)$ defined by the $\mathcal{R}$-strategy, then define a *conditional restraint* (or *restraint vector*) by $\vec{r} = (p, \delta(k)[v])$, where p is the maximum of $\gamma(m-1)$ and the φ-use at $d(m)$ which is defined by the $\mathcal{R}$-strategy, and enumerate $\gamma(m+1)$ into A. Intuitively speaking, we give up the honestification of the agitator $d(m)$ chosen by the $\mathcal{S}$-strategy.

In this case, we are able to preserve a disagreement $\Theta(A; b) = 0 \neq 1 = B(b)$, as will be seen below, so the $\mathcal{S}$-strategy takes no further actions for any k', so long as this disagreement holds.

[The conditional restraint $\vec{r} = (p, q)$ ensures that a number $x \leq q$ can be enumerated into A only if B changes below p.]

Subcase 1b. Otherwise, then enumerate $\gamma(m)$ into A. In fact, in this case, there is no further agitator that is delayed for honestification, so that for any x, if the current B-change b is below $\varphi(d(x))$, then $\gamma(x)$ is enumerated into A.

In either subcase, $\Delta(B; k)$ is set to be undefined.

Case 2. Otherwise, then X must have changed below $\varphi(d(m))$, so $\Gamma(m)$ is automatically undefined, and we continue the procedure of the rectification of $\Delta(B; k)$ exactly the same as above by considering the greatest y with $\gamma(y) \leq \delta(k)$, which must be some $m' < m$, if any.

The crucial point in the module above is that when we get a response in subcase 1a after $d(m)$ is enumerated into D, we enumerate $\gamma(m+1)$, instead of $\gamma(m)$, into A. By the choice of b, we have that $m - 1 < \gamma(m-1) < b$, so $b \geq m+1$, the enumeration of $\gamma(m+1)$ is sufficient to rectify $\Gamma(X, A)$. The conditional restraint $\vec{r} = (p, \delta(k)[v])$ ensures that if B will not change below p, then $\gamma(m)$ is not allowed to be enumerated into A, and then $\Theta(A; b) \downarrow = 0 \neq 1 = B(b)$ will be preserved forever. Notice that $p = \max\{\gamma(m-1), \varphi(d(m))\}$, where $d(m)$ is the agitator defined by the $\mathcal{R}$-strategy. So if B does not change below p, then no lower priority $\mathcal{S}$-strategy can enumerate $\gamma(m')$ for any $m' \leq m$ into A. On the other hand, for any $m' > m$, $\gamma(m') > \delta(k) \geq \theta(b)$, preserving $\Theta(A; b) = 0 \neq 1 = B(b)$ forever.

Suppose that $\gamma(\leq n)[s]$ are bounded. Then if f is built infinitely often , then f is a computable function, and $f =^* B$. Otherwise, let s^- be minimal after which $\gamma(y)[s]$ has reached its limit for each $y \leq n$.

By the module above, every inequality $\Delta(B; k) \neq K(k)$ will eventually be rectified, or the current Δ will be set to be totally undefined. Therefore for a fixed m, there is a stage $s_0 > s^-$, such that for any $s > s_0$ and any k, if $\Delta(B; k) \neq K(k)$ holds at stage s, and $d(m)$ is enumerated into D at stage s, then $m \leq k$.

For a fixed m, let s_1 be a stage greater than s_0 (taken as above) such that $\gamma(b)$ will never be enumerated into A after stage s_1 for any $b \leq m-1$ and such that B will never change below $\varphi(d(m))$ for the $\mathcal{R}$-strategy's agitator $d(m)$. Then if the agitator $d(m)$ of the $\mathcal{S}$-strategy is enumerated at a stage $s > s_1$, then at the next $\mathcal{R}$-expansionary stage $t > s$, subcase 1a occurs (otherwise, either $d(m')$ is enumerated for some $m' < m$, or f is built further, absurd), and we have created and preserved an inequality $\Theta(A; b) \downarrow= 0 \neq 1 = B(b)$ forever.

This argument shows that for a fixed m, the $\mathcal{S}$-strategy enumerates $d(m)$ only finitely often, and it is only strategies with base point less than m that can enumerate $d(m)$. The enumeration of $d(m)$ by the collection of all strategies therefore occurs only finitely often . So $\Gamma(X, A)$ will be total. On the other hand, the conditional restraints of the $\mathcal{S}$-strategies will never make Γ incorrect. Therefore, $\Gamma(X, A) = B$ holds.

By the building and rectification of Δ, if Δ is built infinitely often (notice that B is not computable), then either $\Delta(B)$ is total, in which case, $\Delta(B) = K$, or there is a y such that $\Delta(B; y)$ diverges. In which case, letting m be the least such y, we have that both $d(m)[s]$ and $\varphi(d(m))[s]$ are bounded, but that $\theta\varphi(d(m))[s]$ will be unbounded over the construction, so $\Theta(A) \neq B$, $\mathcal{S}$ is satisfied.

This completes the description of the ALS strategy.

We now build our $\mathcal{S}$-strategy based on the ideas above. Our $\mathcal{S}$-strategy is a localised, and then uniform version of the module above. The differences between our $\mathcal{S}$-strategy and the ALS strategy include: (i) $\Phi(B, X)$ may be partial, and (ii) after an enumeration of some agitator $d(m)$ into D for some m, we must take actions before the next $\mathcal{R}$-expansionary stage, if B changes below the corresponding φ-use. With this in mind, let us develop our $\mathcal{S}$-strategy. It is the first non-trivial $\mathcal{S}$-strategy, with priority ranking given to one $\mathcal{R}$-strategy, and dealing with the injury from the collection of all lower priority $\mathcal{R}$- and $\mathcal{S}$-strategies.

Our $\mathcal{S}$-Strategy: Suppose that τ and α are $\mathcal{R}$- and $\mathcal{S}$-strategies respectively. Let $\tau\hat{\ }\langle 0\rangle \subseteq \alpha$. α will satisfy its $\mathcal{S}$-requirement, $\Theta(A) \neq B$ say, while its priority ranking is given to the $\mathcal{R}$-strategy τ which is building a Turing functional Γ to show that $\Gamma(X, A) = B$, if $\Phi(B, X) = D$. So α will have to deal with the injury arising from the building of Γ. It will work with a fixed *base point* n say. Whenever we define the base point n, we define it afresh. If α is initialised, then the base point n is cancelled.

For every j, τ will define an *agitator* $d_\tau^\tau(j)$ for $\Gamma(j)$. For every $j > n$, the base point of α, α will define an *agitator* $d_\tau^\alpha(j)$ for $\Gamma(j)$. Note that we define $d_\tau^\alpha(j)$ at a stage only if $\Gamma(j)$ is currently undefined, and that whenever we build d_τ^α, we always choose j to be the least $k > n$ for which $d_\tau^\alpha(k)$ is undefined, and define $d_\tau^\alpha(j)$.

Given a strategy β and a number j, if $d_\tau^\beta(j)$ is defined, then we say that $d_\tau^\beta(j)$ is *honest* if and only if either $\gamma(j)$ is undefined, or $B \upharpoonright (\varphi(d_\tau^\beta(j)) + 1)$ has not changed since $\gamma_\tau(j)$ was last created. If $d_\tau^\beta(j)$ is not honest, then we say that it is *dishonest*.

The basic constraint for a node β is that if β is visited at a stage s, and s is β-expansionary, then for every j, if $d_\tau^\beta(j)$ is defined, then it is honest at stage s. This basic constraint will never be violated. However, one of the main points is that we can allow some agitators, $d_\tau^\beta(j)$ say, to be dishonest if we can ensure that either the honestification of $d_\tau^\beta(j)$ is not needed or there is a number, b say, which is less than the permitting marker of $\gamma(j)$ and which will be enumerated into B. In which case, we may allow ourselves to enumerate $\gamma(j)$ into A, given that we can then rehonestify such agitators $d_\tau^\beta(j)$.

In addition, we note that the definition of base points has ensured that for a fixed k, there are only finitely many strategies β which define agitators $d^{\beta}_{\tau}(k)$ during the construction. In so doing, if there is a fixed β such that $d^{\beta}_{\tau}(k)[s]$ becomes unbounded, then β ensures that $\Phi(B, X)$ is partial. Otherwise, let d be the maximal $d^{\beta}_{\tau}(k)$ for all β. Then d comes to a limit, in which case we have that either $\varphi(d)$ becomes unbounded, or $\gamma(k)$ is bounded over the course of the construction.

Intuitively speaking, the $\mathcal{S}$-strategy will build Δ in the same way as above. Whenever an inequality $\Delta(B; k) \neq K(k)$ occurs, we start a cycle of rectification of $\Delta(B; k)$ by enumerating some agitator $d^{\alpha}_{\tau}(m)$ into D. During the course of the rectification, $\Delta(B; k) = 0$ is reserved, while $\Theta(A) \upharpoonright (\varphi(d^{\alpha}_{\tau}(k)) + 1)$ have been preserved since the stage at which $\delta(k)$ was created. This will be realised by defining a sequence of conditional restraints $\vec{r}(\alpha)[s] = (p(\alpha)[s], q(\alpha)[s])$, such that $p(\alpha)[s]$ is decreasing, until B changes below the most recent $p(\alpha)$, or there is no agitator $d^{\alpha}_{\tau}(y)$ available. Let v be the stage at which $\Delta(B; k)$ was created. If we reach a stage at which $\Delta(B; k) \neq K(k)$ is reserved, but there is no agitator $d^{\alpha}_{\tau}(y)$ available, then we have that $\Theta(A)[v] \upharpoonright (\varphi(d^{\alpha}_{\tau}(k))[v] + 1)$ have been preserved since stage v by the sequence of conditional restraints, so that we are going to preserve $\Theta(A)[v] \upharpoonright (\varphi(d^{\alpha}_{\tau}(k))[v] + 1)$ unless B will change below the base marker of α which should be bounded, unless $\Phi(B, X)$ is partial. If we see a B-change below the most recent $p(\alpha)$, the first coordinate of α's conditional restraint, then all agitators which have become dishonest during the procedure of the rectification of α can be rehonestified by enumerating their corresponding γ-markers into A, since all lost permissions are repaid by the B-change below the most recent $p(\alpha)$.

To guarantee that the collection of $\mathcal{S}$-strategies of priority below the $\mathcal{R}$-strategy will not make $\Gamma(X, A)$ partial (unless $\Phi(B, X)$ is partial), we will create and preserve an inequality $\Theta(A; b) \neq B(b)$ for some b, if the response of an agitation is a B-change which is below $\varphi(d^{\alpha}_{\tau}(k))$ and above the maximal φ-use of agitators of priority higher than the one we just enumerated into D. Using this, we can prove by induction that if there is an m such that $d^{\alpha}_{\tau}(m)[s]$ becomes unbounded, then $\Phi(B, X)$ is partial. We now look at the instructions for the $\mathcal{S}$-strategy.

The $\mathcal{S}$-Strategy α: We first analyse the new features of our strategy.

The Base Markers: We define the *base marker of* α by:

$$bm = \varphi(d^{<\alpha}_{\tau}(\leq n))[s]$$

where n is the base point of α.

The main body of α assumes that $\gamma(x)[s]$ is bounded for all $x \leq n$, and α assumes that $\gamma(\leq n)$ will be determined by $\varphi(d^{<\alpha}_{\tau}(\leq n))[s]$. For this reason, we call $\varphi(d^{<\alpha}_{\tau}(\leq n))[s]$ the *base marker of* α. By using the base markers, we allow the collection of all $\mathcal{S}$-strategies of priority below the $\mathcal{R}$-strategy to check whether or not $\Phi(B, X)$ is total, a Π_3-proposition. Therefore α will decide whether or not there are $\beta < \alpha$ and $k \leq n$ such that $\varphi(d^{\beta}_{\tau}(k))[s]$ is unbounded during the course of the construction. We use b to denote the outcome that $\varphi(d^{<\alpha}_{\tau}(\leq n))[s]$ is unbounded as s goes to infinity. We will ensure that if there is a $\beta < \alpha$, and a fixed $y \le n$ such that $\varphi(d^{\beta}_{\tau}(y))[s]$ is unbounded over the construction, then $\Phi(B, X)$ is partial. Therefore τ is no longer active at $\xi \supset \alpha\hat{}\langle b\rangle$, although there are still $\mathcal{R}$-strategies below $\alpha\hat{}\langle b\rangle$.

Of course $\alpha\hat{}\langle b\rangle$ must be expanded to approximate the least $y \le n$ such that $\gamma_{\tau}(y)[s]$ is unbounded. This will provide a *well defined environment* in the sense of believable computations, for lower priority strategies. So we have:

The Rough Outcomes of α: We then define the *rough possible outcomes of* α by

$$b <_{\mathrm{L}} -1 <_{\mathrm{L}} \omega <_{\mathrm{L}} 2$$

where b means that there are $\beta < \alpha$, $k \le n$ such that $\varphi(d^{\beta}_{\tau}(k))[s]$ is unbounded as s goes to infinity; -1 means that there is a computable function f such that $f = B$ (there is no strategy at all below this outcome); 2 means that $l(\Theta(A), B)[s]$ is bounded during the course of the construction; and ω denotes the 'otherwise' case.

If b is the true outcome of α, we can prove that $\Phi(B, X)$ is partial. The reason we call the outcomes defined above rough outcomes, is that later we may refine the outcomes by expanding them into subtrees to find out exactly what progress the strategy makes.

The main body of α assumes that the base marker of α will be bounded during the construction. However this hypothesis may actually fail to hold. Therefore, α will have to deal with possible injury from $\gamma_{\tau'}$-markers for $\mathcal{R}$-strategies $\tau' \supset \alpha\hat{}\langle b\rangle$ due to the fact that we cannot wait until these $\mathcal{R}$-strategies are visited to enumerate their γ-markers into A once B-permissions have been given, and will build a Turing functional $\Delta(B)$ and a computable partial function f.

Before describing the building of Δ, we introduce some properties of the agitators which will be ensured by defining and enumerating agitators:

(i) For $\beta = \tau$ or any $\mathcal{S}$-strategy, we have that for any m, $d^{\beta}_{\tau}(m) < d^{\beta}_{\tau}(m+1)$.

(ii) For any m, and any $\alpha \subset \beta$, $d^{\alpha}_{\tau}(m) < d^{\beta}_{\tau}(m)$.

We ensure that Δ will be built only at even α-expansionary stages. The process of rectification of a $\Delta(B; k) \ne K(k)$ will be started from an even α-expansionary stage, and will be carried out at either odd stages when B changes below the corresponding φ-use, or at the next even α-expansionary stage at which we begin the next cycle of rectification of $\Delta(k)$ for the same k.

Building $\Delta(B; k)$: Wait for a stage at which:

(i) (*Well Ordering*) Let $h^{\alpha}(k) = \max\{\varphi(d^{\le\alpha}_{\tau}(\le k)),\ \varphi(d^{\alpha}_{\tau}(< k)), k\}$. Then $h^{\alpha}(k) < \varphi(d^{\alpha}_{\tau}(k))$,

(ii) $l(\Theta(A), B) > \varphi(d^{\alpha}_{\tau}(k))$. Then:

– Define $\Delta(B; k) \downarrow = K(k)$, with $\delta(k) = \theta\varphi(d^{\alpha}_{\tau}(k))$, and define a *valid use* $\delta^*(k)$ of $\Delta(B; k)$ to be $\delta(k)$.

The valid use $\delta^*(k)$ may be updated later in the construction. In the actual construction, we ensure that $\Delta(B; k)$ becomes undefined if and only if B changes below the valid use $\delta^*(k)$, instead of the value $\delta(k)$. If B changes both above the valid use and below the original $\delta(k)$, then the computation is kept by enumerating some new axiom for $\Delta(B;k)$ automatically.

Suppose that an inequality $\Delta(B;k)\downarrow= 0 \neq 1 = K(k)$ occurs. We will rectify $\Delta(B;k)$ as follows:

Rectifying $\Delta(B;k)\downarrow= 0 \neq 1 = K(k)$.

Let v be the stage at which $\Delta(B;k)$ was created.

During the procedure of the rectification of $\Delta(B;k)$, we have that $\Theta(A)[v] \upharpoonright (\varphi(d^\alpha_\tau(k))[v]+1)$ has been preserved since stage v. That is to say, if $\Delta(B;k)$ is defined, then A has never changed below $\theta\varphi(d^\alpha_\tau(k))[v]$ since stage v.

We consider two cases:

Case 1. There is no $y > n$ such that $\gamma(y) \leq \delta(k)$.

In this case, we know that $\Theta(A) \upharpoonright (\varphi(d^\alpha_\tau(k))[v]+1)$ will not be injured by the γ-uses unless B changes below the base marker of α observed at stage v, i.e., $\varphi(d^{<\alpha}_\tau(\leq n))[v]$, which the main body of α assumes to be bounded during the construction. So we can define $f = B$ on the initial segment $\varphi(d^\alpha_\tau(k))[v]$.

However a new problem, not encountered in the naive module, is that the computations $\Theta(A) \upharpoonright (\varphi(d^\alpha_\tau(k))[v]+1)$ may be injured by $\gamma_{\tau'}(y')$ for some $\tau' \supset \alpha\hat{}\langle b\rangle$ and for some y'. (Notice that now all nodes to the right of $\alpha\hat{}\langle -1\rangle$ are initialised.) Let b^α be the maximal $\gamma_{\tau'}(y')[v]$, where v is the stage at which $\Delta(B;k)$ was created. To solve this problem, we ensure that if there is a b such that $\varphi(d^{<\alpha}_\tau(\leq n))[v] < b \leq b^\alpha[v]$, and b is enumerated into B, then we can create an inequality $\Theta(A;b)\downarrow= 0 \neq 1 = B(b)$, and preserve the computation $\Theta(A;b)\downarrow= 0$ by defining a conditional restraint $\vec{r}(\alpha) = (\varphi(d^{<\alpha}_\tau(\leq n))[v], \delta(k)[v])$.

Of course if the B-change b is below $\varphi(d^{<\alpha}_\tau(\leq n))[v]$, then $\alpha\hat{}\langle b\rangle$ will be visited in a future stage, and if a B-change b is above $b^\alpha[v]$, then no permission has occurred for any $\gamma_{\tau'}(y')[v]$ for any $\tau' \supset \alpha\hat{}\langle b\rangle$, and any y'.

To realise this goal, we require that at stage v, we have that $b^\alpha[v] < \varphi(d^\alpha_\tau(k))[v]$. This means that whenever we define $\Delta(B;k)$, we need the following *well ordering condition*:

(iii) $$\boxed{b^\alpha < \varphi(d^\alpha_\tau(k))}$$

In this way, we are able to deal with the possible injury of the main body of α from honestification of agitators associated with nodes of priority below $\alpha\hat{}\langle b\rangle$.

So in Case 1, we implement the following:

– For any $x \leq \varphi(d^\alpha_\tau(k))[v]$, if $f(x)$ is not defined, then define $f(x) = B(x)$,

– Let $p_*(\alpha) = \max\{b^\alpha[v],\ p_*(\alpha)[s-1] \mid p_*(\alpha)[s-1]\downarrow\}$, $q_*(\alpha) = \delta(k)[v]$, and $g_*(\alpha) = \varphi(d^{<\alpha}_\tau(\leq n))[v]$,

– Initialise all nodes to the right of $\alpha\hat{}\langle -1\rangle$, and

– We say that the rectification of $\Delta(k) \neq K(k)$ is finished, and we now continue to define or rectify Δ at other arguments. Meanwhile we are waiting for a stage $t > s$ at which B changes below $p_*(\alpha)$.

[*Remark*. The definition of $p_*(\alpha)$ ensures that after B has been stable on the base marker of α, then the $p_*(\alpha)$ will be $\leq$-increasing in stages. This ensures that for any

x, if $\Theta(A;x)$ is going to be injured, then we can create and preserve an inequality $\Theta(A;x) \neq B(x)$. (see subcase 1a below)]

If $t > s$ is the least stage at which some $b \leq p_*(\alpha)$ enters B, then:

Subcase 1a. $g_*(\alpha) < b \leq p_*(\alpha)$. Then:
– Set $p(\alpha) \leftarrow g_*(\alpha)$, meaning define $p(\alpha)$ to be the value of $g_*(\alpha)$,
– Set $q(\alpha) \leftarrow q_*(\alpha)$, and
– Create a conditional restraint $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$, and
– Set $g_*(\alpha)$, $p_*(\alpha)$, and $q_*(\alpha)$ to be undefined, if they are defined.
In this case, we say that α *receives special attention*.

Subcase 1b. Otherwise, then
– Set $g_*(\alpha)$, $p_*(\alpha)$, $q_*(\alpha)$, $\vec{r}(\alpha)$, $g(\alpha)$ and f to be undefined, if they are defined. [To set $\vec{r}(\alpha)$ undefined means that both $p(\alpha)$ and $q(\alpha)$ are cancelled.]

In this case, all strategies to the right of $\alpha\hat{\ }\langle b\rangle$ are initialised. We regard f, $g_*(\alpha)$, $p_*(\alpha)$, and $q_*(\alpha)$ as being located at $\alpha\hat{\ }\langle -1\rangle$.

Case 2. There is a $y > n$ such that $\gamma(y) \leq \delta(k)$.

Let s be the current stage. The choice algorithm will pick the next agitator to be used as follows.

THE CHOICE ALGORITHM

1. Let m be the greatest y such that $\gamma(y) \leq \delta(k)[v]$ holds during stage s.
2. Output $d^\alpha_\tau(m)$.

We prove the following:

2.1 LEMMA (*The Choice Lemma*). Let $d^\alpha_\tau(m)$ be the agitator defined by the choice algorithm. Then:

(i) For any y, if $n < y$, $d^\alpha_\tau(y)$ is defined, and $\gamma(y) \leq \delta(k)[v]$ holds during stage s, then

$$h^\alpha(y)[v] \leq h^\alpha(m)[v]$$

where $h^\alpha(x) = \max\{\varphi(d^{\leq\alpha}_\tau(\leq x)),\ \varphi(d^\alpha_\tau(< x)), x\}$.

(ii) If $m \leq k$, then for any y, if $n < y$, $d^\alpha_\tau(y)$ is defined, and $\gamma(y) \leq \delta(k)[v]$ holds during stage s, then:

$$\varphi(d^{\leq\alpha}_\tau(\leq y))[v] \leq \varphi(d^\alpha_\tau(k))[v].$$

Proof. For (i): Let y be such that $n < y$, $d^\alpha_\tau(y)$ is defined, and that $\gamma(y)[s] \leq \delta(k)[v]$. By the choice of m, $y \leq m$. By definition of h^α,

$$h^\alpha(y)[v] \leq h^\alpha(m)[v].$$

(i) follows.

For (ii): Let y be such that $n < y$, $d^{\alpha}_{\tau}(y)$ is defined, and $\gamma(y)[s] \leq \delta(k)[v]$, then $y \leq m \leq k$. By the well ordering at stage v, if $y < k$, then

$$\varphi(d^{\leq\alpha}_{\tau}(\leq y))[v] \leq h^{\alpha}(k)[v] < \varphi(d^{\alpha}_{\tau}(k))[v].$$

From this, (ii) follows.

Lemma 2.1 follows. □

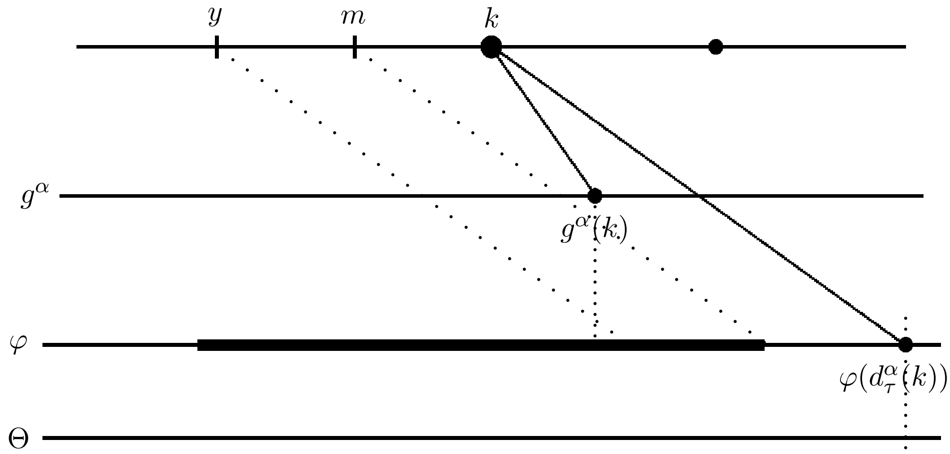

Fig. 1. The well ordering in lemma 2.1.

We now rectify $\Delta(B; k)$ in Case 2 as follows: Let m be the greatest y such that $\gamma(y) \leq \delta(k)[v]$; let $p(\alpha) = \max\{\varphi(d^{\leq\alpha}_{\tau}(y))[v], b^{\alpha}[v], p_*(\alpha) \mid \gamma(y) \leq \delta(k)[v], p_*(\alpha)\downarrow\}$; let $q(\alpha) = \delta(k)[v]$; define the *valid use* $\delta^*(k) = p(\alpha)$; create a *conditional restraint* $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$; if $p(\alpha) \leq \varphi(d^{\alpha}_{\tau}(k))[v]$, then define $g(\alpha)$ by $g(\alpha) = \max\{h^{\alpha}(m)[v],\ p_*(\alpha) \mid p_*(\alpha)\downarrow\}$; enumerate $d^{\alpha}_{\tau}(m)$ into D; and wait for the next step.

Wait for the first stage $t > s$ at which one of the following cases occurs.

Case a. $g(\alpha)$ is defined, and there is a b such that $g(\alpha) < b \leq p(\alpha)$ and b enters B at stage t. Then:

– Set $p(\alpha) \leftarrow g(\alpha)$, and $g(\alpha)$ to be undefined,

– Create a conditional restraint $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$, and

– We say that α *receives special attention*.

Case b. Otherwise, and there is an element $b \leq p(\alpha)$ which enters B at stage t. Then set $g(\alpha)$, $p(\alpha)$, and $q(\alpha)$ to be all undefined, if they are defined.

In either Case a or Case b, $\Delta(B; k)$ becomes undefined.

Case c. Otherwise, and t is an α-expansionary stage. Then we continue the rectification of the inequality $\Delta(B; k) \neq K(k)$.

In this case, $\Gamma(m)$ has been lifted, and $\Delta(B; k) = 0$ is reserved.

To see that the conditional restraint $\vec{r}(\alpha)$ preserves some computations $\Theta(A; b)$, we need the following general notation:

2.2 DEFINITION. Let τ, β be an $\mathcal{R}$-, and an $\mathcal{S}$-strategy respectively, and y be a natural number such that $d^\beta_\tau(y)$ is defined. Then:

(i) We use $\varphi^+(d^\beta_\tau(y))$ to denote the φ-use $\varphi(d^\beta_\tau(y))$ which was observed at the stage at which we specified $\gamma(y)$.

(ii) We define *the valid use* $\varphi^*(d^\beta_\tau(y))$ as

$$\varphi^*(d^\beta_\tau(y)) = \min\{\varphi^+(d^\beta_\tau(y)), p(\xi) \mid \tau \subset \xi \subseteq \beta, \vec{r}(\xi) \downarrow= (p(\xi), q(\xi)),\ \gamma(y) \leq q(\xi)\}.$$

[If $\{\varphi^+(d^\beta_\tau(y)), p(\xi) \mid \tau \subset \xi \subseteq \beta, \vec{r}(\xi) \downarrow= (p(\xi), q(\xi)),\ \gamma(y) \leq q(\xi)\} = \emptyset$, then the valid use $\varphi^*(d^\beta_\tau(y))$ is defined as -1.]

The valid use $\varphi^*(d^\beta_\tau(y))$ specifies the length of initial segment below which B changes will cause the agitator $d^\beta_\tau(y)$ to issue a request to enumerate $\gamma_\tau(y)$ into A.

(iii) We define an *absolute restraint* p^τ for τ by

$$p^\tau = \min\{p(\xi) \mid \xi \subset \tau,\ \xi\hat{\ }\langle 2\rangle \not\subseteq \tau,\ \vec{r}(\xi) \downarrow= (p(\xi), q(\xi))\}.$$

(The exclusion of $\xi\hat{\ }\langle 2\rangle \subseteq \tau$ in the definition of p^τ is harmless due to the initialization of strategies at α-expansionary stages.)

The definition of the value of the absolute restraint ensures that if B does not change below p^τ, then τ must be to the left of the true path.

(iv) For an $\mathcal{S}$-strategy β, we say that $d^\beta_\tau(y)$ *requires attention* at a stage, if there is an element $b \leq \min\{\varphi^*(d^\beta_\tau(y)), p^\tau\}$ which enters B at this stage. In this case, we allow $d^\beta_\tau(y)$ *to receive attention* by enumerating its γ-marker $\gamma_\tau(y)$ into A.

(v) For an $\mathcal{R}$-strategy τ, we say that $d^\tau_\tau(y)$ *requires attention* at a stage, if there is an element $b \leq \min\{\varphi^+_\tau(d^\tau_\tau(y)),\ p^\tau\}$ which enters B at the same stage. In this case, we allow $d^\tau_\tau(y)$ *to receive attention* by enumerating its marker $\gamma_\tau(y)$ into A.

The absolute restraint p^τ ensures that

(1) If p^τ is defined, then τ is not visited, and

(2) If p^τ is defined, then there is no agitator $d^\beta_\tau(y)$ for any β, any y which requires attention.

Suppose that at stage s, we enumerate $d^\alpha_\tau(m)$ into D, and create a *conditional restraint* by $\vec{r}(\alpha) = (p, q)$. Then this means that

(A). For any $\mathcal{S}$-strategy β, and any y, if $\alpha \subseteq \beta$, $d^\beta_\tau(y)$ is defined, and $\gamma(y) \leq q$, then we issue a request that β does not require honestification for agitator $d^\beta_\tau(y)$ unless B changes below $p(\alpha)$, since we have defined the *valid use of agitator* $d^\beta_\tau(y)$ by

$$\varphi^*(d^\beta_\tau(y)) = \min\{\text{old } \varphi^*(d^\beta_\tau(y)), p(\alpha)\}$$

where old $\varphi^*(d^\beta_\tau(y))$ is the valid use of agitator which has been defined before the current conditional restraint $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$ is created.

The role of the valid use $\varphi^*(d^\beta_\tau(y))$ is that if $\gamma(y) \leq q$, then only if B changes below the current valid use $\varphi^*(d^\beta_\tau(y))$, β is allowed to honestify its agitator $d^\beta_\tau(y)$ by enumerating $\gamma(y)$ into A.

(B). For any $\mathcal{R}$-strategy τ', any y', any α', if $\alpha\hat{}\langle\omega\rangle \subseteq \tau'$, $d^{\alpha'}_{\tau'}(y')$ is defined, and $\gamma_{\tau'}(y') \leq q$, then the valid use of agitator $d^{\alpha'}_{\tau'}(y')$ is updated in the same way as in (A) above. In this case, we have defined the absolute restraint of τ', denoted by $p^{\tau'}$, by

$$p^{\tau'} = \min\{\text{old } p^{\tau'}, p(\alpha)\}$$

where old $p^{\tau'}$ is the absolute restraint of τ' defined by this way at previous stages.

Suppose that with the agitation $d^{\alpha}_{\tau}(m)$ at stage s, we have not got any B-change below $p(\alpha)[s]$ by the next α-expansionary stage $t > s$. Then we have that

(i) There is no B-permission for any agitator of nodes $\leq \alpha$ with γ-marker less than or equal to $\delta(k)[v]$ which has occurred during stages $[s, t]$.

(ii) By the conditional restraint $\vec{r}(\alpha)[s]$, there is no agitator of nodes $\not\leq \alpha$ with γ-marker less than or equal to $\delta(k)[v]$ which has received attention during stages $[s, t]$.

By (i) and (ii), $A_s \upharpoonright (\theta\varphi(d^{\alpha}_{\tau}(k))[v] + 1) = A_t \upharpoonright (\theta\varphi(d^{\alpha}_{\tau}(k))[v] + 1)$.

(iii) $\Delta(B; k) = 0$ has been reserved by the definition of the valid use $\delta^*(k)[s] = p(\alpha)[s]$.

(iv) $\Gamma(m)$ has been lifted by an X_τ-change below $\varphi(d^{\alpha}_{\tau}(m))[v]$.

Now we continue the procedure of the rectification of $\Delta(B; k) \neq K(k)$ as above. Since there are only finitely many y such that $\gamma(y) \leq \delta(k)[v]$, there is a stage $s > v$ at which either $\Delta(B; k)$ is rectified by a B-change below the current $p(\alpha) = \delta^*(k)$, or s is α-expansionary, but there is no agitator $d^{\alpha}_{\tau}(y)$ available for agitation, so at stage s, f is built towards to B being computable, and Δ is set to be totally undefined.

With a fixed inequality $\Delta(B; k) \neq K(k)$, the $p(\alpha)[s]$ is created as a decreasing sequence in stages until either B changes below the most recent $p(\alpha)$ (without $p(\alpha)$ immediately being redefined to be less than its previous value), or $\Theta(A)[v]$ up to $\varphi(d^{\alpha}_{\tau}(k))[v]$ has been cleared of all γ-markers, except for the γ-markers determined by the base marker of α. In the former case, all agitators which have been delayed for receiving attention due to α's conditional restraints may require attention by the B-change below $p(\alpha)$; while in the latter case, f is built, and all nodes to the right of $\alpha\hat{}\langle -1\rangle$ are initialised, so that all agitators which have become dishonest due to α's conditional restraints are cancelled — and in this case, $g(\alpha)$, $p(\alpha)$ and $q(\alpha)$ are cancelled. A subtle point is that we cannot cancel any agitator associated with a node below $\alpha\hat{}\langle b\rangle$. By the definition of $p(\alpha)$ at stages at which we enumerate some agitator $d^{\alpha}_{\tau}(m)$, we have that B has never changed below $p(\alpha)$ since stage v. The reason is that before we rectified $\Delta(B; k)$ for the first time, B had never changed below $\delta(k)[v] = \delta^*(k)[v]$, and after α started a procedure to rectify $\Delta(B; k)[v]$, we always have that $p(\alpha) \geq b^{\alpha}[v]$, and that B has not changed below $p(\alpha)$ subsequent to $p(\alpha)$ being defined, since $b^{\alpha}[v]$ is an upper bound of γ-markers of strategies below $\alpha\hat{}\langle b\rangle$ observed at stage v. This suggests the following rules on conditional restraints $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$.

The Conditional Restraints: Overview

The conditional restraint $\vec{r}(\alpha) = (p, q)$ will have the following properties which are essential to the proof of the theorem:

(i) If $\vec{r}(\alpha) = (p, q)$ is created at stage s, and α receives *special attention at stage* s, then there is an element $b > p$ which is enumerated into B at stage s, and $\Theta(A; b)[s] \downarrow = 0 \neq 1 = B(b)$ with $\theta(b) \leq q$, and if $B_s \upharpoonright (p+1) = B \upharpoonright (p+1)$, then we are able to ensure that $A_s \upharpoonright (q+1) = A \upharpoonright (q+1)$.

(ii) If a restraint vector $\vec{r}(\alpha) = (p, q)$ is created, then:

– For any $\mathcal{R}$-strategy τ', if $\alpha \subset \tau'$, then the permitting marker of $\gamma_{\tau'}(y)$ will be less than or equal to p.

– For any $\mathcal{S}$-strategy β, if $\alpha \subseteq \beta$, $d^{\beta}_{\tau}(y)$ is defined, and $\gamma_{\tau}(y) \leq q$, then the valid use $\varphi^*(d^{\beta}_{\tau}(y))$ is defined by

$$\varphi^*(d^{\beta}_{\tau}(y)) = \min\{\text{old } \varphi^*(d^{\beta}_{\tau}(y)), p\}.$$

(iii) If $\vec{r}(\alpha) \downarrow = (p, q)$ is defined, then no element less than q can be enumerated into A.

(iv) If $B \upharpoonright (p+1)$ changes, then $\vec{r}(\alpha) = (p, q)$ is either cancelled, or $\vec{r}(\alpha)$ is redefined to be (p', q') for some p', q', with $p' < p$.

(v) If $\vec{r}(\alpha)[s] = (p, q)$, and $\vec{r}(\alpha)[s+1]$ is undefined, or $\neq \vec{r}(\alpha)[s]$, then one of the following possibilities occurs,

(5a) α is initialised at stage $s+1$,

(5b) B changes below p at stage $s+1$,

(5c) α builds f at stage $s+1$,

(5d) $\alpha\hat{\ }\langle b\rangle$ is visited at stage $s+1$.

We say that (i)–(v) are *conditional restraint rules*, which will be satisfied for any strategy during the course of the construction. Notice that the conditional restraint created at a stage at which we enumerate some agitator $d^{\alpha}_{\tau}(m)$ into D, will never delay the honestification of agitators associated with nodes below $\alpha\hat{\ }\langle b\rangle$, and that a conditional restraint which is created at a stage at which B changes has already created and preserved an inequality between $\Theta(A)$ and B. Therefore if (5c) or (5d) in (v) occurs, then the current $p(\alpha)$, if defined, was specified at some stage we enumerated some agitator $d^{\alpha}_{\tau}(m)$ into D, so that α has never injured any agitator associated with nodes below $\alpha\hat{\ }\langle b\rangle$, while any agitator with nodes to the right of $\alpha\hat{\ }\langle b\rangle$ are cancelled. The conditional restraint rules ensure that:

2.3 LEMMA Let $v < s$ be stages. If $p(\alpha)[t]$ is defined for all $t \in [v, s]$, then

$$p(\alpha)[v] \geq p(\alpha)[v+1] \geq \cdots \geq p(\alpha)[s].$$

Proof. We prove by induction on stages. Let x be such that $v \leq x < s$ and such that $p(\alpha)$ is redefined at stage $x+1$. If $p(\alpha)[x]$ was created at a stage at which α received special attention, then the only case is that $p(\alpha)[x]$ was defined to be $g(\alpha)[x-1]$ (otherwise, $p(\alpha)$ cannot be redefined at stage $x+1$). By the definition of $g(\alpha)$, $g(\alpha)[x-1] \geq p_*(\alpha)[x-1]$. By the choice of x, B has never changed below $p(\alpha)[x]$ since it was created, so the inequality $\Theta(A; b) \neq B(b)$ which was created at the stage at which $p(\alpha)[x]$ was specified, has been preserved by the end of stage x. So the only reason that $p(\alpha)[x+1]$ is redefined at stage $x+1$ is that $p(\alpha)[x+1]$ is defined to be $g_*(\alpha)[x-1]$, which is less than $p_*(\alpha)$, giving $p(\alpha)[x+1] < p(\alpha)[x]$. If $p(\alpha)[x]$ was defined at a stage at which we enumerated some agitator $d^{\alpha}_{\tau}(m)$ into D, then by the definition of $p(\alpha)$, $g(\alpha)$, if $p_*(\alpha)$ is currently defined, then both $p(\alpha)[x]$ and $g(\alpha)[x]$ are greater than or equal to $p_*(\alpha)$, so in any case, we define $p(\alpha)[x+1] \leq p(\alpha)[x]$.

Lemma 2.3 follows.

Lemma 2.3 is the key point to the permitting argument in the present paper.

By the rectification of $\Delta(B; k)$, whenever an inequality $\Delta(B; k) \neq K(k)$ occurs, either the inequality will be eventually rectified, or the current Δ will be reset. Therefore, for a fixed m, there is a stage s_0 say, after which if we rectify $\Delta(B; k) \neq K(k)$ by enumerating $d^{\alpha}_{\tau}(m)$ in to D, then $m \leq k$.

2.4 LEMMA. (i) If $\varphi(d^{\leq\alpha}_{\tau}(\leq n))[s]$ are unbounded as s goes to infinity, then $\Phi(B, X)$ is partial.

(ii) Otherwise, and f is built infinitely often , then $f = B$.

(iii) Otherwise, and $\Delta(B)$ is total, then $\Delta(B) =^* K$.

Proof. (i) follows from an induction argument.

For (ii), let s_0 be minimal such that $\lim_s \varphi(d^{\leq\alpha}_{\tau}(\leq n))[s] = \varphi(d^{\leq\alpha}_{\tau}(\leq n))[s_0]$, s_1 be minimal greater than s_0 such that $B_{s_1} \upharpoonright (\varphi(d^{\leq\alpha}_{\tau}(\leq n))[s_0]+1) = B \upharpoonright (\varphi(d^{\leq\alpha}_{\tau}(\leq n))[s_0]+1)$, and s_3 be minimal greater than s_1 such that $p_*(\alpha)[s_3]$ is defined. Then by the definition of $p_*(\alpha)$, for any $s \geq s_3$, $p_*(\alpha)[s] \leq p_*(\alpha)[s+1]$. For any x, if $f(x)$ is created at a stage $s > s_3$, then by the strategy, $\Theta(A; x) \downarrow = B(x)$ holds at stage s, and by the choice of s_3, $\Theta(A; x)[s]$ will never be injured by any γ-marker after stage s. (ii) follows.

For (iii), by the assumption of (iii), let Δ be the final version of α's Δ. By the strategy, every inequality $\Delta(B; k) \neq K(k)$ will be eventually rectified. (iii) follows.

Lemma 2.4 follows. □

More importantly, we have the following:

2.5 LEMMA. Suppose that $\varphi(d^{\leq\alpha}_{\tau}(\leq n))[s]$ are bounded as s goes to infinity, and that f is built only finitely often . Then for any m, if $h^{\alpha}(m)[s]$ are bounded over the course of the construction, so are $d^{\alpha}_{\tau}(m)[s]$.

Proof. Let s_0 be minimal such that for any $s > s_0$, the following properties hold:

(a) both $b^{\alpha}[s]$ and $h^{\alpha}(m)[s]$ are stable,

(b) f will not be built,

(c) B does not change below $h^{\alpha}(m)$, and

(d) If we enumerate $d^{\alpha}_{\tau}(m)$ to rectify $\Delta(B; k)$ at stage s, then $m \leq k$.

Suppose that we rectify $\Delta(B; k)$ at a stage $s > s_0$ by enumerating $d^{\alpha}_{\tau}(m)$ into D. Let v be the stage at which $\delta(k)$ was created. Let $v > s_0$. By the choice of s_0, and by Lemma 2.1, for any y, if $n < y$, $d^{\alpha}_{\tau}(y)$ is defined, and $\gamma(y) \leq \delta(k)[v]$ holds during stage s, then both $h^\alpha(y)[v] \leq h^\alpha(m)[v]$ and $\varphi^*(d_\tau^{\lneqq\alpha}(y))[v] \leq \varphi(d_\tau^\alpha(k))[v]$ hold. Therefore, if we get a B-change after an agitation at some stage $\geq s$ during the procedure of the rectification of this inequality $\Delta(B;k) \neq K(k)$, then this B-change must be both below $\varphi(d_\tau^\alpha(k))[v]$ and above the current $g(\alpha)$, which is less than or equal to $h^\alpha(m)[v]$, so that we allow α to receive special attention and create and preserve a permanent inequality $\Theta(A;b) \neq B(b)$ for some b. Otherwise, then there will be a stage at which no agitator available for us to rectify $\Delta(B;k)$, so that f is built, contradicting the choice of s_0.

Lemma 2.5 follows. $\square$

Lemma 2.5 ensures that if $d_\tau^\alpha(m)[s]$ are unbounded, then so are $h^\alpha(m)[s]$, we can prove by induction that $\Phi(B, X)$ is partial.

Of course if $d_\tau^\alpha(k)[s]$ are bounded, but $\Delta(B;k)$ diverges, then either $\varphi(d_\tau^\alpha(k))[s]$ are unbounded, or $\theta\varphi(d_\tau^\alpha(k))[s]$ are unbounded. In the former case, $\mathcal{R}$ is satisfied, and in the latter case, $\mathcal{S}$ is satisfied.

As we have mentioned above, we need the following well ordering of agitators, which determines the priority ordering among the agitators.

Well Ordering of Agitators of α

In the analysis of the rectification of $\Delta(B;k)$ above, we have seen that

(1). To deal with the injury of the main body of α from honestification of agitators associated with nodes below $\alpha\hat{\ }\langle b\rangle$, we ensure that we can define $\Delta(B;k)$ at a stage only if the following properties are satisfied:

$$b^\alpha = \max\{\gamma_{\tau'}(y') \mid \alpha \subset \tau', \tau' <_{\mathrm{L}} \alpha\hat{\ }\langle -1\rangle\} < \varphi(d_\tau^\alpha(k)).$$

(2). To ensure that the $\mathcal{S}$-strategies will never make $\Gamma(X, A)$ partial, we require:

$$\varphi(d_\tau^{<\alpha}(\leq k)),\ \varphi(d_\tau^\alpha(< k)), k < \varphi(d_\tau^\alpha(k)).$$

We remark that the well ordering of agitators of α cannot be automatically provided by the $\mathcal{R}$-strategy τ, and that so if either (1) or (2) fails to hold, we need to enumerate $d_\tau^\alpha(k)$ into D.

We outline the rough outcomes of α as follows:

$$b <_{\mathrm{L}} -1 <_{\mathrm{L}} \omega <_{\mathrm{L}} 2.$$

Note that f and Δ are located at $\alpha\hat{\ }\langle -1\rangle$ and $\alpha^\omega = \alpha\hat{\ }\langle\omega\rangle$, respectively.

Suppose that $\varphi(d_\tau^{<\alpha}(\leq n))[s]$ are bounded, that $B \not\leq_{\mathrm{T}} \emptyset$ and that $K \not\leq_{\mathrm{T}} B$. Then by Lemma 2.4, there is a final version of Δ, and $\Delta(B)$ is not total. We use (d, k) to denote that $\Delta(B;k)$ diverges.

The intuition behind the formal definition of possible outcomes will be given in subsection 5.2, and the formal definition of the full outcomes will be given in section 6.

3 Satisfying $\mathcal{S}$-Requirements Below Two $\mathcal{R}$-Strategies

In this section, we describe the strategies to satisfy the following requirements:

$$\mathcal{R}_0,\ \mathcal{R}_1,\ \mathcal{S}_0,\ \mathcal{S}_1,\ \mathcal{S}_2,\ \cdots .$$

3.1 Analysis of An $\mathcal{S}$-Strategy Below Two $\mathcal{R}$-Strategies: Problems and Ideas

Suppose that τ_0, τ_1 and α are $\mathcal{R}_0$-, $\mathcal{R}_1$- and $\mathcal{S}$-strategies respectively. Let $\tau_0 \subset \tau_0\hat{\ }\langle 0\rangle \subseteq \tau_1 \subset \tau_1\hat{\ }\langle 0\rangle \subseteq \alpha$. α assumes that a Turing functional Γ_i is built by τ_i for each $i \in \{0, 1\}$.

α defines a *base point* n_1 *for* Γ_1*, and a base point* n_0 *for* Γ_0 such that $n_0 = \varphi_1^*(d_{\tau_1}^{<\alpha}(\leq n_1))$. For each $j > n_1$, we define an agitator $d_{\tau_1}^{\alpha}(j)$ of α for Γ_1, and for each $j > n_0$, we define an agitator $d_{\tau_0}^{\alpha}(j)$ of α for Γ_0. Now we have two different systems of agitators: for both $j = 0$ and 1,

$$\{d_{\tau_j}^{\alpha}(k) \mid k \in \omega\}.$$

To define the priority ordering of agitators of α, we introduce the following auxiliary functions.

3.1 DEFINITION We define h_j^{α}, and g_j^{α} for both $j = 0, 1$ as follows:

$$h_0^{\alpha}(k) = \max\{\varphi_0^*(d_{\tau_0}^{\beta}(x)),\ \varphi_0^*(d_{\tau_0}^{\alpha}(y)),\ k \mid \beta < \alpha, x \leq k, y < k\}$$

$$g_0^{\alpha}(k) = h_0^{\alpha}(k)$$

$$h_1^{\alpha}(k) = \max\{\varphi_1^*(d_{\tau_1}^{\beta}(x)),\ \ \varphi_1^*(d_{\tau_1}^{\alpha}(y)),\ k \mid \beta < \alpha, x \leq k, y < k\}$$

$$g_1^{\alpha}(k) = \max\{\varphi_0^*(d_{\tau_0}^{\beta}(x)),\ h_1^{\alpha}(k) \mid \beta \leq \alpha, x \leq h_1^{\alpha}(k)\}.$$

We say that $g_j^{\alpha}(m)$ is the *preuse* of $d_{\tau_j}^{\alpha}(m)$. Intuitively speaking, $g_i^{\alpha}(m)$ is the maximal φ-uses of agitators of priority ordering higher than that of $d_{\tau_i}^{\alpha}(m)$. So if B does not change below $g_i^{\alpha}(m)$, then no agitator of priority ordering higher than that of $d_{\tau_i}^{\alpha}(m)$ requires attention.

By definition, for any α, j, and m, if $g_j^{\alpha}(m)[s]$ are unbounded, then we can prove by induction that $\Phi_i(B, X_i)$ is partial for some $i \leq j$.

The key point to the satisfaction of $\mathcal{R}$-requirements is that, for any α, any j, and any m, if $g_j^{\alpha}(m)[s]$ are bounded in the construction, then so are $d_{\tau_j}^{\alpha}(m)[s]$.

α will define Δ and f such that $\Delta(B) = K$, if K is computable in B, and $f = B$ if B is computable. The new difficulty is to build a computation $\Theta(A; x)\downarrow = y$ to be cleared of both Γ_1 and Γ_0.

Let $n_1 = n(\tau_1, \alpha)$ be the base point of α for Γ_1. At first, we will decide whether or not there is a $k \leq n_1$ such that $\varphi_1^*(d_{\tau_1}^{<\alpha}(k))[s]$ becomes unbounded. Let $n_0 = n(\tau_0, \alpha)$

be the base point of α for Γ_0. n_0 will be defined to be $\varphi_1^*(d_{\tau_1}^{<\alpha}(\leq n_1))$. Therefore if $\varphi_1^*(d_{\tau_1}^{<\alpha}(\leq n_1))[s]$ becomes unbounded in the construction, then $\lim_s n_0[s]$ does not exist. And of course α will decide whether or not there is a $k \leq n_0$ such that $\varphi_0^*(d_{\tau_0}^{<\alpha}(k))[s]$ becomes unbounded over the course of the construction.

Firstly we define the rough possible outcomes of α by

$$b <_{\mathrm{L}} -1 <_{\mathrm{L}} \omega <_{\mathrm{L}} 2$$

where the outcomes have the same meaning as that in section 2.

Suppose that b is the right outcome, i.e., $\varphi_0^*(d_{\tau_0}^{<\alpha}(\leq n_0))[s]$ will be unbounded. In this case, we can argue by induction that either $\Phi_0(B, X_0)$ or $\Phi_1(B, X_1)$ is partial. However, in either case, for any node ξ below $\alpha\hat{}\langle b\rangle$, τ_1 is no longer active at ξ, and we have a global win for one of the $\mathcal{R}_0$ and $\mathcal{R}_1$.

This suggests that the main body of α, which assumes that b is not the right outcome, will have to deal with the honestification of agitators $d_{\tau_0}^{\beta}(y)$ for $\beta \supset \alpha\hat{}\langle b\rangle$. Of course, we need to refine the outcome b to guess exactly which φ-use makes $\varphi_0^*(d_{\tau_0}^{\alpha}(\leq n_0))[s]$ unbounded, and to guess the least x such that $\gamma_j(x)[s]$ is unbounded, if $\Gamma_j(X_j, B)$ is partial.

To deal with the injury of α from the collection of nodes below $\alpha\hat{}\langle b\rangle$, we define the following parameters.

3.2 DEFINITION We define b_j^{α} for $j = 0, 1$ and b^{α} by

$$b_0^{\alpha} = \max\{\varphi_0^{+}(d_{\tau_0}^{\beta}(y)) \mid \alpha \subset \beta, \beta <_{\mathrm{L}} \alpha\hat{}\langle -1\rangle, y \in \omega\}$$

$$b_1^{\alpha} = \max\{\gamma_{\tau}(y) \mid \alpha \subset \tau, \tau <_{\mathrm{L}} \alpha\hat{}\langle -1\rangle, y \in \omega\}$$

$$b^{\alpha} = \max\{b_0^{\alpha},\ b_1^{\alpha}\}.$$

α will build Δ and f such that one of the following holds:

- $D \neq \Phi_i(B, X_i)$ for some $i \leq 1$.
- $\Delta(B)$ is total and $\Delta(B) =^* K$.
- f is total and $f =^* B$.
- $\Theta(A) \neq B$.

For each $k > n_0$, we open a cycle k to define $\Delta(B; k)$.

Cycle $k > n_0$: If:
(a) (Well Ordering) $p_*(\alpha)$, b^{α}, $g_1^{\alpha}(k) < \varphi_1(d_{\tau_1}^{\alpha}(k))$, and
(b) $l(\Theta(A), B) > \varphi_1(d_{\tau_1}^{\alpha}(k))$. Then
– define $\Delta(B; k) \downarrow = K(k)$ with $\delta(k) = \theta(\varphi_1(d_{\tau_1}^{\alpha}(k)))$, and
– define $\delta^*(k) = \theta(\varphi_1(d_{\tau_1}^{\alpha}(k)))$.

Suppose that $\Delta(B;k)$ was created at a stage v, and $\Delta(B;k)\downarrow\neq K(k)$ occurs at a stage $s>v$. Now we want to rectify $\Delta(B;k)$ by enumerating some agitator $d^{\alpha}_{\tau_j}(m)$ into D for some j and some m. Our new problem is how to decide which agitator $d^{\alpha}_{\tau_j}(m)$ we can enumerate into D in the two systems of agitators of α, $d^{\alpha}_{\tau_j}(y)$, for all y, and for both $j=0$ and 1. To understand the problem, we outline some rules for the rectification of $\Delta(B;k)$:

(1) We enumerate some agitator $d^{\alpha}_{\tau_i}(m_i)$ into D only at α-expansionary stages.

(2) At every α-expansionary stage, all agitators of α are honest.

(3) During the procedure of the rectification of $\Delta(B;k)$, $\Theta(A)[v]\upharpoonright(\theta\varphi_1(d^{\alpha}_{\tau_1}(k))[v]+1)$ has been preserved by a sequence of conditional restraints $\vec{r}(\alpha)=(p(\alpha),q(\alpha))$.

(4) During the procedure of the rectification of $\Delta(B;k)$, the $p(\alpha)[t]$ is decreasing in stages.

(5) Once $p(\alpha)$ is defined, then we are looking for a B-change below $p(\alpha)$ until the next α-expansionary. To make sure that if B does not change below $p(\alpha)$ by the next α-expansionary stage, then all agitators of α are honest at the next α-expansionary stage, we will define $p(\alpha)=\max\{\varphi_i^*(d^{\leq\alpha}_{\tau_i}(\leq y))[v],b^{\alpha}[v],p_*(\alpha)\mid\gamma_{\tau_i}(y)\leq\delta(k)[v]\}$ at the stage at which we enumerate some agitator of α into D.

This means that whichever agitator is enumerated into D, we always look for a B-change below the maximal φ-uses of agitators of nodes $\leq\alpha$ with corresponding γ-marker less than or equal to $\delta(k)[v]$, and that in this case, we define the valid use $\delta^*(k)$ to be $p(\alpha)$.

In this way, $p(\alpha)$ will be decreasing in stages during the procedure of the rectification of $\Delta(B;k)$.

(6) By the definition of $\Delta(B;k)$, $l(\Theta(A),B)[v]>\varphi_1(d^{\alpha}_{\tau_1}(k))[v]$, so once we have achieved $p(\alpha)\leq\varphi_1(d^{\alpha}_{\tau_1}(k))[v]$, it is possible that a B-change below $p(\alpha)$ allows us to create and preserve an inequality $\Theta(A;b)\downarrow=0\neq1=B(b)$ for some b, if the B-change is above the maximal φ-uses of agitators with priority higher than the agitator we just enumerated into D at the stage we specified the current $p(\alpha)$. That is to say, if at a stage s, we find that $p(\alpha)\leq\varphi_1(d^{\alpha}_{\tau_1}(k))[v]$, and we enumerated an agitator $d^{\alpha}_{\tau_i}(m_i)$ into D, then at the stage at which we find a B-change b such that $g^{\alpha}_i(m_i)[v]<b\leq p(\alpha)$, then we have an inequality $\Theta(A;b)\downarrow=0\neq1=B(b)$, and we are able to preserve this inequality by updating the conditional restraint $\vec{r}(\alpha)=(g^{\alpha}_i(m_i)[v],\delta(k)[v])$.

(7) Item (6) suggests that if we have achieved $p(\alpha)\leq\varphi_1(d^{\alpha}_{\tau_1}(k))[v]$, then we must pick up the agitator $d^{\alpha}_{\tau_i}(m_i)$ of α such that i minimal and $g^{\alpha}_i(m_i)[v]$ maximal, where m_i is the greatest $y>n_i$ such that $\gamma_{\tau_i}(y)\leq\delta(k)[v]$ holds at the current stage, then enumerate $d^{\alpha}_{\tau_i}(m_i)$ into D, and look for a B-change below $p(\alpha)$ until the next α-expansionary stage. If we have got a B-change below $p(\alpha)$, then if this B-change is above $g^{\alpha}_i(m_i)[v]$, then update the conditional restraint by redefining $\vec{r}(\alpha)=(g^{\alpha}_i(m_i)[v],\delta(k)[v])$, otherwise, then we simply set $p(\alpha)$ to be undefined. More importantly, this ensures that if at a stage s, we picked $d^{\alpha}_{\tau_i}(m_i)$ such that $B_s\upharpoonright(g^{\alpha}_i(m_i)[v]+1)=B\upharpoonright(g^{\alpha}_i(m_i)[v]+1)$, and such that $p(\alpha)[s]\leq\varphi_1(d^{\alpha}_{\tau_1}(k))[v]$, then if any response of agitations at stages $\geq s$ during the rectification of this inequality $\Delta(B;k)\neq K(k)$ is a B-change, then this B-change must be above the maximal φ-uses of agitators of priority higher than the one we just enumerated

into D, and then α is allowed to receive special attention, a permanent inequality $\Theta(A; b) \downarrow \neq B(b)$ has been created and preserved. Otherwise, then we will reach a stage at which there is no agitator available, so that $\Theta(A) \upharpoonright (\varphi_1(d^{\alpha}_{\tau_1}(k))[v] + 1)$ is defined and cleared of both Γ_0 and Γ_1 at the base points n_0, n_1 respectively.

(8) Suppose that at stage $s > v$, we define $p(\alpha) \leq \varphi_1(d^{\alpha}_{\tau_1}(k))[v]$, pick up $d^{\alpha}_i(m_i)$ and enumerate $d^{\alpha}_{\tau_i}(m_i)$ into D, and that B changes for the first time at a stage $t > s$ such that there is a (unique) b with $g^{\alpha}_i(m_i)[v] < b \leq p(\alpha)$ which enters B. Then:

For each j, let y_j be the maximal $y > n_j$ such that $\gamma_{\tau_j}(y) \leq \delta(k)[v]$ held at stage s, if there is such a y. By (7), if y_j is defined, then $g^{\alpha}_j(y_j)[v] \leq g^{\alpha}_i(m_i)[v](< b)$. Therefore the updating of conditional restraint $\vec{r}(\alpha)$ at stage t delays the honestification of agitators of α at most the lowest priority agitator $d^{\alpha}_{\tau_j}(y_j)$ for each j. Therefore the conditional restraints of α will never make any Γ_{τ_j} incorrect.

(9) To guarantee the correctness of the strategies, we must make sure that for every j, and every m, there is a stage, s_0 say, after which if we rectify an inequality $\Delta(B; k) \neq K(k)$ by agitating $d^{\alpha}_{\tau_i}(m)$, then $p(\alpha) \leq \varphi_1(d^{\alpha}_{\tau_1}(k))[v]$, where v is the stage at which $\delta(k)$ was specified.

By using (9), we can argue that for every i, and every m, if $g^{\alpha}_i(m)[s]$ is bounded, then so is $d^{\alpha}_{\tau_i}(m)[s]$, and that if $g^{\alpha}_i(m)[s]$ is unbounded, then we can prove by induction that $\Phi_j(B, X_j)$ is partial for some $j \leq i$.

Based on the guidelines above, we rectify $\Delta(B; k)$ as follows.

Suppose that we want to rectify $\Delta(B; k) \downarrow = 0 \neq 1 = K(k)$. Let v be the stage at which $\delta(k)$ was created. In the same spirit as in the first strategy in subsection 2.5, for each $j \leq 1$, let m_j be the greatest $y > n_j$ such that $\gamma_j(y) \leq \delta(k)[v]$. To decide which of the $d^{\alpha}_{\tau_0}(m_0)$ or $d^{\alpha}_{\tau_1}(m_1)$ will be enumerated into D, we investigate the following cases.

3.3 LEMMA . If $m_0 > h^{\alpha}_1(m_1)[v]$, then

(i) For any y, any j, if $d^{\alpha}_{\tau_j}(y)$ is defined, and $\gamma_j(y) \leq \delta(k)[v]$, then

$$g^{\alpha}_j(y)[v] \leq g^{\alpha}_0(m_0)[v].$$

(ii) If $m_0 \leq k$, then for any y, any j, if $d^{\alpha}_{\tau_j}(y)$ is defined, and $\gamma_j(y) \leq \delta(k)[v]$, then

$$\varphi^*_j(d^{\leq\alpha}_{\tau_j}(\leq y))[v] \leq \varphi_1(d^{\alpha}_{\tau_1}(k))[v].$$

Proof. For (i): For every y, if $n_0 < y$, $d^{\alpha}_{\tau_0}(y)$ is defined, and $\gamma_{\tau_0}(y)[s] \leq \delta(k)[v]$, then $y \leq m_0$. Therefore by definition of g^{α}_0, we have

$$g^{\alpha}_0(y)[v] \leq g^{\alpha}_0(m_0)[v].$$

For every y, if $n_1 < y$, $d^{\alpha}_{\tau_1}(y)$ is defined, and $\gamma_{\tau_1}(y)[s] \leq \delta(k)[v]$, then $y \leq m_1$. By definition, we have

$$g^{\alpha}_1(y)[v] = \varphi^*_0 d^{\leq\alpha}_{\tau_0}(\leq h^{\alpha}_1(y))[v]$$

$$\leq \varphi^*_0 d^{\leq\alpha}_{\tau_0}(\leq h^{\alpha}_1(m_1))[v]$$

$$\leq \varphi_0^* d_{\tau_0}^{\leq\alpha}(\leq m_0 - 1)[v] \leq g_0^\alpha(m_0)[v].$$

(i) holds.

For (ii): Let $m_0 \leq k$.

For any y, if $n_0 < y$, $d_{\tau_0}^\alpha(y)$ is defined, and $\gamma_{\tau_0}(y)[s] \leq \delta(k)[v]$, then $y \leq m_0 \leq k$. By the well ordering at stage v, we have the following:

$$\varphi_1(d_{\tau_1}^\alpha(k))[v] > \varphi_0^* d_{\tau_0}^{\leq\alpha}(\leq h_1^\alpha(k))[v]$$

$$\geq \varphi_0^*(d_{\tau_0}^{\leq\alpha}(\leq k))[v] \geq \varphi_0^*(d_{\tau_0}^{\leq\alpha}(\leq y))[v].$$

For any y, if $n_1 < y$, $d_{\tau_1}^\alpha(y)$ is defined, and $\gamma_{\tau_1}(y)[s] \leq \delta(k)[v]$, then $y \leq m_1 \leq h_1^\alpha(m_1)[v] < m_0 \leq k$. By the well ordering at stage v, we have

$$\varphi_1^* d_{\tau_1}^{\leq\alpha}(\leq y))[v] < \varphi_1(d_{\tau_1}^\alpha(k))[v].$$

(ii) holds.

Lemma 3.3 follows. □

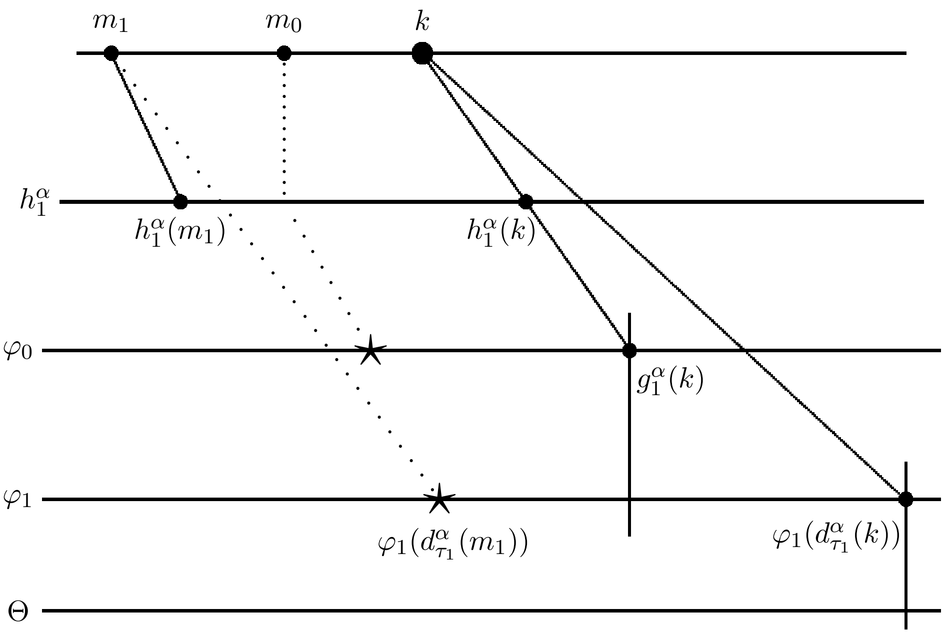

Fig. 2. The case study for $m_0 > h_1^\alpha(m_1)[v]$ in lemma 3.3.

3.4 LEMMA. If $m_0 \leq h_1^\alpha(m_1)[v]$, then

(i) For any y, any j, if $d_{\tau_j}^\alpha(y)$ is defined, and $\gamma_j(y) \leq \delta(k)[v]$, then

$$g_j^\alpha(y)[v] \leq g_1^\alpha(m_1)[v].$$

(ii) If $m_1 \leq k$, then
for any y, any j, if $d^{\alpha}_{\tau_j}(y)$ is defined, and $\gamma_j(y) \leq \delta(k)[v]$, then

$$\varphi^*_j(d^{\leq\alpha}_{\tau_j}(\leq y))[v] \leq \varphi_1(d^{\alpha}_{\tau_1}(k))[v].$$

Proof. For (i): For any y, if $n_1 < y$, $d^{\alpha}_{\tau_1}(y)$ is defined, and $\gamma_{\tau_1}(y)[v] \leq \delta(k)[v]$, then $y \leq m_1$. By definition of g^{α}_j,

$$g^{\alpha}_1(y)[v] \leq g^{\alpha}_1(m_1)[v].$$

By the assumption, for any y, if $n_0 < y$, $d^{\alpha}_{\tau_0}(y)$ is defined, and $\gamma_{\tau_0}(y)[s] \leq \delta(k)[v]$, then

$$y \leq h^{\alpha}_1(m_1)[v].$$

Using this property, we have

$$g^{\alpha}_0(y)[v] = \max\{\varphi^*_0 d^{<\alpha}_{\tau_0}(\leq y)[v],\ \varphi^*_0 d^{\alpha}_{\tau_0}(< y)[v], y\}$$

$$\leq \max\{\varphi^*_0 d^{\leq\alpha}_{\tau_0}(\leq h^{\alpha}_1(m_1))[v],\ h^{\alpha}_1(m_1))[v]\} = g^{\alpha}_1(m_1)[v].$$

(i) holds.

For (ii): Let $m_1 \leq k$.

For any y, if $n_1 < y$, $d^{\alpha}_{\tau_1}(y) \downarrow$, and $\gamma_{\tau_1}(y)[s] \leq \delta(k)[v]$, then $y \leq m_1 \leq k$. By the well ordering at stage v, we have

$$\varphi^*_1(d^{\leq\alpha}_{\tau_1}(\leq y))[v] \leq \varphi_1(d^{\alpha}_{\tau_1}(k))[v].$$

By the assumption, for every y, if $n_0 < y$, $d^{\alpha}_{\tau_0}(y)$ is defined, and $\gamma_{\tau_0}(y)[s] \leq \delta(k)[v]$, then

$$y \leq h^{\alpha}_1(m_1)[v] \leq h^{\alpha}_1(k)[v].$$

By the well ordering at stage v, we have

$$\varphi_1(d^{\alpha}_{\tau_1}(k))[v] > \varphi^*_0 d^{\leq\alpha}_{\tau_0}(\leq h^{\alpha}_1(k))[v] \geq \varphi^*_0 d^{\leq\alpha}_{\tau_0}(\leq y)[v].$$

(ii) holds.

Lemma 3.4 follows. □

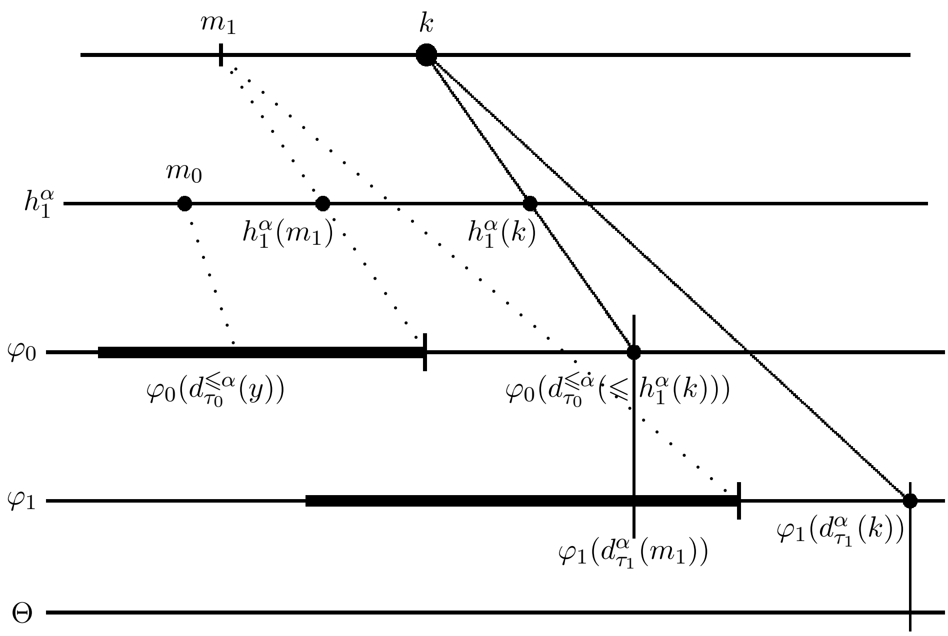

Fig.3 The case study for $m_0 \leq h_1^\alpha(m_1)[v]$ in lemma 3.4.

Based on the case studies, we have the following *choice algorithm* of agitator.

THE CHOICE ALGORITHM. Let s be a stage at which we want to rectify $\Delta(B;k) \neq K(k)$, and let v be the stage at which the current $\Delta(B;k)$ was created.

1. Let m_j be the greatest $y > n_j$ such that $\gamma_j(y) \leq \delta(k)[v]$ holds during stage s, for each $j = 0$, and 1.
2. If m_0 is undefined, or $m_0 \leq h_1^\alpha(m_1)[v]$, then output $d_{\tau_1}^\alpha(m_1)$.
3. Otherwise, then output $d_{\tau_0}^\alpha(m_0)$.

We first prove a property of the choice algorithm.

3.5 LEMMA (*The choice lemma*) Suppose that s is an α-expansionary stage at which we rectify $\Delta(B;k) \neq K(k)$ for some k. Let v be the stage at which $\Delta(B;k)$ was created. If the choice algorithm outputs $d_{\tau_i}^\alpha(m_i)$ at stage s, then both (i) and (ii) below hold.

(i) For any j, any y, if $n_j < y$, $d_{\tau_j}^\alpha(y)$ is defined, and $\gamma_{\tau_j}(y) \leq \delta(k)[v]$ holds during stage s, then

$$g_j^\alpha(y)[v] \leq g_i^\alpha(m_i)[v].$$

(ii) If $m_i \leq k$, then for any j, any y, if $n_j < y$, $d_{\tau_j}^\alpha(y)$ is defined, and $\gamma_{\tau_j}(y) \leq \delta(k)[v]$ holds during stage s, then

$$\varphi^*_{\tau_j}(d^{\leq\alpha}_{\tau_j}(y))[v] \leq \varphi_1(d^{\alpha}_{\tau_1}(k))[v].$$

Proof. By the proof of lemma 3.3 and lemma 3.4.
Lemma 3.5 follows. □

Precisely, $\Delta(B;k) \neq K(k)$ will be rectified as follows:

1. (Defining f) If there is no $y > n_i$ such that $\gamma_{\tau_i}(y) \leq \delta(k)[v]$ for both $i = 0$ and 1, then:

   – Define $f = B$ up to $\varphi_1(d^{\alpha}_{\tau_1}(k))[v]$, on which $\Theta(A)[v]$ have been preserved since stage v.

   [*Remark.* Of course, define $g_*(\alpha)$, $p_*(\alpha)$, and $q_*(\alpha)$ in the same way as in subsection 2.5 to deal with injury from nodes below $\alpha\hat{}\langle b\rangle$.]

2. (Choosing an agitator)– If there is a $y > n_1$ such that $\gamma_{\tau_1}(y) \leq \delta(k)[v]$, then let m_1 be the greatest such y,

   – If m_1 is defined, then let $c_0 = h^{\alpha}_1(m_1)[v]$, else let $c_0 = n_0$,

   – If there is a $y > c_0$ such that $\gamma_{\tau_0}(y) \leq \delta(k)[v]$, then let m_0 be the greatest such y.

3. Let i be the least j such that m_j is defined. Then:

   – Define $p(\alpha) = \max\{\varphi^*_j(d^{\leq\alpha}_{\tau_j}(\leq y))[v], b^{\alpha}[v],\ p_*(\alpha) \mid \gamma_{\tau_j}(y) \leq \delta(k)[v]\}$,

   [*Remark.* Notice that $p(\alpha) \leq \delta(k)[v]$ must hold.]

   – Define $q(\alpha) = \delta(k)[v]$,

   –Define $\delta^*(k) = p(\alpha)$,

   – Create a conditional restraint $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$,

   – If $p(\alpha) \leq \varphi_1(d^{\alpha}_{\tau_1}(k))[v]$, then define $g(\alpha) = g^{\alpha}_i(m_i)[v]$,

   – Enumerate $d^{\alpha}_{\tau_i}(m_i)$ into D, and

   – Wait for the least stage $t > s$ at which either B changes below $p(\alpha)$, or X_{τ_i} changes below $\varphi_i(d^{\alpha}_{\tau_i}(m_i))[v]$.

4. Let t be the least stage $> s$ at which one of the following cases occurs.

   Case 4a. There is a b such that $g(\alpha)\downarrow < b \leq p(\alpha)$ and b enters B. Then:

   – Update the conditional restraint by defining $\vec{r}(\alpha) = (g(\alpha), q(\alpha))$.

   Case 4b. Case3a odes not occur, and there is a $b \leq p(\alpha)$ which enters B at stage t. Then:

   – Set $g(\alpha)$, $p(\alpha)$, and $q(\alpha)$ to be undefined, if they are defined.

   Case 4c. There is an $x \leq \varphi_i(d^{\alpha}_{\tau_i}(m_i))[v]$ which enters X_{τ_i} at stage t, then

   – Set $\Gamma_{\tau_i}(m_i)$ to be undefined.

Therefore the cycle of rectification of $\Delta(B;k) \neq K(k)$ will produce a decreasing sequence $p(\alpha)$, and impose a sequence of conditional restraints $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$, until B changes below the most recent $p(\alpha)$, in which case, $\Delta(B;k)$ is rectified. During the procedure of the rectification, if $p(\alpha) \leq \varphi_1(d^{\alpha}_{\tau_1}(k))[v]$, then we define $g(\alpha)$ to be the maximal φ-uses of agitators of priority higher than that of the agitator we just enumerated into D. If there is a b such that $g(\alpha) < b \leq p(\alpha)$ and b enters B, then we allow α to receive special attention by updating the conditional restraint $\vec{r}(\alpha) = (g(\alpha), q(\alpha))$, in which case, if B will never change below $g(\alpha)$, then $\Theta(A;b) \neq B(b)$ will be preserved forever.

Lemma 3.5 ensures that if at a stage s, say, we picked up an agitator $d^{\alpha}_{\tau_i}(m_i)$ for which $m_i \leq k$ and $B_s \upharpoonright (g^{\alpha}_i(m_i)[v]+1) = B \upharpoonright (g^{\alpha}_i(m_i)[v]+1)$, then in the end, the procedure of the rectification of $\Delta(B;k) \neq K(k)$ will either end at a permanent inequality $\Theta(A;b) \neq B(b)$ for some b, or there is no agitator of α available, in which case, we build f, and reset Δ.

By the rectification of Δ, for any k, once an inequality $\Delta(B;k) \neq K(k)$ occurs, we will reach a stage at which either $\Delta(B;k)$ is rectified, or $\Delta(B)$ is reset. From this we know that for a fixed m_i, there is a stage s_0 such that for any $s > s_0$, if we rectify $\Delta(B;k) \neq K(k)$ and enumerate $d^{\alpha}_{\tau_i}(m_i)$ into D at stage s, then $m_i \leq k$, so by Lemma 3.5, $g(\alpha)$ is defined at stage s. From this, we can easily prove the following:

3.6 LEMMA. Suppose that $\varphi_0(d^{<\alpha}_{\tau_0}(\leq n_0))[s]$ are bounded, that f is built only finitely often , and that $K \not\leq_T B$. Then for every i, every $m > n_i$, if $g^{\alpha}_i(m)[s]$ are bounded, then so are $d^{\alpha}_{\tau_i}(m)[s]$.

Proof. Given i, and $m > n_i$, let s_0 be such that for any $s > s_0$

(a) f is not built at stage s.

(b) Both $g^{\alpha}_i(m)[s]$ and $B \upharpoonright (g^{\alpha}_i(m)[s]+1)$ have reached their limits.

(c) $\varphi_0(d^{<\alpha}_{\tau_0}(\leq n_0))[s]$ has reached its limit.

(d) If $d^{\alpha}_{\tau_i}(m)$ is enumerated into D, and we rectify $\Delta(B;k) \neq K(k)$ at stage s, then $m \leq k$.

Suppose that s_1 is the least stage $s > s_0$ at which we rectify $\Delta(B;k) \neq K(k)$ by enumerating $d^{\alpha}_{\tau_i}(m)$ into D. By the choice of s_0, and by Lemma 3.5, if we get a response of B-change during the procedure of the rectification of the current $\Delta(B;k) \neq K(k)$, then α receives special attention, and a permanent inequality $\Theta(B;b) \neq B(b)$ is created, in which case, $d^{\alpha}_{\tau_i}(m)$ will never be enumerated into D after the permanent inequality is created. Otherwise, then there will be a stage $> s_1$ at which we build f, contradicting the choice of s_0.

Lemma 3.6 follows. □

[*Remark.* The choice algorithm and lemma 3.5, the choice lemma are the crucial ideas in this section and in this proof of the theorem.]

With the intuitions in this subsection, we look at the formal instructions of the $\mathcal{S}$-strategy α in the next subsection.

3.2 Instructions for the $\mathcal{S}$-Strategy α

Suppose that the base marker of α will be bounded during the course of the construction. The main aim of α is to build a p.c. function f, and a Turing functional Δ.

α will proceed as follows:

1. If s is not α-expansionary, then do nothing.
2. If the value of $\varphi_0^*(d_{\tau_0}^{\leq\alpha}(n_0))$ has changed since $\alpha\hat{}\langle b\rangle$ was last visited, then
   – set $p_*(\alpha)$, $g_*(\alpha)$, $p(\alpha)$, $q(\alpha)$, $g(\alpha)$ and $\vec{r}(\alpha)$ to be undefined, if they are defined, and
   – let $\alpha\hat{}\langle b\rangle$ be eligible to act.
3. If $\vec{r}(\alpha)$ is defined, then go to step 9.
4. If there is an $x > n_0$ such that $\Delta(B; x) \downarrow= 0 \neq 1 = K(x)$, then let k be the least x, and go to step 9.
5. Otherwise, then let k be the least $x > n_0$ such that $\Delta(B; x) \uparrow$.
6. (Defining Δ) If:
   (6a) $p_*(\alpha)$, b^α, $g_1^\alpha(k) < \varphi_1(d_{\tau_1}^\alpha(k))$,
   (6b) $\Phi_1(B, X_1; d_{\tau_1}^\alpha(k)) \downarrow= 0 = D(d_{\tau_1}^\alpha(k))$, and
   (6c) $l(\Theta(A), B) > \varphi_1(d_{\tau_1}^\alpha(k))$. Then
   – define $\Delta(B; k) \downarrow= K(k)$ with $\delta(k) = \theta(\varphi_1(d_{\tau_1}^\alpha(k)))$, and
   – define $\delta^*(k) = \theta(\varphi_1(d_{\tau_1}^\alpha(k)))$.
7. (Well Ordering) (a) If either $p_*(\alpha) \geq \varphi_1(d_{\tau_1}^\alpha(k))$, or $b^\alpha \geq \varphi_1(d_{\tau_1}^\alpha(k))$ or $g_1^\alpha(k) \geq \varphi_1(d_{\tau_1}^\alpha(k))$, then enumerate $d_{\tau_1}^\alpha(k)$ into D.
8. Otherwise, then do nothing.
9. If repair$(\alpha) = (i, \tau_i, m_i, k)$, then let $k_0 = k$, otherwise, then let k_0 be the least $x > n_0$ such that $\Delta(B; x) \downarrow= 0 \neq 1 = K(x)$. Then:
   – set $k \leftarrow k_0$.
10. Let v be the stage at which the current $\Delta(B; k)$ was created, and execute the appropriate case below.
    Case 10a. If:
    (a) there is no $y > n_1$ such that $\gamma_1(y) \leq \delta(k)$, and
    (b) there is no $y > n_0$ such that $\gamma_0(y) \leq \delta(k)$, then:
    – for every $x \leq \varphi_1(d_{\tau_1}^\alpha(k))[v]$, if $f(x)$ is undefined, define $f(x) \downarrow= B(x)$,
    – initialise all nodes ξ with $\alpha\hat{}\langle -1\rangle <_{\mathrm{L}} \xi$,

– let $g_*(\alpha) = h_0^\alpha h_1^\alpha(n_1)[v]$,

– let $p_*(\alpha) = \max\{p_*(\alpha)[s-1],\ g_*(\alpha),\ b^\alpha[v] \mid p_*(\alpha)[s-1]\downarrow\}\}$,

– let $q_*(\alpha) = \delta(k)$,

– create a restraint vector $\vec{r}_*(\alpha) = (p_*(\alpha), q_*(\alpha))$ via (v,k),

– if B changes below $p_*(\alpha)$, then α requires special attention,

– set $g(\alpha)$, $p(\alpha)$, $q(\alpha)$ and $\vec{r}(\alpha)$ to be undefined, and

– set Δ to be totally undefined.

[*Remark.* (1). Let v_1 be the least α-expansionary stage after we get the disagreement $\Delta(B;k) \neq K(k)$. Then step 10 occurs st stage v_1. By the definition of $\Delta(B;k)$ at stage v, $B_v \upharpoonright (b^\alpha[v]+1) = B_{v_1} \upharpoonright (b^\alpha[v]+1)$. By the definition of $p(\alpha)$ at stages $\geq v_1$, we have that $B_v \upharpoonright (b^\alpha[v]+1) = B_s \upharpoonright (b^\alpha[v]+1)$. In case 10a, we transfer $b^\alpha[v]$ to $p_*(\alpha)$ to record the upper bound of all permitting markers of agitators associated with nodes both below α and to the left of $\alpha\hat{}\langle -1\rangle$ that may injure the computations $\Theta(A)[v]$ up to $\varphi_1(d^\alpha_{\tau_1}(k))[v]$ $(> b^\alpha[v])$. The role of $g_*(\alpha)$ and $p_*(\alpha)$ is to allow α to receive special attention whenever we find an element b bigger than $g_*(\alpha)$ and less than $p_*(\alpha)$ which is enumerated into B, then we are able to preserve an inequality $\Theta(A;b)\downarrow = 0 \neq 1 = B(b)$ by creating a restraint vector $\vec{r}(\alpha) = (g_*(\alpha), \delta(k))$, unless B has changed below $g_*(\alpha)$. From now on we ensure that any permitting marker of an agitator associated with strategies $\beta \supseteq \alpha\hat{}\langle b\rangle$ or $\beta \supseteq \alpha\hat{}\langle\omega\rangle$ must not be greater than $g_*(\alpha)$. This works because if B will never change, then α has preserved a permanent inequality $\Theta(A;b)\downarrow \neq B(b)$.

(2). One of the differences between this strategy and the $\mathcal{S}$-strategy in subsection 2.5 is as follows. In the case of subsection 2.5, before B changes below $g_*(\alpha)$, none of the parameters associated with nodes both below α and to the left of $\alpha\hat{}\langle -1\rangle$ changes. In this case, there maybe some node β with $\alpha\hat{}\langle b\rangle \subseteq \beta$, and some $y \leq n_0$ such that $d^\beta_{\tau_0}(y)$ is defined. In this case, although no node below $\alpha\hat{}\langle b\rangle$ is visited, whenever we define $\Gamma_0(y)$, we define $\varphi^*_0(d^\beta_{\tau_0}(y))$, and actually, this is done by the $\mathcal{R}_0$-strategy τ_0. The point is that if any such $\gamma_0(y)$ was created after stage v at which we created the current $\delta(k)$, then $\gamma_0(y)$ must be greater than $\delta(k)$, so that the honestification of $d^\beta_{\tau_0}(y)$ has never been delayed by the restraint vectors defined during the course of the rectification of $\Delta(B;k)$.

(3). The above analysis indicates that if α is visited at stage s, and s is α-expansionary, then the honestification of agitators of nodes below $\alpha\hat{}\langle b\rangle$ has never been injured by α, so that if step 2 occurs, we cancel $p_*(\alpha)$ and $p(\alpha)$ (if they are defined), and allow $\alpha\hat{}\langle b\rangle$ to act.]

Case 10b. If:

(a) there is no $y > n_1$ such that $\gamma_1(y) \leq \delta(k)$, and

(b) there is a $y > n_0$ such that $\gamma_0(y) \leq \delta(k)$, then

– let m_0 be the greatest $y > n_0$ such that $\gamma_0(y) \leq \delta(k)$,

– define $p(\alpha) = \max\{\varphi_i^*(d_{\tau_i}^{\leq\alpha}(\leq y))[v],\ b^\alpha[v],\ p_*(\alpha) \mid \gamma_i(y) \leq \delta(k),\ i = 0, 1, p_*(\alpha)\downarrow\}$,

– define the valid $\delta(k)$-use by $\delta^*(k) = p(\alpha)$,

– $q(\alpha) = \delta(k)$,

– create a restraint vector $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$ via (v, k),

– define $\text{repair}(\alpha) = (0, \tau_0, m_0, k)$,

– if $p(\alpha) \leq \varphi_1(d_{\tau_1}^\alpha(k))[v]$, then

- define $g(\alpha) = \max\{p_*(\alpha),\ g_0^\alpha(m_0)[v] \mid p_*(\alpha)\downarrow\}$.

– enumerate $d_{\tau_0}^\alpha(m_0)$ into D.

Otherwise, then

- Let m_1 be the greatest y such that $\gamma_1(y) \leq \delta(k)$.

Case 10c. If there is a $y > h_1^\alpha(m_1)[v]$ such that $\gamma_0(y) \leq \delta(k)$, then

– let m_0 be the greatest y such that $\gamma_0(y) \leq \delta(k)$,

– define $p(\alpha) = \max\{\varphi_i^*(d_{\tau_i}^{\leq\alpha}(\leq y))[v],\ b^\alpha[v],\ p_*(\alpha) \mid \gamma_i(y) \leq \delta(k),\ i = 0, 1, p_*(\alpha)\downarrow\}$,

– define the valid $\delta(k)$-use by $\delta^*(k) = p(\alpha)$,

– define $q(\alpha) = \delta(k)$,

– create a restraint vector $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$ via (v, k),

– define $\text{repair}(\alpha) = (0, \tau_0, m_0, k)$,

– if $p(\alpha) \leq \varphi_1(d_{\tau_1}^\alpha(k))[v]$, then

- define $g(\alpha) = \max\{p_*(\alpha),\ g_0^\alpha(m_0)[v] \mid p_*(\alpha)\downarrow\}$,

– enumerate $d_{\tau_0}^\alpha(m_0)$ into D.

Case 10d. Otherwise. Then

– define $p(\alpha) = \max\{\varphi_i^*(d_{\tau_i}^{\leq\alpha}(\leq y))[v],\ b^\alpha[v],\ p_*(\alpha) \mid \gamma_i(y) \leq \delta(k),\ i = 0, 1,\ p_*(\alpha)\downarrow\}$,

– define the valid $\delta(k)$-use by $\delta^*(k) = p(\alpha)$,

– define $q(\alpha) = \delta(k)$,

– create a restraint vector $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$ via (v, k),

– define $\text{repair}(\alpha) = (1, \tau_1, m_1, k)$,

– if $p(\alpha) \leq \varphi_1(d_{\tau_1}^\alpha(k))[v]$, then

- let $g(\alpha) = \max\{p_*(\alpha),\ g_1^\alpha(m_1)[v] \mid p_*(\alpha)\downarrow\}$,

– enumerate $d^{\alpha}_{\tau_1}(m_1)$ into D.

We say that α *requires special attention at stage* s, if either (1) or (2) below holds,

(1) $g_*(\alpha)\downarrow$, and there is an element b such that $g_*(\alpha) < b \leq p_*(\alpha)$ and b is enumerated into B at stage s.

(2) $g(\alpha)$ is defined, and there is an element b such that $g(\alpha) < b \leq p(\alpha)$ and b is enumerated into B at stage s.

Suppose that B is enumerated at odd stages, that at an odd stage, there is a unique element which is enumerated into B, and that s is an odd stage. Let b be the element which is enumerated into B at stage s. Then execute the following instructions:

11. (Receiving Special Attention) If α requires special attention at stage s, then let α receive special attention by one of the following cases.

    Case 11a. (1) holds at stage s. Then

    – set $p(\alpha) \leftarrow g_*(\alpha)$,

    – set $q(\alpha) \leftarrow q_*(\alpha)$,

    – create a restraint vector $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$,

    – set $g(\alpha)$ to be undefined, if it is defined, and

    – set $p_*(\alpha)$, $g_*(\alpha)$, and $q_*(\alpha)$ to be undefined, if they are defined.

    Case 11b. (2) occurs. Then

    – define $p(\alpha) = g(\alpha)$,

    – define $q(\alpha) = \delta(k)$, and

    – create a restraint vector $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$, and

    – set $g(\alpha)$ to be undefined, if it is defined.

12. (Updating Valid Use) For any τ, β, and y:

    (i) If $d^{\beta}_{\tau}(y)$ is defined, then define $\varphi^*_{\tau}(d^{\beta}_{\tau}(y))$ by

$$\varphi^*_{\tau}(d^{\beta}_{\tau}(y)) = \min\{\varphi^+_{\tau}(d^{\beta}_{\tau}(y)),\ p(\xi) \mid \tau \subset \xi \subseteq \beta,\ \vec{r}(\xi)\downarrow = (p(\xi), q(\xi)),\ \gamma_{\tau}(y) \leq q(\xi)\}.$$

    Let $x = \gamma_{\tau}(y)$.

    (ii) Define $m_{\tau}(x)$ by

$$m_{\tau}(x) = \max\{\varphi^*_{\tau}(d^{\beta}_{\tau}(y)) \mid d^{\beta}_{\tau}(y)\downarrow\}.$$

    (iii) Define p^{τ} by

$$p^\tau = \min\{p(\xi) \mid \xi \subset \tau, \vec{r}(\xi) \downarrow= (p(\xi), q(\xi))\}.$$

(iv) Define the *permitting marker of* x by

$$m(x) = \min\{m_\tau(x),\ p^\tau\}.$$

13. (Enumerating A) For every $x = \gamma_\tau(y)$ for some τ, and y, if $b \leq m(x)$, then

    – enumerate x into A.

14. (Dropping Restraint) For any β:

    – if $p(\beta) \downarrow\geq b$, then set $g(\beta)$, $p(\beta)$, $q(\beta)$, and $\vec{r}(\beta)$ to be undefined, if they are defined, and

    – if $p_*(\beta) \downarrow\geq b$, then set $g_*(\beta)$, $p_*(\beta)$, $q_*(\beta)$, $g(\beta)$, $p(\beta)$, $q(\beta)$, and $\vec{r}(\beta)$ to be undefined, if any.

The possible outcomes will be analysed in subsection 5.2, and defined in section 6 below.

4 Rules and Principles: Overview of the Proof

In the last two sections, we developed an $\mathcal{S}$-strategy working below one and two $\mathcal{R}$-strategies respectively. In each case, we developed a complicated permitting system to satisfy both $\mathcal{R}$-, and $\mathcal{S}$-requirements as well as the permitting requirement. Essentially, an agitator may be delayed so as to be honestified by a decreasing sequence of permitting markers. However during the course of the construction, at any stage, there are finitely many $\mathcal{S}$-strategies, each of which produces a decreasing sequence of permitting markers. We want to convince ourselves the permitting argument in the last two sections will still work in the general case. In this section, we outline some rules for the permitting marker systems which ensure that the permitting markers system works.

4.1 The Permitting Markers Systems

Permitting Markers. Before moving on to the general strategies, let us introduce some principles for the permitting markers, which will be useful throughout the paper.

Suppose that τ is an $\mathcal{R}$-strategy. Then for a natural number k, we use $d^\tau_\tau(k)$ to denote the agitator $d(k)$ defined by τ. Such an agitator $d^\tau_\tau(k)$ is called τ's own agitator. To satisfy $\mathcal{R}$, we ensure that τ's own agitators are always honest at the stages at which τ is visited. On the other hand, for a fixed k, there are finitely many $\mathcal{S}$-strategies α which define agitators $d^\alpha_\tau(k)$ for $\Gamma_\tau(k)$.

Then $\gamma_\tau(y)$ will be determined by the collection of all agitators $d^\beta_\tau(y)$. We use Q^y_τ to denote the set of all nodes β, including τ, such that $d^\beta_\tau(y)$ is defined.

We will ensure that $Q^y_\tau[s]$ is $\subseteq$-decreasing according to stages, i.e., if both $Q^y_\tau[s]$ and $Q^y_\tau[s+1]$ are defined, then $Q^y_\tau[s] \supseteq Q^y_\tau[s+1] \supset \emptyset$. [Notice that $\tau \in Q^y_\tau$ for almost every y.]

Conditional Restraint Let α be an $\mathcal{S}$-strategy. During the course of the construction, we may create a conditional restraint $\vec{r}(\alpha)$. The conditional restraint $\vec{r}(\alpha)$ will have the following properties:

(i) If both $\vec{r}(\alpha)[s] = (p(\alpha)[s], q(\alpha)[s])$ and $\vec{r}(\alpha)[s+1] = (p(\alpha)[s+1], q(\alpha)[s+1])$ are defined, then

$$p(\alpha)[s] \geq p(\alpha)[s+1].$$

(ii) If $\vec{r}(\alpha)[s] = (p(\alpha)[s], q(\alpha)[s])$ is defined, and there is a $b \leq p(\alpha)[s]$ which is enumerated into B at stage $s+1$, then either $\vec{r}(\alpha)[s+1]$ is undefined, or $p(\alpha)[s+1] < p(\alpha)[s]$.

(iii) If $\vec{r}(\alpha)[s]$ is defined, and $\vec{r}(\alpha)[s+1]$ is not defined, then one of the (3a)–(3b) below occurs,

(3a) α is initialised at stage $s+1$, and

(3b) There is an element $b \leq p(\alpha)[s]$ which is enumerated into B at stage $s+1$.

In either case, we have that for any agitator β, any τ, and any y, if $\alpha \subset \beta$, then either $d^\beta_\tau(y)$ is not defined at the end of stage $s+1$, or $\gamma_\tau(y) > q(\alpha)[s]$, or $p(\alpha)[s] > \varphi^+_\tau(d^\beta_\tau(y))$.

We say that (i)–(iii) are *conditional restraint rules.* Notice that the conditional restraints defined in sections 2 and 3 satisfy these properties. For every $\mathcal{S}$-strategy α, the conditional restraint rules will always be satisfied during the course of the construction.

4.1 DEFINITION. (Valid Use of an Agitator) For an agitator $d^\alpha_\tau(y)$, we define a *valid use of the agitator* $d^\alpha_\tau(y)$. Whenever we create $\gamma_\tau(y)$, we have that for all α, if $d^\alpha_\tau(y)$ is defined, then $\Phi(B, X; d^\alpha_\tau(y)) \downarrow= 0 = D(d^\alpha_\tau(y))$. Then we define $\Gamma_\tau(X, A; k) \downarrow= B(k)$ with $\gamma_\tau(y)$ fresh. And for each α, if $d^\alpha_\tau(y)$ is defined, then we define $\varphi^+(d^\alpha_\tau(y)) = \varphi(d^\alpha_\tau(y))$ to denote the $\varphi(d^\alpha_\tau(y))$-use at the stage at which $\gamma(y)$ is created, and define the *valid use* $\varphi^*_\tau(d^\alpha_\tau(y))$ of $d^\alpha_\tau(y)$ by $\varphi^*_\tau(d^\alpha_\tau(y)) = \varphi^+_\tau(d^\alpha_\tau(y))$.

During the course of the construction, we may update the valid use of $d^\alpha_\tau(y)$ by Definition 2.2.

[*Remark.* Notice that $\varphi^*(d^\tau_\tau(y))$ will always be $\varphi^+(d^\tau_\tau(y))$.]

This means that the B-honestification of $d^\alpha_\tau(y)$ is *delayed by* $\varphi^*(d^\alpha_\tau(y))$. Then $d^\alpha_\tau(y)$ *requires honestification* only if there is an element $b \leq \varphi^*(d^\alpha_\tau(y))$ which is enumerated into B.

By the conditional restraint rules outlined above, we have:

For any τ, any α and any y, if both $d^\alpha_\tau(y)$ and $\gamma_\tau(y)$ are defined, then

$\varphi^*_\tau(d^\alpha_\tau(y))[s]$ will be decreasing in stages. This means that for any s, if $\gamma_\tau(y)$ is defined, then for the stage v say, at which $\gamma_\tau(y)$ was created, we have:

$$\varphi^*_\tau(d^\alpha_\tau(y))[v] \geq \varphi^*_\tau(d^\alpha_\tau(y))[v+1] \geq \cdots \geq \varphi^*_\tau(d^\alpha_\tau(y))[s].$$

4.2 DEFINITION. (τ-Permitting Marker) However $\gamma_\tau(y)$ needs to be enumerated into A only if there is an agitator $d^\alpha_\tau(y)$ for some $\alpha \in Q^y_\tau$ which requires B-honestification. Therefore the *τ-permitting marker of* $\gamma_\tau(y)\downarrow = x$ is defined by

$$m_\tau(x) = \max\{\varphi^*_\tau(d^\alpha_\tau(y)) \mid d^\alpha_\tau(y)\downarrow\}.$$

A key point to our permitting argument is that for every $x = \gamma_\tau(y)$ for some τ, and some y, the τ-permitting marker $m_\tau(x)[s]$ will be decreasing in stages. This ensures that if x is delayed to be enumerated into A, then either x will prove to be undesirable in which case it is cancelled by initialisation, or B changes below the most recent τ-permitting marker $m_\tau(x)$, in which case, x is allowed to be enumerated into A.

To prove that the τ-permitting markers $m_\tau(x)[s]$ are decreasing in stages, we introduce the following:

4.3 LEMMA. (Minimax Lemma) Suppose that $Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_n \supset \emptyset$ is a sequence of finite sets Q_j of nodes, $j = 1, 2, \cdots, n$, and that for every j, and every node $\alpha \in Q_j$, there is a sequence $a^\alpha_1, \cdots, a^\alpha_j$ such that $a^\alpha_1 \geq a^\alpha_2 \geq \cdots \geq a^\alpha_j$.

For every $j = 1, 2, \cdots, n$, we define a_j by $a_j = \max\{a^\alpha_j \mid \alpha \in Q_j\}$. Then:

$$a_1 \geq a_2 \geq \cdots \geq a_n.$$

Proof. We prove the Lemma by induction on j. Clearly the Lemma holds for $j = 1$. Suppose by induction that it holds for all j' with $1 \leq j' \leq j < n$. By the assumption of the Lemma, for every $\alpha \in Q_{j+1}$, we have

$$a^\alpha_j \geq a^\alpha_{j+1}.$$

By definition of a_x, we have

$$\begin{aligned} a_j &\geq \max\{a^\alpha_j \mid \alpha \in Q_{j+1}\} \\ &\geq \max\{a^\alpha_{j+1} \mid \alpha \in Q_{j+1}\} = a_{j+1}. \end{aligned}$$

Lemma 4.3 follows. □

Suppose that $\Gamma_\tau(B, X_\tau; y)$ is defined at stage s. Let v be the stage at which $\gamma_\tau(y)$ was created. For each $j \in [v, s]$, let Q_j be the set of nodes α such that $d^\alpha_\tau(y)$ is defined at stage j. Then we have that $Q_v \supseteq Q_{v+1} \supseteq \cdots \supseteq Q_s$. For every $j \in [v, s]$, and for every $\alpha \in Q_j$, let $a^\alpha_j = \varphi^*_\tau(d^\alpha_\tau(y))[j]$. By the property of valid uses, we have that $a^\alpha_1 \geq a^\alpha_2 \geq \cdots \geq a^\alpha_j$ hold for all j, and all $\alpha \in Q_j$. By the definition of m_τ, for $x = \gamma_\tau(y)$, $m_\tau(x)[j] = \max\{a^\alpha_j \mid \alpha \in Q_j\}$, therefore by the Minimax Lemma, Lemma 4.3, we have the following:

$$m_\tau(x)[v] \geq m_\tau(x)[v+1] \geq \cdots \geq m_\tau(x)[s].$$

Therefore, τ-permitting markers are decreasing in stages.

By definition of $m_\tau(x)$, for $x = \gamma_\tau(y)$, we have:

There is an element $b \leq m_\tau(x)$ which enters B if and only if there is a node α (including τ itself) such that $d^\alpha_\tau(y)$ is defined, and there is an element $b \leq \varphi^*_\tau(d^\alpha_\tau(y))$ such that b is enumerated into B at the same stage, in which case, we say that $d^\alpha_\tau(y)$ *requires attention*, and let the agitator $d^\alpha_\tau(y)$ receive attention by enumerating its γ-marker $\gamma_\tau(y) = x$ into A.

However the above enumeration may be further delayed by an *absolute restraint* p^τ, which is defined by $p^\tau = \min\{p(\alpha) \mid \alpha \subset \tau, \vec{r}(\alpha) \downarrow= (p(\alpha), q(\alpha))\}$. A point is that if p^τ is defined, then τ is not eligible to be specified to act.

4.4 DEFINITION. (Permitting marker of a γ-use) Then for every $x = \gamma_\tau(y)$, the *permitting marker of* x is defined by

$$m(x) = \min\{m_\tau(x),\ p^\tau\}.$$

Suppose that $\Gamma_\tau(X_\tau, A; y)$ is defined. Let v be the stage at which $\gamma_\tau(y)$ was created. Then for $x = \gamma_\tau(y)$, the following property holds:

$$m(x)[v] \geq m(x)[v+1] \geq \cdots \geq m(x)[s].$$

This means that the permitting marker $m(x)$ is decreasing in stages, for every $x = \gamma_\tau(y)$ for some τ, y.

Then A will be enumerated as follows:

4.5 DEFINITION. (B-Honestification) Let $x = \gamma_\tau(y)$ for some τ, and some y. Then x is enumerated into A at a stage $s+1$ if and only if $B_s \upharpoonright (m(x)+1) \neq B_{s+1} \upharpoonright (m(x)+1)$ holds during stage $s+1$.

In the next subsection, we introduce some general principles which will be useful in the description of the general strategies and the full construction.

4.2 Basic Principles

4.6 DEFINITION. The construction will proceed by stages. At odd stages, we enumerate B, and we allow certain strategies to *receive attention*, if they *require attention.* And in even stages, we proceed a tree construction as usual. We say that a node ξ is *visited at stage* s, if s is even, and ξ is specified to be eligible to act at some substage t of stage s.

4.7 DEFINITION (Visiting Rules). If ξ is visited at stage s ($= 2n+2$ some n), then

(i) Every $\xi' \subset \xi$ is visited at stage s.

(ii) For every $\mathcal{S}$-strategy $\alpha \subset \xi$, we have:

(2a) if $\alpha\hat{\ }\langle 2\rangle \not\subseteq \xi$, then $\vec{r}(\alpha)$ is not defined after α is visited at stage s, but it is possible that $\vec{r}_*(\alpha)$ is defined.

(2b) if $\alpha\hat{\ }\langle b\rangle \subseteq \xi$, then both $\vec{r}_*(\alpha)$ and $\vec{r}(\alpha)$ are undefined after α is visited at stage s.

Before we enumerate any element into A at an odd stage s, we first allow certain $\mathcal{S}$-strategies to receive *special attention*. This will create and preserve some inequality $\Theta_\alpha(A;b)\downarrow\neq B(b)$, for some α and some b.

To satisfy $\mathcal{R}$, we ensure that the construction will satisfy the following:

4.8 DEFINITION ($\mathcal{R}$-Principle). Let τ be an $\mathcal{R}$-strategy. Then if τ is visited at stage s, then for any y, if $d^\tau_\tau(y)$ is defined, then $d^\tau_\tau(y)$ is honest at stage s.

The $\mathcal{R}$-principle ensures that if τ is an $\mathcal{R}$-strategy, and $\Gamma_\tau(X_\tau, A)$ is total, then $\Gamma_\tau(X_\tau, A) = B$. $\mathcal{R}$ is satisfied.

In addition, to make sure that an $\mathcal{S}$-strategy as we described in the last two sections are well defined, we need the following *$\mathcal{S}$-Principle*.

4.9 DEFINITION ($\mathcal{S}$-Principle). Given an $\mathcal{S}$-strategy α, if α is visited at stage s, and s is α-expansionary, then

– for any $\mathcal{R}$-strategy τ, and any y, if $d^\alpha_\tau(y)$ is defined, then $d^\alpha_\tau(y)$ is honest at stage s.

The restraint vector rules and the visiting rules will be useful in proving that both $\mathcal{R}$- and $\mathcal{S}$-principles will be satisfied during the course of the construction.

5 General Strategies

In this section, we describe and analyse the general strategies, which are necessary for us to build the formal definition of possible outcomes, and priority tree in section 6, and helpful to build the full construction in section 7.

5.1 A General $\mathcal{R}$-Strategy

Generally, an $\mathcal{R}$-strategy τ say, will build a Turing functional Γ_τ to show that $\Gamma_\tau(X_\tau, A) = B$. We fix an auxiliary set $D^\tau_\tau = \omega^{[i]}$ for some $i \geq 0$. For every $j \geq 0$, τ will define an agitator $d^\tau_\tau(j)$ for $\Gamma_\tau(j)$. $d^\tau_\tau(j)$ will be chosen from D^τ_τ. At any stage s, the agitators of τ will be a finite sequence of the form $d^\tau_\tau(0) < d^\tau_\tau(1) < \cdots < d^\tau_\tau(k)$..

In addition, τ will have to define agitators for certain $\mathcal{S}$-strategies α. Given an $\mathcal{S}$-strategy α, we say that τ is *active at* α, if $\tau\hat{\ }\langle 0\rangle \subseteq \alpha$ and α assumes that Γ_τ has not been destroyed yet. If τ is active at α, then we fix an auxiliary set $D^\alpha_\tau = \omega^{[i]}$ for some $i \geq 0$. α will work with a fixed *base point* $n(\tau, \alpha)$ for Γ_τ. For every $j > n(\tau, \alpha)$, we define an agitator $d^\alpha_\tau(j)$ of α for $\Gamma_\tau(j)$. $d^\alpha_\tau(j)$ will be chosen from D^α_τ. Of course, $d^\alpha_\tau(j)$ will be defined to satisfy the following property: $d^\alpha_\tau(n+1) < d^\alpha_\tau(n+2) < \cdots$, where $n = n(\tau, \alpha)$.

Given k, we define Q^k_τ by $Q^k_\tau = \{\tau,\ \alpha \mid n(\tau,\alpha)\downarrow< k\}$. Notice that Q^k_τ is a variable depending on stages, and that $Q^k_\tau[s] \subseteq Q^{k+1}_\tau[s]$ holds for all k and all s. The latter ensures that for any k, if $\gamma(k)$ is injured, then so is $\gamma(k+1)$, if any.

The honestification caused by an X_τ-change will proceed as follows.

5.1 DEFINITION. (X-Honestification of τ) (i) We say that $d^\alpha_\tau(j)$ *requires X-honestification at stage s*, if v is the stage at which $\gamma_\tau(j)$ was created, and there is an $x \leq \varphi_\tau(d^\alpha_\tau(j))[v]$ which is enumerated into X_τ at stage s.

(ii) We say that $\gamma_\tau(j)$ *requires X-honestification at stage s*, if there is an α such that $d^\alpha_\tau(j)$ requires X-honestification at stage s.

If $\gamma_\tau(j)$ requires X-honestification, then

- Set $\Gamma_\tau(X_\tau, A; j)$ to be undefined.

We require that the action of X-honestification will be executed automatically.

The basic constraint of τ is to satisfy the $\mathcal{R}$-principle prescribed in definition 4.8.

The *B-Honestification* is the same as that prescribed in Definition 4.5.

We recall that we don't need a separate step to rectify Γ_τ since the honestification of agitators $d^\tau_\tau(k)$ for all k has already ensured that if $\Gamma_\tau(X_\tau, A)$ is total, then $\Gamma_\tau(X_\tau, A) = B$.

The Building of Γ_τ. The Turing functional Γ_τ will be built as follows:

1. Let j be the least k such that $\Gamma_\tau(X_\tau, A; k)\uparrow$, set $d^\beta_\tau(k')$ are undefined for all β and for all $k' > j$.

   Let $Q^j_\tau = \{\tau, \alpha \mid \alpha \text{ is an } \mathcal{S} - \text{strategy},\ \tau \text{ is active at } \alpha \text{ and } n(\tau,\alpha)\downarrow < j\}$.

2. (Defining agitators) If there is a strategy $\beta \in Q^j_\tau$ such that either $d^\beta_\tau(j)$ is not defined, or $d^\alpha_\tau(j) \in D$, then let α be the $<$-least such β, set $d^{\alpha'}_\tau(j)$ to be undefined for all $\alpha' \supset \alpha$, if they are defined, and define $d^\alpha_\tau(j)$ to be the least element y satisfying the following conditions:

   (a) $y > d^{\alpha'}_\tau(j)$ for all $\alpha' < \alpha$ and $\alpha' \in Q^j_\tau$,

   (b) $y > d^\alpha_\tau(j')$ for all $j' < j$,

   (c) $y >$ old $d^\alpha_\tau(j)$,

   (d) $y > j$, and

   (e) $y \in D^\alpha_\tau$.

3. If:

   (a) for every $\alpha \in Q^j_\tau$, $d^\alpha_\tau(j)$ is defined, and

   (b) for every $\alpha \in Q^j_\tau$, $l(D, \Phi_\tau(B, X_\tau)) > d^\alpha_\tau(j)$, then:

   – define $\Gamma_\tau(X_\tau, A; j)\downarrow = B(j)$ with $\gamma_\tau(j)$ fresh,

   – for every $\alpha \in Q^j_\tau$, define the *valid use* of $d^\alpha_\tau(j)$ by $\varphi^*_\tau(d^\alpha_\tau(j)) = \varphi_\tau(d^\alpha_\tau(j))$, and define $\varphi^+_\tau(d^\alpha_\tau(j)) = \varphi_\tau(d^\alpha_\tau(j))$.

It is easy to see that if there are infinitely many τ-expansionary stages, then Γ_τ will be built infinitely often.

5.2 A General $\mathcal{S}$-Strategy

A general $\mathcal{S}$-strategy will satisfy its $\mathcal{S}$-requirement, $\Theta(A) \neq B$ say, while its priority ordering is given to finitely many $\mathcal{R}$-strategies, $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_l$ say, which are building Turing functionals $\Gamma_1, \Gamma_2, \cdots, \Gamma_l$ respectively. Suppose W.L.O.G. that τ_j is an $\mathcal{R}_j$-strategy, for $j \in \{1, 2, \cdots, l\}$, and that α is an $\mathcal{S}$-strategy. The $\mathcal{S}$-strategy α will work with a *base point* $n_j = n(\tau_j, \alpha)$ for Γ_j for every $j \in \{1, 2, \cdots, l\}$.

For α, we define some auxiliary functions h_j^α, f_j^α, and g_j^α for every j with $0 \leq j \leq l$.

5.2 DEFINITION. (i) For every $j \in \{1, 2, \cdots, l\}$, we define h_j^α, f_j^α, and g_j^α by

$$h_j^\alpha(x) = \max\{\varphi_j^*(d_{\tau_j}^\beta(y)),\ \varphi_j^*(d_{\tau_j}^\alpha(z)),\ x \mid \beta < \alpha,\ y \leq x, z < x\}$$

for all $x \in \omega$,

$$f_j^\alpha(x) = \max\{\varphi_j^*(d_{\tau_j}^\beta(y)),\ x \mid \beta \leq \alpha, y \leq x\}$$

$$g_1^\alpha(x) = h_1^\alpha(x)$$

$$g_j^\alpha(x) = f_1^\alpha \cdots f_{j-1}^\alpha h_j^\alpha(x).$$

Notice that $g_j^\alpha(x)$ is actually the maximal φ-uses of agitators with priority higher than that of $d_{\tau_j}^\alpha(x)$. We have to ensure that if $g_j^\alpha(x)[s]$ are bounded, then so is $d_{\tau_j}^\alpha(x)[s]$. On the other hand, the unboundedness of $g_{\tau_j}^\alpha(x)[s]$ allows us to prove inductively that $\Phi_{\tau_i}(B, X_{\tau_i})$ is partial for some $i \leq j$.

The base points n_j will satisfy the following properties:

(i) Whenever we define n_l, we define it afresh.

(ii) For every j with $1 < j \leq l$, $n_{j-1} = h_{j-1}^\alpha(n_j)$.

For every $y > n_i$, we define an agitator $d_{\tau_i}^\alpha(y)$ of α for Γ_{τ_i}. The well ordering of agitators among the l systems of agitators will be as follows. At the stage at which we define $\Delta(B; k)$, we require that

Well Ordering: $p_*(\alpha),\ b^\alpha,\ g_l^\alpha(k) < \varphi_{\tau_l}(d_{\tau_l}^\alpha(k))$.

As before, whenever we define $\Delta(B; k)$, we define both $\delta^*(k)$ and $\delta(k)$ to be $\theta\varphi_{\tau_l}(d_{\tau_l}^\alpha(k))$. Suppose that $\Delta(B; k) \downarrow= 0 \neq 1 = K(k)$. Then α will rectify the $\Delta(B; k)$ by enumerating some of its agitators $d_{\tau_j}^\alpha(m)$ into D. Let $p(\alpha)$ be the maximal $\varphi_{\tau_j}^*(d_{\tau_j}^{\leq\alpha}(\leq m))$ such that $m > n_j$, and $\gamma_{\tau_j}(m) \leq \delta(k)$. In the same way as in sections 2 and 3, after we enumerate an agitator $d_{\tau_j}^\alpha(m)$ into D, we are looking for either a B-change below $p(\alpha)$, or an X_{τ_j} change below $\varphi_{\tau_j}(d_{\tau_j}^\alpha(m))$. However our new problem is how to decide an agitator $d_{\tau_j}^\alpha(m)$ to enumerate into D? As we analysed in section 2 and section 3, we will have to ensure that if $m \leq k$, and B has been correct on the initial segment $g_{\tau_j}^\alpha(m)$, then there will be no B-change in response with agitation associated with this inequality $\Delta(B; k) \neq K(k)$, unless an inequality $\Theta(A; b) \neq B(b)$ will be created and preserved forever. This is fundamental, because it makes sure that the $\mathcal{S}$-strategies will never make any $\Gamma_{\tau_j}(B, X_{\tau_j})$ partial, unless some $\Phi_{\tau_i}(B, X_{\tau_i})$ is partial for some $i \leq j$. This suggests that we will choose and enumerate into D the agitator $d^{\alpha}_{\tau_j}(m)$ with the following properties:

(1) $m > n_j$,

(2) $\gamma_{\tau_j}(m) \leq \delta(k)$,

(3) $g^{\alpha}_j(m)[v]$ is maximal, where v is the stage at which $\delta(k)$ was created, and

(4) If $m \leq k$, then for all i, all $y > n_i$, if $\gamma_{\tau_i}(y) \leq \delta(k)$, then $\varphi_{\tau_i}(d^{\leq\alpha}_{\tau_i}(\leq y))[v] \leq \varphi_{\tau_l}(d^{\alpha}_{\tau_l}(k))[v]$.

Notice that at the stage at which we define $\Delta(B; k)$, we have that $l(\Theta(A), B) > \varphi_{\tau_l}(d^{\alpha}_{\tau_l}(k))$. Suppose that at a stage s we rectify $\Delta(B; k)$, that we enumerate $d^{\alpha}_{\tau_j}(m)$ into D, and that $B_s \upharpoonright (g^{\alpha}_j(m) + 1) = B \upharpoonright (g^{\alpha}_j(m) + 1)$. (3) ensures that during the procedure of the rectification of $\Delta(B; k)$ from stage s, once we get a B-change with an agitation of some remaining agitators, we will create and preserve an inequality $\Theta(A; b) \downarrow = 0 \neq 1 = B(b)$ for some b, and this inequality will be preserved forever. Therefore if we are not able to create such an inequality in the procedure of the rectification of $\Delta(B; k)$, we will reach a stage at which we build $f = B$ for one more time, in which case Δ is set to be totally undefined.

The well ordering of agitators which is executed at stages at which we build Δ, will ensure that once an inequality $\Delta(B; k) \downarrow = 0 \neq 1 = K(k)$ occurs, we can always choose an agitator $d^{\alpha}_{\tau_j}(m)$ with the properties (1)–(4) prescribed as above, unless for any i, and any y, if $y > n_i$, then $\gamma_{\tau_i}(y) > \delta(k)$, in the latter case, we build $f = B$.

We now describe an algorithm to decide the agitator to be enumerated into D.

THE CHOICE ALGORITHM. Let $v < s$. Suppose that $\Delta(B; k)$ was created at stage v, that $\Delta(B; k) \downarrow = 0 \neq 1 = K(k)$, and that we rectify $\Delta(B; k) \neq K(k)$ at an α-expansionary stage s. Then we implement the following algorithm to choose the agitator.

(1) Set:

– $c_l \leftarrow n_l$, and

– $j \leftarrow l$.

(2j) If there is a $y > c_j$ such that $\gamma_{\tau_j}(y) \leq \delta(k)$, then

– define m_j to be the greatest such y,

– if m_j is defined, then define $c_{j-1} = h^{\alpha}_j(m_j)[v]$, and

– if m_j is undefined, then define $c_{j-1} = f^{\alpha}_j(c_j)[v]$.

(3j) If $j = 1$, then go on to step (4), if $j > 1$, then set $j \leftarrow (j-1)$, and go back to step (2j).

(4) Let i be the least j such that m_j is defined. Output agitator $d^{\alpha}_{\tau_i}(m_i)$.

We prove some properties of the choice algorithm.

5.3 LEMMA. The choice algorithm terminates, unless there are no i, and y such that $y > n_i$, and $\gamma_{\tau_i}(y) \leq \delta(k)$ holds during stage s.

Proof. We prove by induction that for each j, if m_i is undefined, for every $i \in \{j, j+1, \cdots, l\}$, then $c_{j-1} = n_j$. For $j = l$, if m_l is not defined, then $c_{l-1} =$

$\varphi_l^* d_{\tau_l}^{\leq\alpha}(\leq n_l) = \varphi_l^* d_{\tau_l}^{<\alpha}(\leq n_l) = n_{l-1}$. The second equality holds because $d_{\tau_l}^{\alpha}(\leq n_l)$ are undefined, so non-available.

Suppose that the Lemma holds for all $j' > j$. Then $c_{j-1} = \varphi_j^* d_{\tau_j}^{\leq\alpha}(\leq n_j)$, since n_j is the base point of α for Γ_{τ_j}, so $d_{\tau_j}^{\alpha}(\leq n_j)$ are not defined. $c_{j-1} = n_{j-1}$ follows.

Lemma 5.3 holds.

5.4 LEMMA. Let $v < s$. $\Delta(B;k)$ was defined at stage v. Suppose that the choice algorithm outputs $d_{\tau_j}^{\alpha}(m_j)$ at stage s. Then:

(i) For any $i < j$, any y, if $n_i < y$, $d_{\tau_i}^{\alpha}(y)$ is defined, and and $\gamma_{\tau_i}(y) \leq \delta(k)$ holds during stage s, then

$$g_i^{\alpha}(y)[v] \leq g_j^{\alpha}(m_j)[v].$$

(ii) If $m_j \leq k$, then for any $i < j$, and y, if $n_i < y$, $d_{\tau_i}^{\alpha}(y)$ is defined, and $\gamma_{\tau_i}(y) \leq \delta(k)$ holds during stage s, then

$$\varphi_i^*(d_{\tau_i}^{\leq\alpha}(\leq y))[v] < \varphi_l(d_{\tau_l}^{\alpha}(k))[v].$$

5.5 LEMMA. Suppose that the choice algorithm outputs an agitator $d_{\tau_j}^{\alpha}(m_j)$. We have:

(i) For all i and y if $i \geq j$, $y > n_i$, and $\gamma_{\tau_i}(y) \leq \delta(k)$ holds at stage s, then $g_i^{\alpha}(y)[v] \leq g_j^{\alpha}(m_j)[v]$.

(ii) If $m_j \leq k$, then for any $i \geq j$, and any y, if $n_i < y$ and $\gamma_{\tau_i}(y) \leq \delta(k)$ holds at stage s, then $\varphi_{\tau_i}^*(d_{\tau_i}^{\alpha}(y))[v] \leq \varphi_{\tau_l}(d_{\tau_l}^{\alpha}(k))[v]$.

Lemmas 5.4 and 5.5 can be easily verified by extending the arguments given in section 3. The detailed proof will be given in the verification in section 8.

Combining Lemmas 5.4 and 5.5, we have:

5.6 LEMMA. (*The Choice Lemma*) If the choice Algorithm of Agitation outputs $d_{\tau_j}^{\alpha}(m_j)$ at stage s, then

(i) For any i, any y, if $n_i < y$, $d_{\tau_i}^{\alpha}(y)$ is defined, and $\gamma_{\tau_i}(y) \leq \delta(k)$ holds at stage s, then:

$$g_i^{\alpha}(y)[v] \leq g_j^{\alpha}(m_j)[v].$$

(ii) If $m_j \leq k$, then for any i, y, if $n_i < y$, $d_{\tau_i}^{\alpha}(y)$ is defined, and $\gamma_{\tau_i}(y) \leq \delta(k)$ holds at stage s, then

$$\varphi_{\tau_i}^*(d_{\tau_i}^{\leq\alpha}(\leq y))[v] \leq \varphi_{\tau_l}(d_{\tau_l}^{\alpha}(k))[v].$$

Lemma 5.3. ensures that if there is an inequality $\Delta(B;k) \neq K(k)$, then either we build $f = B$ or there will be an agitator $d_{\tau_j}^{\alpha}(m)$ which is available for enumeration into D. Lemma 5.6 ensures that for any j, and any $m > n_j$, if $d_{\tau_j}^{\alpha}(m)$ is enumerated after $B \upharpoonright (g_j^{\alpha}(m) + 1)$ has been stable at a stage we rectify $\Delta(B;k)$ with $m \leq k$, then during the course of the rectification of this $\Delta(B;k)$, either we will create and preserve a permanent inequality $\Theta(A;b)\downarrow=0\neq 1=B(b)$ for some b, or $f=B$ will be built in which case, Δ will be set to be totally undefined. This also ensures that every inequality $\Delta(B;k)\downarrow=0\neq 1=K(k)$ will either be rectified or be cancelled due to the reset of Δ. Therefore, for fixed j and m, there is a stage s_0 say, after which if we rectify an inequality $\Delta(B;k)\neq K(k)$ by enumerating $d^{\alpha}_{\tau_j}(m)$ into D, we have that $m\leq k$.

We conclude from the above argument that for fixed j,m, if $m>n_j$, and $g^{\alpha}_j(m)[s]$ are bounded, then so is $d^{\alpha}_{\tau_j}(m)[s]$. On the other hand if $g^{\alpha}_j(m)[s]$ are unbounded, we can prove by induction that there is an $i\leq j$, such that $\Phi_{\tau_i}(B,X_{\tau_i})$ is partial, a global win for $\mathcal{R}_i$.

The choice algorithm and Lemma 5.6, *the choice lemma* are the crucial new ideas for an $\mathcal{S}$-strategy and the proof of the theorem.

The same as before, we need to consider the base marker of α. We say that $\varphi^*_1(d^{<\alpha}_{\tau_1}(\leq n_1))$ is the *base marker* of α. At first, α will decide whether or not there is a base marker of α which becomes unbounded during the course of the construction. As before we use b to denote that there is a base marker $\gamma_j(k)$ for some $k\leq n_j$ which will be unbounded during the course of the construction, and that b is the leftmost outcome of α.

Decision for Base Markers. We use b to denote that $\varphi^*_1(d^{<\alpha}_{\tau_1}(\leq n_1))[s]$ are unbounded during the course of the construction. By the definition of n_j for j with $1\leq j\leq l$, if b is the correct outcome, then there is a $j\leq l$ such that $\Phi_j(B,X_{\tau_j})$ is partial. In any case, τ_l is no longer active at ξ for any ξ with $\alpha\hat{\ }\langle b\rangle\subset\xi$. However for $j<l$, it is possible that there are $\mathcal{S}$-strategies β such that $\alpha\hat{\ }\langle b\rangle\subseteq\beta$, and $d^{\beta}_{\tau_j}(y)$ are defined. To deal with the possible injury from nodes below $\alpha\hat{\ }\langle b\rangle$, we introduce the following parameters.

Define $b^{\alpha}_0=\varphi^+_1(d^{<\alpha}_{\tau_1}(\leq n_1))$, the base marker of α.

If $1\leq j<l$, then define b^{α}_j by

$$b^{\alpha}_j=\max\{\varphi^+_{\tau_j}(d^{\beta}_{\tau_j}(y))\mid\alpha\hat{\ }\langle b\rangle\subseteq\beta,y\in\omega\}.$$

Define b^{α}_l by

$$b^{\alpha}_l=\max\{\gamma_\tau(y)\mid\tau\supset\alpha\hat{\ }\langle b\rangle,y\in\omega\}.$$

Suppose that there are only finitely many stages at which the base marker b^{α}_0 changes. Then α will build a p.c. function f and a Turing functional Δ.

Notice that before we define $\Delta(B;k)$, we require that for all $j=1,2,\cdots,l$, $b^{\alpha}_j<\varphi_{\tau_j}(d^{\alpha}_{\tau_l}(k))$. Therefore, the well ordering of α at stages at which we define $\Delta(B;k)$ is the following:

$$\max\{p_*(\alpha),\ b^{\alpha}_j,g^{\alpha}_l(k)\mid j=1,2,\cdots,l\}<\varphi_{\tau_l}(d^{\alpha}_{\tau_l}(k)).$$

And using Lemma 5.6, we have

5.7 LEMMA. If:

(a) The base markers of α are bounded,

(b) $\lim_s \vec{r}(\alpha)[s]$ does not exists, and

(c) f is built only finitely often , then:

– for every i, j with $1 \leq j \leq i \leq l$, and for every $m > n_i$, if we have that $\lim_s b_j^\alpha[s] \downarrow= b_j^\alpha < \omega$ exists for each j with $1 \leq j \leq i$, and $\lim_s g_i^\alpha(m)[s] \downarrow= g_i^\alpha(m) < \omega$, then $\lim_s d_{\tau_i}^\alpha(m)[s] \downarrow= d_{\tau_i}^\alpha(m) < \omega$.

Proof. Let s_0 be minimal such that:

(a) for every $j \in \{1, 2, \cdots, i\}$, $\lim_s b_j^\alpha[s] \downarrow= b_j^\alpha[s_0]$, and

(b) f will never be built at any stage $s \geq s_0$.

Given j and $m > n_j$, let s_1 be minimal $> s_0$ such that $\lim_s g_j^\alpha(m)[s] \downarrow= g_j^\alpha(m)[s_1]$, and let s_2 be minimal $> s_1$ such that $B_{s_2} \upharpoonright (g_j^\alpha(m)[s_1] + 1) = B \upharpoonright (g_j^\alpha(m)[s_1]+1)$. Suppose to the contrary that $d_{\tau_j}^\alpha(m)[s]$ becomes unbounded. Therefore there are infinitely many stages at which repair$(\alpha) = (j, \tau_j, m, k)$ is created for some k. By the rectification of Δ, we can choose s_3 to be minimal $> s_2$ such that for any s, k, if repair$(\alpha) = (j, \tau_j, m, k)$ is created at stage s, then $m \leq k$. Let s_4 be minimal $> s_3$ at which repair$(\alpha) = (j, \tau_j, m, k)$ is created for some k.

By Lemma 5.4 and Lemma 5.5, for any i, and any y, the following properties hold at stage s_4,

(1) If $n_i < y$, and $\gamma_{\tau_i}(y) \leq \delta(k)$, then $\varphi_{\tau_i}(d_{\tau_i}^{\leq\alpha}(\leq y))[v] \leq \varphi_{\tau_l}(d_{\tau_l}^\alpha(k))[v]$.

(2) If $n_i < y$ and $\gamma_{\tau_i}(y) \leq \delta(k)$, then $g_i^\alpha(y)[v] \leq g_j^\alpha(m)$.

By the choice of s_2, $B_{s_2} \upharpoonright (g_j^\alpha(m)[v]+1) = B \upharpoonright (g_j^\alpha(m)[v]+1)$, therefore for any i and y, if $n_i < y$ and $\gamma_{\tau_i}(y)[s_4] \leq \delta(k)$, then $B_{s_4} \upharpoonright (g_{\tau_i}^\alpha(y)[v]+1) = B \upharpoonright (g_{\tau_i}^\alpha(y)[v]+1)$. By (1) and by the strategy, for any $s \geq s_4$, if we rectify $\Delta(B; k)$ at stage s, then we define both $p(\alpha)$ and $g(\alpha)$, so if any of these agitations gets a response of B-change below the current $p(\alpha)$, then α receives special attention, and then create and preserve a permanent inequality $\Theta(A; b) \downarrow= 0 \neq B(b)$ for some b. Otherwise, then during the course of the rectification of $\Delta(B; k)$ after (including) stage s_4, we will always get an X-change, so that some $\Gamma_{\tau_i}(y)$ has been lifted to be above $\delta(k)$, so that in the end, there will be no such agitator available, we reach a stage at which f is built and at the same time Δ is reset. A contradiction.

Lemma 5.7 follows.

We now analyse the possible outcomes of α.

Possible Outcomes of the $\mathcal{S}$-Strategy α

First, α will decide whether or not there is a base marker $\gamma_i(k)$ for some i, k such that $\gamma_i(k)[s]$ becomes unbounded during the course of the construction. For Γ_l, the base markers of α for Γ_l are $\gamma_l(0), \gamma_l(1), \cdots, \gamma_l(n_l)$, where $n_l = n(\tau_l, \alpha)$ is the base point of α for Γ_l. By the definition of n_l, for every $k \leq n_l$, $\gamma_l(k)$ will be determined by $\varphi_l(d_{\tau_l}^{<\alpha}(k))$, where $d_{\tau_l}^{<\alpha}(k) = \max\{d_{\tau_l}^\beta(k) \mid \beta < \alpha\}$. For $\beta <_{\mathrm{L}} \alpha$, we assume that $d_{\tau_l}^\beta(k)[s]$ is bounded during the course of the construction, because β is to the left of the true path TP. Suppose that $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_m$ comprise all $\beta \subset \alpha$ such that τ_l is active at β and $k < n(\tau_l, \beta)$, the base point of β for τ_l. So we only need to check if there is a $j \in \{1, 2, \cdots, m\}$ such that $d_{\tau_l}^{\beta_j}(k)[s]$ is unbounded during the course of the construction. We use $(d_{\tau_l}^{\beta_j}(k))$ to denote the outcome that $d_{\tau_l}^{\beta_j}(k)[s]$ is unbounded.

If for every $j \in \{1, 2, \cdots, m\}$, $d^{\beta_j}_{\tau_l}(k)[s]$ is bounded, then $\lim_s d^{<\alpha}_{\tau_l}(k)[s] \downarrow= d < \omega$, and $\varphi_l(d)[s]$ becomes unbounded. Of course in this case we have that $\gamma_l(k)[s]$ becomes unbounded.

We use $(\varphi_l(d^{<\alpha}_{\tau_l}(k))$ to denote the outcome that $\varphi_l(d^{<\alpha}_{\tau_l}(k))[s]$ are unbounded as s goes to infinity. However we must refine the possible outcome $(\varphi_l(d^{<\alpha}_{\tau_l}(k))$ as follows:

$$(d^{\beta_1}_{\tau_l}(k)) <_{\mathrm{L}} (d^{\beta_2}_{\tau_l}(k)) <_{\mathrm{L}} \cdots <_{\mathrm{L}} (d^{\beta_m}_{\tau_l}(k)) <_{\mathrm{L}}$$
$$(\varphi, l, \tau_l, p, 0) <_{\mathrm{L}} (\varphi, l, \tau_l, p, 1) <_{\mathrm{L}} \cdots <_{\mathrm{L}} (\varphi, l, \tau_l, p, k)$$

where the possible outcomes $(d^{\beta_j}_{\tau_l}(k))$ will be determined by β_j, and $(\varphi, l, \tau_l, p, j)$ means that j is the least x such that $\gamma_{\tau_l}(x)[s]$ will be unbounded.

For every $j \in \{1, 2, \cdots, l\}$, we have to decide whether or not there is a $k \leq n_j = n(\tau_j, \alpha)$ such that $\varphi^*_j(d^{<\alpha}_{\tau_j}(k))[s]$ becomes unbounded during the course of the construction.

Therefore for every $k \leq n(\tau_j, \alpha)$, $\gamma_j(k)$ will be determined by $\varphi_j(d^{<\alpha}_{\tau_j}(k))$. Of course we assume that if $\beta <_{\mathrm{L}} \alpha$, then $d^{\beta}_{\tau_j}(k)[s]$ is bounded.

Suppose that $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_m$ lists all $\beta \subset \alpha$ such that τ_j is active at β, and $n(\tau_j, \beta) < k$. Just as before, the possible outcomes of $(\varphi_j(d^{<\alpha}_{\tau_j}(k))$ are as follows:

$$(d^{\beta_1}_{\tau_j}(k)) <_{\mathrm{L}} (d^{\beta_2}_{\tau_j}(k)) <_{\mathrm{L}} \cdots <_{\mathrm{L}} (d^{\beta_m}_{\tau_j}(k)) <_{\mathrm{L}}$$
$$(\varphi, j, \tau_j, p, 0) <_{\mathrm{L}} (\varphi, j, \tau_j, p, 1) <_{\mathrm{L}} \cdots .$$

Now we define the possible outcomes of $\alpha\hat{\ }\langle b\rangle$ as follows:

$$(\varphi_l(d^{<\alpha}_{\tau_l}(0)) <_{\mathrm{L}} (\varphi_l(d^{<\alpha}_{\tau_l}(1)) <_{\mathrm{L}} \cdots <_{\mathrm{L}}$$
$$\cdots$$
$$(\varphi_1(d^{<\alpha}_{\tau_1}(0)) <_{\mathrm{L}} (\varphi_1(d^{<\alpha}_{\tau_1}(1)) <_{\mathrm{L}} \cdots <_{\mathrm{L}}$$

And define the possible outcomes of α by

$$b <_{\mathrm{L}} -1 <_{\mathrm{L}} \omega <_{\mathrm{L}} 2.$$

We now examine the possible outcomes of $\alpha\hat{\ }\langle\omega\rangle$. Suppose that $\alpha\hat{\ }\langle\omega\rangle$ is on the true path. We look at the following cases:

Case 1. f is built infinitely many times.

In this case, f is total, and $f =^* B$, contradicting the hypothesis of the theorem.

Case 2. Otherwise, and $\Delta(B)$ is total.

Then, $\Delta(B) =^* K$ and $K \leq_{\mathrm{T}} B$, again contradicting the hypothesis of the theorem.

Therefore $\Delta(B)$ is partial. We use (d,k) to denote that $\Delta(B;k)$ diverges. So the possible outcomes of $\alpha\hat{}\langle\omega\rangle$ are defined as follows:

$$(d, n_1+1) <_{\mathrm{L}} (d, n_1+2) <_{\mathrm{L}} \cdots ,$$

where n_1 is the base point of α for Γ_1.

To analyse the possible outcomes of $\alpha\hat{}\langle\omega\rangle\hat{}\langle(d,k)\rangle$, we first investigate the possible outcomes of $(d^{\alpha}_{\tau_j}(k))$, which means that $d^{\alpha}_{\tau_j}(k)[s]$ becomes unbounded during the course of the construction.

By Lemma 5.7, either $b^{\alpha}_j[s]$ does not exist, or $\lim_s g^{\alpha}_j(k)[s]$ does not exist. Note that $g^{\alpha}_j(k) = f^{\alpha}_1 \cdots f^{\alpha}_{j-1} h^{\alpha}_j(k)$.

Therefore the possible outcomes of $(d^{\alpha}_{\tau_j}(k))$ are as follows:

$$\begin{gathered}(b^{\alpha}_j) <_{\mathrm{L}} (\varphi_j(d^{<\alpha}_{\tau_j}(k)) <_{\mathrm{L}} \\ (\varphi_{j-1}(d^{\leq\alpha}_{\tau_{j-1}}(0)) <_{\mathrm{L}} (\varphi_{j-1}(d^{\leq\alpha}_{\tau_{j-1}}(1)) <_{\mathrm{L}} \cdots <_{\mathrm{L}} \\ \cdots \\ (\varphi_1(d^{\leq\alpha}_{\tau_1}(0)) <_{\mathrm{L}} (\varphi_1(d^{\leq\alpha}_{\tau_1}(1)) <_{\mathrm{L}} \cdots .\end{gathered}$$

The possible outcomes of $(\varphi_j(d^{\alpha}_{\tau_j}(k))$ are as follows:

$$(d^{\alpha}_{\tau_j}(k)) <_{\mathrm{L}} (\varphi, j, \tau_j, p, 0) <_{\mathrm{L}} (\varphi, j, \tau_j, p, 1) <_{\mathrm{L}} \cdots .$$

Finally, we define the possible outcomes of $\alpha\hat{}\langle\omega\rangle\hat{}\langle(d,k)\rangle$ as follows:

$$(\varphi_1(d^{\alpha}_{\tau_1}(k)) <_{\mathrm{L}} (\varphi_2(d^{\alpha}_{\tau_2}(k)) <_{\mathrm{L}} \cdots <_{\mathrm{L}} (\varphi_l(d^{\alpha}_{\tau_l}(k)) <_{\mathrm{L}} (\theta, e(\alpha), \alpha, p)$$

where $(\theta, e(\alpha), \alpha, p)$ means that $\Theta_\alpha(A)$ is partial.

The above gives the possible outcomes of α as an expanded subtree.

6 The Priority Tree T

In this section we build the tree of strategies. We use lower case Greek letters $\alpha, \beta, \gamma, \cdots$ to denote nodes of the priority tree.

6.1 DEFINITION (i) We define *the priority ranking of the requirements* as follows:

$$\mathcal{R}_0 < \mathcal{S}_0 < \mathcal{R}_1 < \mathcal{S}_1 < \mathcal{R}_2 < \mathcal{S}_2 < \cdots .$$

For requirements $\mathcal{X}$, $\mathcal{Y}$, we use $\mathcal{X} < \mathcal{Y}$ to denote that the priority ranking of $\mathcal{X}$ is higher than that of $\mathcal{Y}$.

(ii) *The possible outcomes of an $\mathcal{R}$-strategy* are $0 <_{\mathrm{L}} 1$, denoting infinitary and finitary actions respectively.

6.2 DEFINITION. Given a node ξ:

(i) We say that ξ is a *φ-node*, if $\xi = \xi^-\hat{}\langle(\varphi, e(\tau), \tau, p, k)\rangle$ for some $\mathcal{R}$-strategy τ and some k.

If $\xi = \xi^{-}\hat{}\langle(\varphi, e(\tau), \tau, p, k)\rangle$, then
– define $i^{\xi} = e(\tau)$,
– define $\tau^{\xi} = \tau$, and
– define $k^{\xi} = k$.

(ii) We say that ξ is a *θ-node*, if $\xi = \xi^{-}\hat{}\langle(\theta, e(\alpha), \alpha, p)\rangle$ for some $\mathcal{S}$-strategy α. If $\xi = \xi^{-}\hat{}\langle(\theta, e, \alpha, p)\rangle$, then we define:
– $e^{\xi} = e$, and
– $\alpha^{\xi} = \alpha$.

6.3 DEFINITION. Given a node ξ:

(i) We say that an $\mathcal{R}_e$-strategy τ *is active at* ξ, if:
(a) $\tau\hat{}\langle 0\rangle \subseteq \xi$, and
(b) there is no φ-node β, such that $\tau \subset \beta \subseteq \xi$ and $i^{\beta} \leq e = e(\tau)$.

(ii) We say that an $\mathcal{S}$-strategy α *is active at* ξ, if:
(a) $\alpha \subset \xi$, and
(b) $\alpha\hat{}\langle 2\rangle \not\subseteq \xi$.

6.4 DEFINITION. Given a node ξ:

(i) We say that ξ is an *i-node*, if $\xi = \xi^{-}\hat{}\langle i\rangle$ for some $i \in \{-1, 0, 1, 2\}$.

(ii) We say that ξ is a *d-node*, if $\xi = \xi^{-}\hat{}\langle(d^{\alpha}_{\tau}(k))\rangle$ for some $\mathcal{R}$-strategy τ, some $\mathcal{S}$-strategy α, and some $k \in \omega$.

(iii) We say that ξ is a *$\varphi(d)$-node*, if we have $\xi = \xi^{-}\hat{}\langle(\varphi_{\tau}(d^{\alpha}_{\tau}(k)))\rangle$ or $\xi = \xi^{-}\hat{}\langle(\varphi_{\tau}(d^{\leq\alpha}_{\tau}(k)))\rangle$ for some $\mathcal{R}$-strategy τ and $\mathcal{S}$-strategy α, and some $k \in \omega$.

6.5 DEFINITION. Given a node ξ, and an $\mathcal{R}$-strategy τ with $\tau\hat{}\langle 0\rangle \subseteq \xi$, we say that τ is *Σ^0_3-injured at* ξ, if there is a φ-node δ such that $\tau \subset \tau\hat{}\langle 0\rangle \subseteq \delta \subseteq \xi$ and such that $\tau^{\delta} = \tau$.

6.6 DEFINITION. Given a node ξ:

(i) If $\xi = \xi^{-}\hat{}\langle(\varphi_{\tau}(d^{\tau}_{\tau}(k)))\rangle$ for some $\mathcal{R}$-strategy τ, and some k, then the *possible outcomes of* ξ are as follows:

$$(\varphi, e, \tau, p, 0) <_{\mathrm{L}} (\varphi, e, \tau, p, 1) <_{\mathrm{L}} \cdots,$$

where $e = e(\tau)$.

(ii) If $\xi = \xi^{-}\hat{}\langle(\varphi_{\tau}(d^{\leq\alpha}_{\tau}(k)))\rangle$ for some $\mathcal{R}$-strategy τ, some $\mathcal{S}$-strategy α, and some $k \in \omega$, then:

(a) suppose that $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_m$ lists all $\beta \subseteq \alpha$ such that $k > n(\tau, \beta)$ and τ is active at β,

(b) the *possible outcomes* of ξ are defined as follows:

$$(d^{\beta_1}_{\tau}(k)) <_{\mathrm{L}} (d^{\beta_2}_{\tau}(k)) <_{\mathrm{L}} \cdots <_{\mathrm{L}} (d^{\beta_m}_{\tau}(k)) <_{\mathrm{L}}$$
$$(\varphi, e, \tau, p, 0) <_{\mathrm{L}} (\varphi, e, \tau, p, 1) <_{\mathrm{L}} \cdots,$$

where $e = e(\tau)$.

(iii) If $\xi = \xi^{-\frown}\langle(\varphi_\tau(d^\alpha_\tau(k)))\rangle$ for some τ, α, k, and $\alpha \neq \tau$, then the *possible outcomes* of ξ are as follows:

$$(d^\alpha_\tau(k)) <_{\mathrm{L}} (\varphi, e, \tau, p, 0) <_{\mathrm{L}} (\varphi, e, \tau, p, 1) <_{\mathrm{L}} \cdots,$$

where $e = e(\tau)$.

(iv) If $\xi = \xi^{-\frown}\langle(d^\alpha_\tau(k))\rangle$ for some τ, α and k, then:

(a) suppose that $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_l$ lists all $\mathcal{R}$-strategies which are active at α,

(b) suppose that $\tau = \tau_j$ for some $j \in \{1, 2, \ldots, l\}$,

(c) then the *possible outcomes* of ξ are defined as follows:

$$\begin{gathered}(b^\alpha_{\tau_j}) <_{\mathrm{L}} (\varphi_{\tau_j}(d^{<\alpha}_{\tau_j}(k))) <_{\mathrm{L}} \\ (\varphi_{\tau_{j-1}}(d^{<\alpha}_{\tau_{j-1}}(0))) <_{\mathrm{L}} (\varphi_{\tau_{j-1}}(d^{<\alpha}_{\tau_{j-1}}(1))) <_{\mathrm{L}} \cdots <_{\mathrm{L}} \\ \cdots \\ (\varphi_{\tau_1}(d^{<\alpha}_{\tau_1}(0))) <_{\mathrm{L}} (\varphi_{\tau_1}(d^{<\alpha}_{\tau_1}(1))) <_{\mathrm{L}} \cdots .\end{gathered}$$

where $b^\alpha_{\tau_j} = \max\{\varphi^+_{\tau_j}(d^\beta_{\tau_j}(y)) \mid \alpha^\frown\langle b\rangle \subseteq \beta\}$, so $b^\alpha_{\tau_j}$ will be expanded into

$$(\varphi, e(\tau_j), \tau_j, p, 0) <_{\mathrm{L}} (\varphi, e(\tau_j), \tau_j, p, 1) <_{\mathrm{L}} \cdots .$$

(v) If $\xi = \xi^{-\frown}\langle(\varphi, e, \tau, p, k)\rangle$ for some $e = e(\tau)$, τ and k, and there is an $\mathcal{R}$-strategy τ' such that $\tau \subset \tau' \subset \tau'^\frown\langle 0\rangle \subseteq \xi$, and that τ' is not Σ^0_3-injured at ξ, then:

(a) let τ^* be the shortest such τ', and let $e^* = e(\tau^*)$,

(b) define the *possible outcome* of ξ as follows:

$$(\varphi, e^*, \tau^*, p, 0) <_{\mathrm{L}} (\varphi, e^*, \tau^*, p, 1) < \cdots .$$

6.7 DEFINITION. Given an $\mathcal{S}$-strategy α, suppose that $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_l$ lists all $\mathcal{R}$-strategies which are active at α. Then:

(i) The *possible outcomes* of α are as follows:

$$\begin{gathered}(\langle b\rangle^\frown\varphi_{\tau_l}(d^{<\alpha}_{\tau_l}(0))) <_{\mathrm{L}} (\langle b\rangle^\frown\varphi_{\tau_l}(d^{<\alpha}_{\tau_l}(1))) <_{\mathrm{L}} \cdots <_{\mathrm{L}} \\ \cdots \\ (\langle b\rangle^\frown\varphi_{\tau_1}(d^{<\alpha}_{\tau_1}(0))) <_{\mathrm{L}} (\langle b\rangle^\frown\varphi_{\tau_1}(d^{<\alpha}_{\tau_1}(1))) <_{\mathrm{L}} \cdots <_{\mathrm{L}} \\ -1 <_{\mathrm{L}} \omega <_{\mathrm{L}} 2.\end{gathered}$$

(ii) The *possible outcomes* of $\alpha^\omega = \alpha^\frown\langle\omega\rangle$ are as follows:

$$(d, 0) <_{\mathrm{L}} (d, 1) <_{\mathrm{L}} (d, 2) <_{\mathrm{L}} \cdots .$$

(iii) The *possible outcomes* of $\alpha\hat{~}\langle\omega\rangle\hat{~}\langle(d,k)\rangle$ are as follows:

$$(\varphi_{\tau_1}(d^{\alpha}_{\tau_1}(k))) <_{\mathrm{L}} (\varphi_{\tau_2}(d^{\alpha}_{\tau_2}(k))) <_{\mathrm{L}} \cdots <_{\mathrm{L}} (\varphi_{\tau_l}(d^{\alpha}_{\tau_l}(k))) <_{\mathrm{L}} (\theta, e, \alpha, p),$$

where $e = e(\alpha)$.

6.8 DEFINITION. Given a node ξ:

(i) We say that ξ is a *real strategy* if one of the following conditions holds:

(a) ξ is the root node,

(b) ξ is an i-node for some $i \in \{0, 1, 2\}$,

(c) ξ is a θ-node, and $\xi = \xi^{-}\hat{~}\langle(\varphi, e, \tau, p, k)\rangle$ and for any $\mathcal{R}$-strategy τ', if $\tau \subset \tau' \subset \tau'\hat{~}\langle 0\rangle \subset \xi$, then τ' is Σ^0_3-injured at ξ.

(ii) We say that ξ is a *decision strategy*, if ξ is not a -1-node and not a real strategy.

We now define the satisfaction along a node.

6.9 DEFINITION. Given a node ξ:

(i) We say that $\mathcal{R}_e$ is *satisfied at* ξ if there is an $\mathcal{R}_e$-strategy τ such that either

(a) $\tau\hat{~}\langle 1\rangle$, or

(b) $\tau\hat{~}\langle 0\rangle \subseteq \xi$, there is no φ-node β such that $\tau \subset \beta \subseteq \xi$ and $i^{\beta} < e$, and there is a φ-node β such that $\tau \subset \beta \subseteq \xi$ and $i^{\beta} = e$.

(ii) We say that $\mathcal{R}_e$ *is active at* ξ, if $\mathcal{R}_e$ is not satisfied at ξ and there is an $\mathcal{R}_e$-strategy τ such that $\tau \subset \tau\hat{~}\langle 0\rangle \subseteq \xi$ and such that there is no φ-node β such that $\tau \subset \beta \subseteq \xi$ and $i^{\beta} \leq e$.

(iii) We say that $\mathcal{S}_e$ is *satisfied at* ξ if either there is an $\mathcal{S}_e$-strategy α such that $\alpha\hat{~}\langle 2\rangle \subseteq \xi$ or there is a θ-node β such that $\beta \subseteq \xi$ and $e^{\beta} = e$.

Then the priority tree is defined as follows.

6.10 DEFINITION. (i) Define the root node as an $\mathcal{R}_0$-strategy.

(ii) The immediate successors of a node are the possible outcomes of the corresponding strategy.

(iii) A decision strategy will not work on any requirement.

(iv) A real strategy ξ say, will work on the highest priority requirement which is not satisfied, and not active at ξ.

If ξ works on $\mathcal{R}_e$ for some e, then ξ is an $\mathcal{R}$-strategy with $e(\xi) = e$, and if ξ works on S_e for some e, then ξ is an $\mathcal{S}$-strategy with $e(\xi) = e$.

Using definition 6.10, we build the priority tree T as follows.

6.11 DEFINITION. (i) The priority tree T is defined as a c.e. set of all nodes which appear during the course of the construction.

(ii) At the stage at which a node ξ is mentioned for the first time in the construction, it is enumerated into T simultaneously and automatically.

We now prove some properties of the structure of the priority tree T.

6.12 PROPOSITION. Given a node ξ, suppose that $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_l$ are all $\mathcal{R}$-strategies $\tau \subset \xi$ which are active at ξ. Let $e_j = e(\tau_j)$, $j \in \{1, 2, \cdots, l\}$. Then $e_1 < e_2 < \cdots < e_l$.

Proof. Suppose to the contrary that the proposition fails to hold for ξ. Let $i < l$ be such that $e_1 < e_2 < \cdots < e_i$ but $e_i \geq e_{i+1}$. Because τ_i is active at ξ, for every α, if $\tau_i \subset \alpha \subseteq \xi$, then τ_i is active at α. By Definition 6.10, $e_i \neq e_{i+1}$. So $e_{i+1} < e_i$. By Definition 6.10, and by the choice of τ_{i+1}, $\mathcal{R}_{e_{i+1}}$ is active at τ_i, and there is a φ-node β such that $\tau_i \subset \beta \subseteq \tau_{i+1}$ with $i^\beta \leq e_{i+1} < e_i$. τ_i is not active at τ_{i+1}, and so τ_i is not active at ξ, contradicting the choice of τ_i.

Proposition 6.12 follows. $\square$

6.13 PROPOSITION. (Finite Injury Along Any Infinite Path Proposition) Suppose that P is an infinite path through the priority tree T. Then for any $\mathcal{R}$- or $\mathcal{S}$-requirement $\mathcal{X}$, there is a node $\xi_0 \in P$ such that either:

(i) $\mathcal{X}$ is satisfied at ξ for every ξ with $\xi_0 \subseteq \xi \in P$, or

(ii) $\mathcal{X}$ is active at ξ for every ξ with $\xi_0 \subseteq \xi \in P$.

Proof. The proof is by induction on the priority ranking of the requirements. For $\mathcal{R}_0$: Let λ be the root node. If $\lambda\hat{\ }\langle 1\rangle \in P$, then by Definition 6.9, for every ξ, if $\lambda\hat{\ }\langle 1\rangle \subseteq \xi \in P$, and so $\mathcal{R}_0$ is satisfied at ξ. If there is a φ-node β such that $\lambda\hat{\ }\langle 0\rangle \subseteq \beta \in P$ with $i^\beta = 0$, then by Definition 6.9, for any ξ, if $\beta \subseteq \xi \in P$, then $\mathcal{R}_0$ is satisfied at ξ. Otherwise, for any ξ, if $\lambda\hat{\ }\langle 0\rangle \subseteq \xi \in P$, then $\mathcal{R}_0$ is active at ξ.

Suppose by induction that the Proposition holds for all $\mathcal{X}' < \mathcal{X}$. Let ξ_0 be the shortest real strategy $\in P$ such that for every $\mathcal{X}' < \mathcal{X}$, either

(a) $\mathcal{X}'$ is satisfied at ξ for all ξ with $\xi_0 \subseteq \xi \in P$, or

(b) $\mathcal{X}'$ is active at ξ for all ξ with $\xi_0 \subseteq \xi \in P$.

By the choice of ξ_0 and by Definition 6.10, there is an $\mathcal{X}$-strategy $\subseteq \xi_0$. Let α be the longest $\mathcal{X}$-strategy $\beta \subseteq \xi_0$. By the choice of ξ_0 and α, α is the longest $\mathcal{X}$-strategy $\in P$. Now there are two cases:

Case 1. $\mathcal{X} = \mathcal{R}_e$ for some e.

The proof for this case is the same as that for the case of $\mathcal{X} = \mathcal{R}_0$.

Case 2. $\mathcal{X} = \mathcal{S}_e$ for some e.

By the choice of α, there are only two subcases:

Subcase 2a. $\alpha\hat{\ }\langle 2\rangle \in P$.

By Definition 6.9, for any ξ, if $\alpha\hat{\ }\langle 2\rangle \subseteq\in P$, then $\mathcal{S}_e$ is satisfied at ξ.

Subcase 2b. There is a $\beta = \alpha\hat{\ }\langle\omega\rangle\hat{\ }\langle(d,k)\rangle\hat{\ }\langle(\theta, e, \alpha, p)\rangle \in P$ for some k.

Then by Definition 6.9, for any ξ, if $\beta \subseteq \xi \in P$, then $\mathcal{S}_e$ is satisfied at ξ.

This completes the inductive proof.

Proposition 6.13 follows. $\square$

7 The Construction

In this section, we use the priority tree T built in the last section to describe the full construction. First we list some parameters which may be associated with a strategy.

For an $\mathcal{R}$-strategy τ, we introduce the following parameters and notations:

7.1 DEFINITION. Let τ be an $\mathcal{R}$-strategy. Then

(i) We define a *boundary*, $b(\tau)$ say. Whenever we define $b(\tau)$, we define it afresh, and for every $x \leq b(\tau)$, we define $\Gamma_\tau(X_\tau, A; x) \downarrow = 0$ with $\gamma_\tau(x) = -1$.

(ii) For every $j > b(\tau)$, we define an *agitator* d of τ for $\Gamma_\tau(X_\tau, A; j)$, denoted by $d^\tau_\tau(j)$.

(iii) We say that an agitator $d^\tau_\tau(j)$ is *honest*, if either $\gamma_\tau(j)$ is undefined or $B \upharpoonright (\varphi^+_\tau(d^\tau_\tau(j)) + 1)$ has not changed since $\gamma_\tau(j)$ was last created, and *dishonest*, otherwise.

(iv) We say that s is τ-*expansionary*, if:

(a) for any α, any j, if $d^\alpha_\tau(j) \downarrow$, then $l(D, \Phi_\tau(B, X_\tau))[s] > d^\alpha_\tau(j)$, and

(b) $l(D, \Phi_\tau(B, X_\tau))[s] > l(D, \Phi_\tau(B, X_\tau))[v]$ for all $v < s$ such that $s^- < v < s$ and τ was visited at stage v, with s^- = the stage at which τ was last initialised.

7.2 DEFINITION. Given an $\mathcal{S}$-strategy α, suppose that $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_l$ are all $\mathcal{R}$-strategies $\tau \subset \alpha$ which are active at α. Then the following parameters and notations are associated with α:

(i) Let $e_i = e(\tau_i, \alpha) = e(\tau_i)$, $1 \leq i \leq l$.

(ii) Define a *base point* of α for Γ_{τ_l}, denoted by $n_l(\alpha) = n(\tau_l, \alpha)$.

(iii) For every j, $1 < j \leq l$, if $n_j \downarrow$, then define $n_{j-1}(\alpha) = n(\tau_{j-1}, \alpha) = h^\alpha_j(n_j)$, where $h^\alpha_j(x) = \max\{\varphi^*_{\tau_j}(d^\beta_{\tau_j}(y)),\ \varphi^*_{\tau_j}(d^\alpha_{\tau_j}(z)),\ x \mid \beta < \alpha, y \leq x,\ z < x\}$.

(iv) For every $j \in \{1, 2, \cdots, l\}$, we say that n_j is the *base point* of α for Γ_{τ_j}.

(v) For every $j \in \{1, 2, \cdots, l\}$, and every $k > n_j$, we define an *agitator* $d^\alpha_{\tau_j}(k)$ of α for $\Gamma_{\tau_j}(k)$.

(vi) If $\Delta_\alpha(n_1 + 1) \downarrow$, we always assume that for every $x \leq n_1$ we have $\Delta_\alpha(x) \downarrow = 0$ with $\delta_\alpha(x) = -1$.

(vii) During the course of the construction, a *conditional restraint or restraint vector* $\vec{r}(\alpha)$ may be defined to be (p, q) for some $p, q \in \omega$.

(viii) If $d^\alpha_{\tau_j}(y)$ and $\gamma_{\tau_j}(y)$ are defined, let v be the stage at which $\gamma_{\tau_j}(y)$ was created, and

– define $\varphi^+_{\tau_j}(d^\alpha_{\tau_j}(y)) = \varphi_{\tau_j}(d^\alpha_{\tau_j}(y))[v]$.

(ix) If $d^{\alpha}_{\tau_j}(y)$ is defined, then define

$\varphi^*_{\tau_j}(d^{\alpha}_{\tau_j}(y)) = \min\{\varphi^+_{\tau_j}(d^{\alpha}_{\tau_j}(y)), p(\beta) \mid \tau_j \subseteq \beta \subseteq \alpha, \vec{r}(\beta) \downarrow= (p(\beta), q(\beta)),\ \gamma_{\tau_j}(y) \leq q(\beta)\}$.

(x) Define $b^{\alpha}_l = \max\{\gamma_{\tau}(y) \mid \alpha \subset \tau, \tau <_{\mathrm{L}} \alpha\hat{\ }\langle -1\rangle,\ y \in \omega\}$.

For j with $1 \leq j < l$, define b^{α}_j by

$b^{\alpha}_j = \max\{\varphi^+_{\tau_j}(d^{\beta}_{\tau_j}(y)) \mid \alpha \subset \beta,\ \beta <_{\mathrm{L}} \alpha\hat{\ }\langle -1\rangle\}$,

define

$b^{\alpha}_0 = \max\{\varphi^+_{\tau_1}(d^{\beta}_{\tau_1}(y)) \mid \beta < \alpha,\ y \leq n_1\}$,

and define

$b^{\alpha} = \max\{b^{\alpha}_j \mid j = 0, 1, \cdots, l\}$.

7.3 DEFINITION. (i) Suppose that Z is the c.e. set of all pairs (τ, τ) and (τ, α) with τ an $\mathcal{R}$-strategy, α an $\mathcal{S}$-strategy and with τ active at α.

(ii) Suppose that (τ, β) is the $(i+1)$-th element which is enumerated into Z. Then we set $D^{\beta}_{\tau} = \omega^{[i]}$.

Then any element of the form $d^{\beta}_{\tau}(j)$ will be chosen from D^{β}_{τ}.

7.4 DEFINITION. Given a node ξ:

(i) If ξ is a φ-node and $\xi = \xi^-\hat{\ }\langle(\varphi, e, \tau, p, k)\rangle$ for $e = e(\tau)$ and for some τ, k, then let $\gamma^{\xi} = \gamma_{\tau}(k)$.

(ii) Define a parameter $u^{\xi} = \min\{\gamma^{\beta} \mid \beta \subseteq \xi, \beta \text{ is a } \varphi - \text{node}\}$. We call u^{ξ} the *upper bound of* ξ.

Suppose that $\xi = \alpha$ is an $\mathcal{S}_e$-strategy. Then:

(iii) We say that $\Theta_e(A; x) \downarrow= y$ is α-*believable*, if $\theta_e(x) < u^{\xi}$.

(vi) Define the *length of agreement function* $l(\alpha) = \max\{x \mid (\forall y < x)[\Theta(A; y) \downarrow= B(y)$ via an $\alpha-$believable computation$]\}$.

(v) We say that s is α-*expansionary*, if $l(\alpha)[s] > l(\alpha)[v]$ for all $v < s$ at which α was visited.

During the course of the construction, we require that the use functions of Φ, Θ, Γ and Δ satisfy certain properties.

7.5 DEFINITION (Use Rules).

(i) Given a Turing functional Φ, the use function $\varphi(B, X)$ will satisfy the following properties. For any x, s:

(a) If $\varphi(x+1)[s] \downarrow$, then $\varphi(x)[s] \downarrow< \varphi(x+1)[s]$.

(b) If $\varphi(x)[s] \downarrow$, then $x \leq \varphi(x)[s]$.

(c) If both $\varphi(x)[s]$ and $\varphi(x)[s+1]$ are defined, then $\varphi(x)[s] \le \varphi(x)[s+1]$.

(d) If $\varphi(x)[s] \downarrow$, then $\varphi(x)[s+1] \uparrow$ if and only if either $B_s \upharpoonright (\varphi(x)[s]+1) \neq B_{s+1} \upharpoonright (\varphi(x)[s]+1)$ or $X_s \upharpoonright (\varphi(x)[s]+1) \neq X_{s+1} \upharpoonright (\varphi(x)[s]+1)$.

We say that (i) (a)– (d) above are *φ-rules*.

(ii) Given a Turing functional Θ, the use function $\theta(A)$ will satisfy the following properties: For any x, s,

(a) If $\theta(x+1)[s] \downarrow$, then $\theta(x)[s] \downarrow < \theta(x+1)[s]$.

(b) If $\theta(x)[s] \downarrow$, then $x \le \theta(x)[s]$.

(c) If both $\theta(x)[s]$ and $\theta(x)[s+1]$ are defined, then $\theta(x)[s] \le \theta(x)[s+1]$.

(d) If $\theta(x)[s] \downarrow$, then

$$\theta(x)[s+1] \uparrow \Longleftrightarrow A_s \upharpoonright (\theta(x)[s]+1) \neq A_{s+1} \upharpoonright (\theta(x)[s]+1).$$

We say that (ii) (a)– (d) are *θ-rules.*

(iii) Given an $\mathcal{R}$-strategy τ, the use function $\gamma_\tau(X_\tau, A)$ of Γ_τ will satisfy the following properties: For any $x > b(\tau)$ and any s,

(a) If $\gamma_\tau(x+1)[s] \downarrow$, then $\gamma_\tau(x)[s] \downarrow < \gamma_\tau(x+1)[s]$.

(b) Whenever $\gamma_\tau(x)$ is defined, we define it afresh.

(c) If both $\gamma_\tau(x)[s]$ and $\Gamma_\tau(x)[s+1]$ are defined, then we have $\gamma_\tau(x)[s] \le \gamma_\tau(x)[s+1]$.

(d) If $\gamma_\tau(x)[s] \downarrow$, and there is a strategy α such that $d^\alpha_\tau(x)[s] \downarrow$ and $X_{\tau,v} \upharpoonright (\varphi_\tau(d^\alpha_\tau(x)[v]+1) \neq X_{\tau,s+1} \upharpoonright (\varphi_\tau(d^\alpha_\tau(x))[v]+1)$, then $\gamma_\tau(x)[s+1]$ is undefined, where v is the stage at which $\gamma_\tau(x)[s]$ was created.

(e) If $\gamma_\tau(x)[s] \downarrow$ and $\gamma_\tau(x)[s+1] \uparrow$, then either

$$A_s \upharpoonright (\gamma_\tau(x)[s]+1) \neq A_{s+1} \upharpoonright (\gamma_\tau(x)[s]+1), \qquad \text{or}$$
$$X_{\tau,s} \upharpoonright (\gamma_\tau(x)[s]+1) \neq X_{\tau,s+1} \upharpoonright (\gamma_\tau(x)[s]+1).$$

(f) If $\Gamma_\tau(X_\tau, A; x)[s] \downarrow = 0 = B_s(x) \neq B_{s+1}(x)$, then there is a $v > s$ such that $\gamma_\tau(x)[s] \neq \gamma_\tau(x)[v]$.

We say that (iii) (a)– (f) are *γ-rules.*

(iv) Given an $\mathcal{S}$-strategy α, the use function $\delta_\alpha(B)$ will satisfy the following properties: For any $x > n_1(\alpha)$, and any s, we define $\delta(x)$, and $\delta^*(x)$ with the following properties:

(a) $\delta(x) \downarrow$ if and only if $\delta^*(x)$ is defined.

(b) if $\delta^*(x) \downarrow$, then $\delta^*(x) \le \delta(x)$.

(c) Let v be the stage at which the current $\delta(x)$ was created. Then:

$$\delta^*(x)[v] \geq \delta^*(x)[v+1] \geq \cdots .$$

(d) $\delta(x)$ becomes undefined, if and only if there is an element $b \leq \delta^*(x)$ which is enumerated into B.

(e) If there is an element b with $\delta^*(x) < b \leq \delta(x)$ which is enumerated into B, then we redefine $\Delta(B; x)$ to be the same value with the same use as it was last defined.

We say that $\delta^*(x)$ is the *valid use of* $\delta(x)$. We call (a)–(e) above δ-*rules.*

During the course of the construction, some strategies may be initialised.

7.6 DEFINITION (Initialisation).

(i) If an $\mathcal{R}$-strategy τ is initialised, then

(a) the Turing functional Γ_τ is set to be totally undefined,

(b) the boundary $b(\tau)$ is cancelled, and

(c) for any $j > b(\tau)$, $d^\tau_\tau(j)$ is cancelled.

(ii) If an $\mathcal{S}$-strategy α is initialised, then

(a) for every $i \in \{1, 2, \cdots, l\}$, $n_i(\alpha)$ is cancelled,

(b) for any i, j, $d^\alpha_i(j)$ is cancelled,

(c) Δ_α, which is located at $\alpha\hat{}\langle\omega\rangle$, is set to be totally undefined,

(d) f_α, which is located at $\alpha\hat{}\langle -1\rangle$, is set to be totally undefined,

(e) $\vec{r}(\alpha)$ is cancelled (if it is defined), and

(f) repair(α) is cancelled (if it is defined),

(g) $p_*(\alpha)$, $q_*(\alpha)$, and $g_*(\alpha)$ are cancelled, if they are defined.

Based on the rules of the permitting marker system prescribed in Definitions 4.1–4.4, we introduce the following:

7.7 DEFINITION. Given an $\mathcal{R}$-strategy τ, for every $j > b(\tau)$,
(i) we define Q^j_τ by

$$Q^j_\tau = \{\tau, \alpha \mid n(\tau, \alpha)\downarrow < j\}.$$

(ii) At a stage s, if $\gamma_\tau(j)$ is defined, let v be the stage at which $\gamma_\tau(j)$ was created, then we define:
– an X-*use* of $\gamma_\tau(j)$ by

$$x^j_\tau = \max\{\varphi^+_\tau(d^\alpha_\tau(j)) \mid \alpha \in Q^j_\tau\}$$

– for each $\alpha \in Q^j_\tau$, we define the *valid use of agitator* $d^\alpha_\tau(j)$ by Definition 7.2 (ix).

(iii) define the *τ-permitting marker of* $\gamma_\tau(j)$ by

$$m_\tau(\gamma_\tau(j)) = \max\{\varphi^*_\tau(d^\alpha_\tau(j)) \mid \alpha \in Q^j_\tau\}.$$

(iv) Define the *absolute restraint of* τ by
$p^\tau = \min\{p(\alpha),\ p(\beta) \mid \alpha\hat{}\langle\omega\rangle \subseteq \tau, \vec{r}(\alpha)\downarrow= (p(\alpha), q(\alpha)), \beta\hat{}\langle b\rangle \subseteq \tau, \vec{r}(\beta)\downarrow= (p(\beta), q(\beta))$ was created at an odd stage$\}$.

[*Remark.* (i) When $\vec{r}(\alpha)$ is created at an even stage, all nodes ξ with $\alpha\hat{}\langle 2\rangle \leq \xi$ are initialised.

(ii) If $\vec{r}(\alpha) = (p, q)$ is created at an even stage, then for any $\mathcal{R}$-strategy τ, any y, if $\gamma_\tau(y) \leq q$, then $p \geq \gamma_\tau(y)$.

So the p^τ here plays the same role as that required in Definition 4.4.]

(v) For $j > b(\tau)$, then for $x = \gamma_\tau(j)$, we define the *permitting marker of* x by

$$m(x) = \min\{p^\tau,\ m_\tau(x)\}.$$

7.8 DEFINITION. Given an $\mathcal{S}$-strategy α and a stage s:

(i) We say that an agitator $d^\alpha_{\tau_i}(j)$ of α for $\Gamma_{\tau_i}(j)$ for some i, j is *honest at stage* s, if either $\gamma_{\tau_i}(j)$ is undefined or $B \upharpoonright (\varphi^+_{\tau_i}(d^\alpha_{\tau_i}(j)) + 1)$ has not changed since the stage v at which $\gamma_{\tau_i}(j)$ was created. And is *dishonest at stage* s otherwise.

(ii) We say that α *requires special attention at stage* s, if either (8a) or (8b) below holds:

(8a) $g_*(\alpha)$ is defined, and there is an element b such that $g_*(\alpha) < b \leq p_*(\alpha)$ and b is enumerated into B at stage s.

(8b) $g(\alpha)$ is defined, and there is an element b such that $g(\alpha) < b \leq p(\alpha)$ and b is enumerated into B at stage s.

If α requires special attention at stage s, then α *receives special attention* as follows.

Case 1. (8a) occurs. Then
– set $p(\alpha) \leftarrow g_*(\alpha)$,
– set $q(\alpha) \leftarrow q_*(\alpha)$, and
– create a restraint vector $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$, and
– set $\vec{r}_*(\alpha)$ to be undefined.

Case 2. (8b) occurs. Then
– set $p(\alpha) \leftarrow g(\alpha)$,
– redefine $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$.

7.9 DEFINITION. Suppose that $\gamma_\tau(y)$ is defined, and $\alpha \in Q^y_\tau$.

(i) We say that the agitator $d^\alpha_\tau(y)$ *requires attention at stage* s, if there is an element $b \leq \varphi^*_\tau(d^\alpha_\tau(y))$ which has been enumerated into B since the current valid use $\varphi^*_\tau(d^\alpha_\tau(y))$ was created;

(ii) We say that $\gamma_\tau(y)$ *requires attention at stage* s, if there is an $\alpha \in Q^y_\tau$ such that $d^\alpha_\tau(y)$ requires attention at stage s.

7.10 DEFINITION (X-Honestification) Given an $\mathcal{R}$-strategy τ, the X_τ-honestification will be executed automatically as prescribed in Definition 5.1.

Given an $\mathcal{S}$-strategy α, suppose that $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_l$ are all $\mathcal{R}$-strategies $\tau \subset \alpha$ which are active at α. Then the following parameters and notations are associated with α:

7.11 DEFINITION. For every $j \in \{1, 2, \cdots, l\}$, we define for all x,
h_j^α, f_j^α, and g_j^α as the same as that in Definition 5.2.

Note that the priority tree T and the c.e. set Z in 7.1 will be built automatically. Before describing the construction, we fix an enumeration of a given c.e. set.

Suppose that the c.e. sets B and X_τ for all $\mathcal{R}$-strategies τ are enumerated as follows.

(i) The c.e. set B will be enumerated by $\{B_s \mid s \in \omega\}$ in such a way that $B_0 = \emptyset$, $|B_{2n+1} - B_{2n}| = 1$, and $B_{2n+2} = B_{2n+1}$, for all $n \in \omega$.

Given an $\mathcal{R}$-strategy τ:

(ii) We say that s is a τ-*stage*, if $s = 2n$ for some $n > 0$, and τ is visited at stage s.

(iii) X_τ will be enumerated only at τ-stages, and at a τ-stage s, there is at most one element which is enumerated into X_τ precisely at stage s.

(iv) If $X_{\tau,s} \neq X_{\tau,s-1}$ and s is a τ-stage, then let $x(\tau, s)$ be the unique element of $X_{\tau,s} - X_{\tau,s-1}$.

To distinguish actions at even stages from those at odd stages, we assume the conventions described in Definition 4.6.

We now describe the construction.

7.12 DEFINITION The Full Construction will proceed as follows.

Stage $s = 0$. Set $A = D = T = Z = \emptyset$.

Stage $s = 2n + 1$. Let b be the unique element $x \in B_s - B_{s-1}$. Then run the following program:

1. (Initialisation) If there is an $\mathcal{S}$-strategy β such that $\vec{r}(\beta) \downarrow = (p, q)$ for some p, q such that $b \leq p$, then

   – let α be the least such β,

   – initialise all nodes ξ with $\alpha\hat{\ }\langle 2\rangle \leq \xi$, and

   – go back to step 1.

2. (Receiving special attention) If there is an $\mathcal{S}$-strategy β such that β requires special attention at stage s, then let α be the $<$-least such β, and let α *receive special attention* as follows:

   Case 2a. If $g_*(\alpha) < b \leq p_*(\alpha)$, then

   – set $p(\alpha) = g_*(\alpha)$, $q(\alpha) = q_*(\alpha)$,

– let (v, k) be such that $\vec{r}_*(\alpha)$ was defined via (v, k),

– create a restraint vector $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$, and we say that $\vec{r}(\alpha)$ is defined via (v, k)

– set $g_*(\alpha)$, $p_*(\alpha)$, and $q_*(\alpha)$ to be undefined,

– set $g(\alpha)$ to be undefined, if it is defined, and

– initialise all nodes ξ with $\alpha\hat{\ }\langle -1\rangle <_{\mathrm{L}} \xi$, and go back to step 2.

Case 2b. Otherwise. Then:

– let $k,\ v$ be such that $\vec{r}(\alpha)$ was defined via (v, k),

– define $p(\alpha) = g(\alpha),\ \ q(\alpha) = \delta(k)$,

– define $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$ via (v, k),

– set $g(\alpha)$ to be undefined,

– set repair(α) to be undefined, and

– initialise any δ with $\alpha\hat{\ }\langle 2\rangle \leq \delta$ and go back to step 2.

3. (Updating Permitting Markers) In increasing order of x, for every x, if $x = \gamma_\tau(y)$ for some τ, some y, then update the permitting marker $m(x)$ of x as prescribed in definition 7.7.

   Given an $\mathcal{R}$-strategy τ, we say that τ *requires attention at stage* s, if there is a y such that $\gamma_\tau(y)$ requires attention at stage s.

4. (Honestification and rectification) If there is an $\mathcal{R}$-strategy τ' such that τ' requires attention at stage s, then:

   – let τ be the $<$-least such τ',

   – for every y, and for $x = \gamma_\tau(y)$ if $b \leq m(x)$, then enumerate $\gamma_\tau(y)$ into A,

   – we say that τ receives attention at stage s,

   – initialise any δ with $\tau\hat{\ }\langle 0\rangle <_{\mathrm{L}} \delta$ and go back to step 4.

5. (Dropping restraints) There are two cases.

   Case 5a. There is an $\mathcal{S}$-strategy β such that $\vec{r}(\beta)\downarrow= (p, q)$ for some p, q and $b \leq p$. Then:

   – let α be the $<$-least such β,

   – if $\vec{r}(\alpha)$ is defined, then set $\vec{r}(\alpha)$ to be undefined.

   Case 5b. There is an $\mathcal{S}$-strategy β such that $p_*(\beta)$ is defined, and $b \leq p_*(\beta)$. Then:

   – let α be the $<$-least such β,

   – set $g_*(\alpha), p_*(\alpha)$, $q_*(\alpha)$, $g(\alpha)$, and $\vec{r}(\alpha)$ to be all undefined, if any.

   In either case, go back to step 5.

6. Otherwise, go to stage $s + 1$.

Stage $s = 2n + 2$. We say that ξ is *visited at stage* s, if it is *eligible to act at a substage* t *of stage* s. First, we allow the root node λ to be eligible to act at substage $t = 0$.

Substage t. Let ξ be eligible to act at substage t of stage s. If $t = s$, then initialise any δ with $\xi <_{\mathrm{L}} \delta$ and go to stage $s + 1$. If $t < s$, then there are three cases:

Case 1. $\xi = \tau$ is an $\mathcal{R}$-strategy. Then run the following:

Program τ.

1. If $b(\tau)$ is undefined, then define it afresh, initialise all $\delta \not\leq \tau$ and go to stage $s + 1$.

   Suppose that $b(\tau)$ is defined.

2. (X-Honestification) If there is a $k > b(\tau)$ and an $\alpha \in Q^k_\tau$ such that

   (a) $\gamma_\tau(k) \downarrow$,

   (b) $d^\alpha_\tau(k)$ is defined, and

   (c) $X_{\tau,v} \upharpoonright (\varphi_\tau(d^\alpha_\tau(k))[v] + 1) \neq X_{\tau,s} \upharpoonright (\varphi_\tau(d^\alpha_\tau(k))[v] + 1)$, where v is the stage at which $\gamma_\tau(k)$ was created, then:

   – let j be the least such k,

   – set $\Gamma_\tau(X_\tau, A; x)$ to be undefined for all $x \geq j$,

   – initialise any δ with $\tau\hat{}\langle 0\rangle <_{\mathrm{L}} \delta$ and go to stage $s + 1$.

3. If s is not τ-expansionary, then let $\tau\hat{}\langle 1\rangle$ be eligible to act next.

4. Otherwise. Then:

   – let k be the least x such that $\Gamma_\tau(X_\tau, A; x) \uparrow$,

   – set $d^\beta_\tau(k')$ to be undefined for all β and for all $k' > k$, and

   – let $Q^k_\tau = \{\tau, \alpha \mid n(\tau, \alpha) \downarrow < k\}$.

5. (Defining Γ_τ) If:

   (a) for every $\beta \in Q^k_\tau$, $d^\beta_\tau(k) \downarrow \notin D$, and

   (b) for every $\beta \in Q^k_\tau$, $\Phi_\tau(B, X_\tau; d^\beta_\tau(k)) \downarrow = 0 = D(d^\beta_\tau(k))$, then:

   – define $\Gamma_\tau(X_\tau, A; k) \downarrow = K(k)$ with $\gamma_\tau(k)$ fresh,

   – for every $\alpha \in Q^k_\tau$, define $\varphi^*_\tau(d^\alpha_\tau(k)) = \varphi^+_\tau(d^\alpha_\tau(k)) = \varphi_\tau(d^\alpha_\tau(k))$, and

   – let $\tau\hat{}\langle 0\rangle$ be eligible to act next.

6. (Defining an agitator) If there is a $\beta \in Q^k_\tau$ such that either $d^\beta_\tau(k)$ is undefined, or $d^\beta_\tau(k) \downarrow \in D$, then:

   – let α be the $<$-least such β,

   – let $d^\beta_\tau(k)$ be undefined for all $\beta \supset \alpha$, and

   – define $d^\alpha_\tau(k)$ to be the least y satisfying the following conditions:

(a) $y \in D^{\alpha}_{\tau}$, and $y \notin D$,

(b) $y > d^{\gamma}_{\tau}(k)$ for all $\gamma < \beta$ with $\gamma^{\gamma}_{\tau}(k) \downarrow$,

(c) $y > d^{\alpha}_{\tau}(k')$ for all $k' < k$ such that $d^{\alpha}_{\tau}(k') \downarrow$, and

(d) $y >$ old $d^{\alpha}_{\tau}(k)$

(e) $y > k$,

– and return to step 6.

7. Otherwise. Then initialise any δ with $\tau\hat{}\langle 0\rangle <_{\mathrm{L}} \delta$ and go to stage $s+1$.

Case 2. $\xi = \alpha$ is an $\mathcal{S}$-strategy. Then run the following:

Program α.

1. If s is not α-expansionary, then let $\alpha\hat{}\langle 2\rangle$ be eligible to act next.

   Suppose that s is α-expansionary.

2. If b^{α}_{0} has changed since $\alpha\hat{}\langle b\rangle$ was last visited, then

   – if $\vec{r}(\alpha)$ is defined, then set it to be undefined,

   – if $p_*(\alpha)$ is defined, then set $g_*(\alpha)$, $p_*(\alpha)$, and $q_*(\alpha)$ to be undefined, and

   – let $\alpha\hat{}\langle b\rangle$ be eligible to act next.

3. If $\vec{r}(\alpha)$ is defined, then go to step 12.

4. If there is an $x > n_1 = n(\tau_1, \alpha)$ such that $\Delta_{\alpha}(B; x) \downarrow = 0 \neq 1 = K(x)$, then go to step 12.

5. If $l = 0$, then:

   – for every x, if $\Theta_{e(\alpha)}(A; x) \downarrow = B(x)$ via an α-believable computation and $f_{\alpha}(x) \uparrow$, then define $f_{\alpha}(x) = B(x)$,

   – initialise all $\delta \not\leq \alpha$ and go to stage $s+1$.

6. (Defining the base point) If n_l is not defined, then

   – define $n_l = n(\tau_l, \alpha)$ to be fresh,

   – initialise any δ with $\delta \not\leq \alpha$ and go to stage $s+1$.

7. Otherwise, let k be the least $x > n_1$ such that $\Delta_{\alpha}(B; x) \uparrow$.

8. If:

   (8a) $p_*(\alpha)$, b^{α}, $g^{\alpha}_{l}(k) < \varphi_{\tau_l}(d^{\alpha}_{\tau_l}(k))$, and

   (8b) $l(\Theta_{\alpha}(A), B) > \varphi_{\tau_j}(d^{\alpha}_{\tau_j}(y))$, then:

   – define $\Delta_{\alpha}(B; k) \downarrow = K(k)$ with $\delta_{\alpha}(k) = \theta_{\alpha}(\varphi_{\tau_l}(d^{\alpha}_{\tau_l}(k)))$, and

   – define the valid use $\delta^*(k)$ of $\delta_{\alpha}(k)$ by $\delta^*_{\alpha}(k) = \delta_{\alpha}(k)$,

   – initialise all nodes ξ with $\alpha\hat{}\langle\omega\rangle <_{\mathrm{L}} \xi$, and go to stage $s+1$.

9. If $\theta_\alpha(\varphi_{\tau_l}(d^\alpha_{\tau_l}(k)))$ has changed since $\alpha\hat{}\langle\omega\rangle\hat{}\langle(d,k)\rangle$ was last either initialised or visited, then let $\alpha\hat{}\langle\omega\rangle\hat{}\langle(d,k)\rangle$ be eligible to act next.

10. (Well ordering) Otherwise, if either $p_*(\alpha) \geq \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))$, or $b^\alpha \geq \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))$, or $g^\alpha_l(k) \geq \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))$, then:
    – enumerate $d^\alpha_{\tau_l}(k)$ into D,
    – initialise all nodes ξ with $\alpha\hat{}\langle 2\rangle \leq \xi$, and go to stage $s+1$.

11. Otherwise, then do nothing.

12. If repair$(\alpha) = (i, \tau_i, m_i, k)$, then let $k_0 = k$, otherwise, then let k_0 be the least $x > n_1$ such that $\Delta_\alpha(B; x) \downarrow = 0 \neq 1 = K(x)$, and go on to step 13.

13. Then:
    – set $k \leftarrow k_0$,
    – set $c^\alpha_l = n_l$,
    – set $j \leftarrow l$, and go to step 14j.

14j. (Defining c^α_{j-1}) Then, if there is a $y > c^\alpha_j$ such that $\gamma_{\tau_j}(y) \leq \delta_\alpha(k)$, then define m_j to be the greatest such y, and define $c^\alpha_{j-1} = h^\alpha_j(m_j)$. If $m_j \uparrow$, then define $c^\alpha_{j-1} = f^\alpha_j(c^\alpha_j)$.

15j. Then:
    – if $j = 1$, then proceed to on to step 16,
    – if $j \neq 1$, then set $j \leftarrow j-1$, and go back to step 14j.

16. (Building f_α) If there is no $x \in \{1, 2, \cdots, l\}$ such that $m_x \downarrow$, then:
    – let v be the stage at which the current $\Delta_\alpha(B; k)$ was created,
    – for every $x \leq \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v]$, if $f_\alpha(x)$ is undefined, then define $f_\alpha(x) \downarrow = B(x)$,
    – set $g(\alpha)$, $p(\alpha)$, and $q(\alpha)$ to be undefined, if they are defined,
    – let $g_*(\alpha) = h^\alpha_1 \cdots h^\alpha_l(n_l)[v]$, the maximal base marker observed at stage v,
    – let $p_*(\alpha) = \max\{p_*(\alpha)[s-1],\ b^\alpha_j[v] \mid p_*(\alpha)[s-1] \downarrow,\ j = 0, 1, 2, \cdots, l\}$,
    – let $q_*(\alpha) = \delta(k)$,
    – define $\vec{r}_*(\alpha) = (p_*(\alpha), q_*(\alpha))$, and we say that it is defined via (v, k),
    – set Δ_α to be totally undefined, and,
    – initialise all δ such that $\alpha\hat{}\langle -1\rangle <_{\mathrm{L}} \delta$ and go to stage $s+1$.

17. (Defining repair(α)/creating a restraint vector) Otherwise, let v be the stage at which the current $\delta_\alpha(k)$ was created:
    – let i be the least x such that $m_x \downarrow$,
    – define $p(\alpha) = \max\{b^\alpha_j[v],\ \varphi_{\tau_j}(d^{\leq\alpha}_{\tau_j}(\leq y))[v],\ p_*(\alpha)[v] \mid \gamma_{\tau_j}(y) \leq \delta_\alpha(k),\ j = 0, 1, 2, \cdots, l,\ p_*(\alpha)[v] \downarrow\}$,

– define $q(\alpha) = \delta_\alpha(k)$,

– create a restraint vector $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$, and we say that it is defined via (v, k),

– define $\delta^*(k) = p(\alpha)$,

– (re)define $\text{repair}(\alpha) = (i, \tau_i, m_i, k)$,

– if $p(\alpha) \leq \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v]$, then

- define $g(\alpha) = \max\{g^\alpha_i(m_i)[v], p_*(\alpha)[v] \mid p_*(\alpha)[v] \downarrow\}$.

– enumerate $d^\alpha_{\tau_i}(m_i)$ into D, and

– initialise any δ with $\alpha\hat{\ }\langle\omega\rangle <_{\mathrm{L}} \delta$ and go to stage $s+1$.

Case 3. ξ is a decision strategy. Let ξ^- be the longest node $\xi' \subset \xi$. There are then 7 cases:

Case 3a. $\xi = \xi^-\hat{\ }\langle(\varphi, e, \tau, p, k)\rangle$ for some e, τ, k. Then:

– let τ^* be the longest τ' such that $\tau \subset \tau' \subseteq \xi$, and such that τ' is not Σ^0_3-injured at ξ, the existence of τ^* is guaranteed by definition 6.8,

– let k^* be the least x such that $\gamma_{\tau^*}(x)$ has changed since $\xi\hat{\ }\langle(\varphi, e^*, \tau^*, p, x)\rangle$ was last either initialised or visited, where $e^* = e(\tau^*)$, if there is such an x. And otherwise let k^* be undefined,

– if $k^* \downarrow$, then let $\xi\hat{\ }\langle(\varphi, e(\tau^*), \tau^*, p, k^*)$ be eligible to act next,

– if $k^* \uparrow$, then initialise any δ with $\xi <_{\mathrm{L}} \delta$ and go to stage $s+1$.

Case 3b. $\xi = \xi^-\hat{\ }\langle(\varphi_\tau(d^\tau_\tau(k))\rangle$ for some $\mathcal{R}$-strategy τ and some k. Let $e = e(\tau)$. Then:

– if there is an x such that $\gamma_\tau(x)$ has changed since $\xi\hat{\ }\langle(\varphi, e, \tau, p, x)\rangle$ was last either initialised or visited, then let y be the least such x, and let $\xi\hat{\ }\langle(\varphi, e, \tau, p, y)\rangle$ be eligible to act next,

– otherwise, then initialise any δ with $\xi <_{\mathrm{L}} \delta$ and go to stage $s+1$.

Case 3c. $\xi = \xi^-\hat{\ }\langle(\varphi_\tau(d^\alpha_\tau(k))\rangle$ for some $\alpha \neq \tau$ and some k. Let $e = e(\tau)$. Then:

– if $d^\alpha_\tau(k)$ has been enumerated into D since $\xi\hat{\ }\langle(d^\alpha_\tau(k))\rangle$ was last either initialised or visited, then let $\xi\hat{\ }\langle(d^\alpha_\tau(k))\rangle$ be eligible to act next,

– otherwise, if there exists some x such that $\gamma_\tau(x)$ has changed since $\xi\hat{\ }\langle(\varphi, e, \tau, p, x)\rangle$ was last either initialised or visited, then let y be the least such x, and let $\xi\hat{\ }\langle(\varphi, e, \tau, p, y)\rangle$ be eligible to act next,

– otherwise, initialise any δ with $\xi <_{\mathrm{L}} \delta$ and go to stage $s+1$.

Case 3d. $\xi = \xi^-\hat{\ }\langle(\varphi_\tau(d^{\leq\alpha}_\tau(k)))\rangle$ for some $\tau \neq \alpha$ and some k.

Suppose that $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_m$ lists all $\mathcal{S}$-strategies $\beta \subseteq \alpha$ such that $n(\tau, \beta) \downarrow < k$. Let $e = e(\tau)$. Then:

– if there is an $i \in \{1, 2, \cdots, m\}$ such that $d^{\beta_i}_\tau(k)$ has been enumerated into D since $\xi\hat{\ }\langle(d^{\beta_i}_\tau(k))\rangle$ was last either initialised or visited, then let j be the greatest such i, and let $\xi\hat{\ }\langle(d^{\beta_j}_\tau(k))\rangle$ be eligible to act next,

– otherwise, if there is an x such that $\gamma_\tau(x)$ has changed since $\xi\hat{\ }\langle(\varphi, e, \tau, p, x)\rangle$ was last either initialised or visited, then let y be the least x, and let $\xi\hat{\ }\langle(\varphi, e, \tau, p, y)\rangle$ be eligible to act next,

– otherwise, initialise any δ with $\xi <_{\mathrm{L}} \delta$ and go to stage $s+1$.

Case 3e. $\xi = \xi^{-}\hat{\ }\langle(d^{\alpha}_{\tau}(k))\rangle$ for some $\tau \neq \alpha$ and some k.

Suppose that $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_l$ lists all $\mathcal{R}$-strategies $\tau \subset \alpha$ which are active at α, and that $\tau = \tau_j$ for some $j \in \{1, 2, \cdots, l\}$.

Then:

– if b^{α}_{j} has changed since $\xi\hat{\ }\langle(b^{\alpha}_{j})\rangle$ was last either initialised or visited, then let $\xi\hat{\ }\langle(b^{\alpha}_{j})\rangle$ be eligible to act next,

– if $\varphi_{\tau_j}(d^{\leq\alpha}_{\tau_j}(k))$ has changed since $\xi\hat{\ }\langle(\varphi_{\tau_j}(d^{\leq\alpha}_{\tau_j}(k)))\rangle$ was last either initialised or visited, then let $\xi\hat{\ }\langle(\varphi_{\tau_j}(d^{\leq\alpha}_{\tau_j}(k)))\rangle$ be eligible to act next,

– otherwise, if there is an i with $1 \leq i < j$ and an x such that $\varphi_{\tau_i}(d^{\leq\alpha}_{\tau_i}(x))$ has changed since $\xi\hat{\ }\langle(\varphi_{\tau_i}(d^{\leq\alpha}_{\tau_i}(x)))\rangle$ was last either initialised or visited, then let i_0 be the greatest such i, and let x_0 be the least x corresponding to i_0, and let $\xi\hat{\ }\langle(\varphi_{\tau_{i_0}}(d^{\leq\alpha}_{\tau_{i_0}}(x_0)))\rangle$ be eligible to act next,

– otherwise, then initialise any δ with $\xi <_{\mathrm{L}} \delta$ and go to stage $s+1$.

Case 3f. $\xi = \alpha\hat{\ }\langle\omega\rangle\hat{\ }\langle(c, k)\rangle$ for some α and for some k. Let $e = e(\alpha)$.

Suppose that $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_l$ lists all $\mathcal{R}$-strategies $\tau \subset \alpha$ which are active at α.

Then:

– if there is a $j \in \{1, 2, \cdots, l\}$ such that $\varphi_{\tau_j}(d^{\alpha}_{\tau_j}(k))$ has changed since $\xi\hat{\ }\langle(\varphi_{\tau_j}(d^{\alpha}_{\tau_j}(k)))\rangle$ was last either initialised or visited, then let i be the greatest such j, and let $\xi\hat{\ }\langle(\varphi_{\tau_i}(d^{\alpha}_{\tau_i}(k)))\rangle$ be eligible to act next,

– otherwise, then let $\xi\hat{\ }\langle(\theta, e, \alpha, p)\rangle$ be eligible to act next.

Case 3g. If $\xi = \alpha\hat{\ }\langle b\rangle$. Then:

– let i be the greatest j such that $\varphi_{\tau_j}(d^{\leq\alpha}_{\tau_j}(y))$ has changed since $\xi\hat{\ }\langle(\varphi_{\tau_j}(d^{\leq\alpha}_{\tau_j}(y)))\rangle$ was last visited for some $y \leq n_j$,

– let m be the least corresponding y,

–let $\xi\hat{\ }\langle(\varphi_{\tau_i}(d^{\leq\alpha}_{\tau_i}(m)))\rangle$ be eligible to act next.

This completes the description of the construction.

8 The Verification

In this section, we verify that the construction satisfies all the requirements.

Recall the following convention: Given a node β and a stage s:

(i) We say that β *is visited at stage* s, if $s = 2n+2$ for some $n \in \omega$, and β is eligible to act at a substage t of stage s.

(ii) We say that β *receives attention at stage* s, if $s = 2n+1$ for some $n \in \omega$, and β is specified to act during stage s.

First, we observe some basic facts of the construction.

8.1 PROPOSITION. (Basic Facts) Given a node ξ:

(i) For any stage s, If s is even, and ξ is visited at stage s, then every $\xi' \subset \xi$ is visited at stage s,

(ii) For any $\mathcal{S}$-strategy α, if $\vec{r}(\alpha) \downarrow = (p, q)$ holds at the beginning of stage s, and there is an element $b \leq p$ which enters B during stage s, then one of the (2a)–(2c) below occurs:

(2a) α is initialised at stage s,

(2b) $\vec{r}(\alpha)$ is redefined to be (p', q') for some p', q' with $p' < p$ during stage s, and

(2c) $\vec{r}(\alpha)$ is cancelled during stage s.

(iii) If there is an α such that $\alpha \subset \xi$, $\alpha\hat{\ }\langle 2\rangle \not\subseteq \xi$, $\vec{r}(\alpha)$ is defined at the end of stage s, then ξ is not visited at stage s.

Proof. This Proposition is straightforward from the construction. $\square$

8.2 PROPOSITION. (Computation Preservation Proposition) Let α be an $\mathcal{S}$-strategy, and $s > v$ be stages, and k be a number:

(i) If $\vec{r}(\alpha)[s]$ is defined, and it was defined at an even stage via (v, k), then

$$A_v \upharpoonright (\delta(k)[v] + 1) = A_s \upharpoonright (\delta(k)[v] + 1)$$

(ii) If $\vec{r}_*(\alpha)$ is defined, and for v, k, $\vec{r}_*(\alpha)$ was defined via (v, k), then

$$A_v \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1) = A_s \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1).$$

(iii) If $\vec{r}(\alpha)$ is defined at stage s, and it was created at an odd stage via (v, k), then

(3a)

$$A_v \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1) = A_s \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1).$$

(3b) There is an element $b \leq \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v]$ such that $\Theta(A; b) \downarrow = 0 \neq 1 = B(b)$ with $\theta(b) \leq \delta(k)[v]$.

Proof. For (i): Suppose that v_n be the even stage at which $\vec{r}(\alpha)$ was defined via (v, k), and that $v_1 < \cdots < v_n$ are all stages $x \in (v, v_n]$ at which step 17 of α occurred. Then

$$v < v_1 < \cdots < v_n \leq s.$$

For $j = 1$. By the construction, step 8 of α occurred at stage v, so $\delta^*(k)[v] = \delta(k)[v] = \theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v]$ is defined. Since $B_v \upharpoonright (\delta(k)[v] + 1) = B_{v_1} \upharpoonright (\delta(k)[v] + 1)$, we have $A_v \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1) = A_{v_1} \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[k] + 1)$.

Suppose by induction that for $j \geq 1$, we have that

$$A_v \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1) = A_{v_j} \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1).$$

By the construction at stage v_j, $p(\alpha)[v_j] = \max\{b^\alpha[v],\ \varphi^*_{\tau_i}(d^{\leq\alpha}_{\overline{\tau_i}}(\leq y))[v],\ p_*(\alpha) \mid \gamma_{\tau_i}(y)[v_j] \leq \delta_\alpha(k)[v],\ i = 1, 2, \cdots, l,\ p_*(\alpha) \downarrow\}$, $q(\alpha) = \delta(k)[v]$, and $\vec{r}(\alpha)[v_j] = (p(\alpha)[v_j], q(\alpha))$ are created at stage v_j.

If $j < n$, then by the choice of v_j, $B_{v_j} \upharpoonright (p(\alpha)[v_j] + 1) = B_{v_{j+1}} \upharpoonright (p(\alpha)[v_j] + 1)$ holds. By the definition of $p(\alpha)[v_j]$, there is no permission for agitators of strategies $\leq \alpha$ with corresponding γ-markers less than or equal to $\delta(k)[v]$ that has occurred during stages $[v_j, v_{j+1}]$, and by the conditional restraint $\vec{r}(\alpha)[v_j]$, during stages $[v_j, v_{j+1}]$, the honestification of agitators of strategies $\not\leq \alpha$ has never enumerated any γ-marker less than or equal to $\delta(k)[v]$ into A. Therefore

$$A_v \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1) = A_{v_{j+1}} \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1).$$

We conclude from this induction that

$$A_v \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1) = A_{v_n} \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1).$$

Since v_n is an even stage, and that $\vec{r}(\alpha)$ is created at stage v_n. By the choice of s, we have that $B_{v_n} \upharpoonright (p(\alpha)[v_n] + 1) = B_s \upharpoonright (p(\alpha)[v_n] + 1)$. This ensures that for any agitator associated with a node $\leq \alpha$, if its γ-marker less than or equal to $\delta(k)[v]$, then the agitator has not received attention during stages $[v_n, s]$, and every agitator associated with nodes $\not\leq \alpha$, has been delayed to be honestified due to the conditional restraint $\vec{r}(\alpha)[v_n]$. Therefore

$$A_v \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1) = A_s \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1).$$

(i) follows.

For (ii): Let s_* be the stage at which $\vec{r}_*(\alpha)$ was created via (v, k). Then $v < s_* \leq s$. By (i) we have that

$$A_v \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1) = A_{s_*} \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1).$$

By the construction, step 16 of α occurs at stage s_*, so $g_*(\alpha) < p_*(\alpha)$, $q_*(\alpha)$, and $\vec{r}_*(\alpha)$ are created at stage s_*, and all nodes to the right of $\alpha\hat{\ }\langle -1 \rangle$ are initialised at stage s_*. By the choice of s, $B_{s_*} \upharpoonright (p_*(\alpha)[s_*] + 1) = B_s \upharpoonright (p_*(\alpha)[s_*] + 1)$. By the definition of $g_*(\alpha)[s_*]$ and $p_*(\alpha)[s_*]$, there is no agitator of nodes $< \alpha$ with γ-marker less than or equal to $\delta(k)[v]$ which has required attention during stages $[s_*, s]$, and there is no agitator of nodes below α with γ-marker less than or equal to $\delta(k)[v]$ which has required attention during stages $[s_*, s]$. Therefore

$$A_v \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1) = A_s \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1).$$

(ii) follows.

For (iii): Let $s_0 + 1$ be the odd stage at which the current $\vec{r}(\alpha)$ was created via (v, k). By (i) and (ii), we have that

$$A_v \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1) = A_{s_0} \upharpoonright (\theta_\alpha \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v] + 1).$$

By steps 8, 16, and 17, there is an element $b \leq \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v]$ such that $b \in B_{s_0+1} - B_{s_0}$. By the construction, both $p(\alpha)$ and $q(\alpha) = \delta(k)[v]$ have been created at stage s_0+1. By the choice of s, $B_{s_0+1} \upharpoonright (p(\alpha)[s_0+1]+1) = B_s \upharpoonright (p(\alpha)[s_0+1]+1)$. This ensures that

(3a) $A_v \upharpoonright (\theta_\alpha\varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v]+1) = A_s \upharpoonright (\theta_\alpha\varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v]+1)$, and that

(3b) $\Theta_\alpha[v](A;b)\downarrow= 0 \neq 1 = B_s(b)$ holds for some $b \leq \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v]$.

(iii) follows.

This establishes Proposition 8.2. □

Secondly we investigate some properties of the conditional restraints, the valid uses, the τ-permitting markers, absolute restraints and the permitting markers.

8.3 PROPOSITION. (Conditional Restraints Proposition) Given an $\mathcal{S}$-strategy α and a stage s, suppose that $\vec{r}(\alpha)[s] = (p(\alpha)[s], q(\alpha)[s])$ is defined. Then:

(i) If $s+1$ is even and $\vec{r}(\alpha)[s+1]\downarrow= (p(\alpha)[s+1], q(\alpha)[s+1])$ is redefined, then $p(\alpha)[s] \geq p(\alpha)[s+1]$.

(ii) If $s+1$ is odd, and $\vec{r}(\alpha)[s+1] = (p(\alpha)[s+1], q(\alpha)[s+1])$ is redefined, then $p(\alpha)[s+1] < p(\alpha)[s]$.

(iii) If $\vec{r}(\alpha)[s+1]\downarrow= (p(\alpha)[s+1], q(\alpha)[s+1])$, then $p(\alpha)[s+1] \leq p(\alpha)[s]$.

(iv) If $\vec{r}(\alpha)[s]\downarrow$ was created at an odd stage and $\vec{r}(\alpha)[s+1] \neq \vec{r}(\alpha)[s]$, then either (4a) or (4b) below occurs,

(4a) α is initialised at stage $s+1$,

(4b) $B_s \upharpoonright (p(\alpha)+1) \neq B_{s+1} \upharpoonright (p(\alpha)+1)$.

(v) If $\vec{r}(\alpha)$ was created at an even stage via (v,k) for some v, k, and $\vec{r}(\alpha)(\alpha)[s+1] \neq \vec{r}(\alpha)[s]$, then one of the (5a)–(5d) below holds,

(5a) α is initialised at stage $s+1$,

(5b) $\vec{r}(\alpha)[s+1]\downarrow= (p', q(\alpha)[s])$ is created for some p' with $p' < p(\alpha)[s]$ via the same (v,k),

(5c) $B_s \upharpoonright (p(\alpha)[s]+1) \neq B_{s+1} \upharpoonright (p(\alpha)[s]+1)$, and

(5d) $B_v \upharpoonright (b^\alpha[v]+1) = B_{s+1} \upharpoonright (b^\alpha[v]+1)$, and either $\alpha\hat{\ }\langle b\rangle$ is visited at stage $s+1$, or f_α is built at step 16 of the $\mathcal{S}$-strategy α.

Proof. For (i), if $\vec{r}(\alpha)[s]$ was defined at an odd stage via (v,k) for some v,k, then by Proposition 8.2 (iii), there is an element b such that $\Theta_\alpha(A;b)\downarrow= 0 \neq 1 = B(b)$ has been created and preserved by the end of stage s. Since $s+1$ is even, $s+1$ is not α-expansionary, which means that if α is visited at stage $s+1$, so is $\alpha\hat{\ }\langle 2\rangle$. Therefore $\vec{r}(\alpha)[s]$ was created at an even stage via (v,k) for some v, k. By the choice of $s+1$, step 3 of α occurs, and so does step 17 of α. Since $\vec{r}(\alpha)[s+1]$ is redefined via the same (v,k), we have that $p(\alpha)[s+1] \leq p(\alpha)[s]$. (i) follows.

For (ii), we consider two cases.

Case 1. $\vec{r}(\alpha)[s]$ was created at an odd stage s_1, say, via (v,k) for some v,k.

If case 2a occurs at stage s_1, then by the construction, both $g_*(\alpha)$ and $g(\alpha)$ are cancelled at stage s_1, if they are defined. By Proposition 8.2 (iii), there is no α-expansionary stage in $[s_1, s+1]$. Therefore $\vec{r}(\alpha)$ will not be redefined at stage $s+1$. Suppose that case 2b occurs at stage s_1. By the construction at stage s_1, $g(\alpha)$ is cancelled at stage s_1, and by Proposition 8.2 (iii), there is no α-expansionary stages in $[s_1, s+1]$. Therefore the only possibility is that $s+1$ is odd and case 2a occurs at stage $s+1$. Let s_2 be such that the current $\vec{r}_*(\alpha)$ was created at stage s_2 via (v,k) for some v, k. If $s_1 \leq s_2 \leq s$, then by the construction at stage s_2, $\vec{r}(\alpha)[s_1]$ is cancelled. A contradiction. Therefore $s_2 < s_1 \leq s$. By the construction, step 16 of α occurs at stage s_2, and so both $\vec{r}(\alpha)$ and Δ_α are cancelled at stage s_2. By the construction, $\vec{r}(\alpha)[s_1]$ is defined to be the $g(\alpha)$ observed at the beginning of stage s_1, which was defined after stage s_2. By the definition of this $g(\alpha)$, we have that $g(\alpha) \geq p_*(\alpha)[s_2] > g_*(\alpha)[s_2]$. Therefore at the end of stage $s+1$, we have $p(\alpha)[s+1] \leq p(\alpha)[s]$. (ii) follows in case 1.

Case 2. $\vec{r}(\alpha)[s]$ was created at an even stage s_1 via (v, k) for some v, k.

If case 2a occurs at stage $s+1$, then let s_2 be such that the current $\vec{r}_*(\alpha)$ was created at stage s_2 via (v', k') for some v', k'. By the construction, step 16 occurs at stage s_2, therefore $s_2 < s_1 \leq s$. By the construction at stage s_1, $p(\alpha)[s_1] \geq g_*(\alpha)[s_2]$. Therefore $p(\alpha)[s+1] \leq p(\alpha)[s]$. If case 2b occurs at stage $s+1$, then $p(\alpha)[s+1] = g(\alpha)[s_1]$. By definition, we have that $g(\alpha)[s_1] < p(\alpha)[s_1]$. So in either case, we have that $p(\alpha)[s+1] \leq p(\alpha)[s]$.

(ii) in case 2 follows. So does (ii).

For (iii), it follows from both (i) and (ii).

For (iv), let s_1 be such that $\vec{r}(\alpha)[s]$ was created at an odd stage s_1 via (v, k) for some v, k.

Suppose that α has not been initialised during stages $[s_1, s+1]$, and that $\vec{r}(\alpha) \neq \vec{r}(\alpha)[s]$. Then either $\vec{r}(\alpha)$ is cancelled at stage $s+1$ or redefined. In the former case, $B_s \upharpoonright (p(\alpha)[s]+1) \neq B_{s+1} \upharpoonright (p(\alpha)[s]+1)$. In the latter case, case 2a occurs at stage $s+1$, and case 2b occurred at stage s_1. Let s_2 be the stage at which the current $\vec{r}_*(\alpha)$ was created. Then $s_2 < s_1 \leq s$. By the construction, Δ_α is reset to be totally undefined at stage s_2, and $p(\alpha)[s_1] = g(\alpha)[s']$ for some s' with $s_2 < s' < s_1$. By the definition of $g(\alpha)$ in step 17 of α, $g(\alpha)[s'] \geq p_*(\alpha)[s_2]$. By the choice of $s+1$, there is an element $b \leq p_*(\alpha)[s_2]$ which enters B at stage $s+1$. Therefore $B_s \upharpoonright (p(\alpha)[s]+1) \neq B_{s+1} \upharpoonright (p(\alpha)[s]+1)$. (iv) follows.

For (v), let s_1 be an even stage at which $\vec{r}(\alpha)[s]$ was created via (v, k) for some v, k. Suppose that (5a)–(5c) fail to hold. Then the only possibility is that step 2 of α occurs at stage $s+1$. By the definition of $p(\alpha)$ during stages $(v, s]$, we have that for any t with $v < t \leq s$, if $p(\alpha)[t]\downarrow$, then $p(\alpha)[t] \geq b^\alpha[v]$, if $p(\alpha)[t]$ is undefined, then $\delta^*(k)[t] = \delta(k)[v]$. Notice that $\vec{r}(\alpha)[s]$ is kept at stage $s+1$, we have that

$$B_v \upharpoonright (b^\alpha[v]+1) = B_{s+1} \upharpoonright (b^\alpha[v]+1).$$

(v) follows.

This completes the proof of Proposition 8.3. $\square$

8.4 PROPOSITION. (Valid Use Proposition) Given τ, α, x, y, v, s, if $\Gamma_\tau(X_\tau, A; y)\downarrow$ at stage s, $\gamma_\tau(y)$ was created at stage $v \leq s$, $\gamma_\tau(y)[v] \notin A_s$, then:

(i) If $d^\alpha_\tau(y)$ is defined,

$$\gamma_\tau(y)[v] > \varphi^*_\tau(d^\alpha_\tau(y))[v] \geq \varphi^*_\tau(d^\alpha_\tau(y))[v+1] \geq \cdots \geq \varphi^*_\tau(d^\alpha_\tau(y))[s],$$

(ii) If $d^\tau_\tau(y)$ if defined, then

$$\gamma_\tau(y)[v] > \varphi^*_\tau(d^\tau_\tau(y))[v] = \cdots = \varphi^*_\tau(d^\tau_\tau(y))[s] = \varphi^+_\tau(d^\tau_\tau(y))[v].$$

Proof. We prove the proposition by induction on $t = s$. Let $t = v$. By the building of Γ, whenever we define $\gamma_\tau(y)$, we define it as fresh. So $\gamma_\tau(y)[v] > \varphi_\tau^+(d_\tau^\beta(y))$ for any β such that $d_\tau^\beta(y)$ is defined. Both (i) and (ii) hold for $t = v$.

Suppose by induction that both (i) and (ii) hold for all t' with $v \le t' < t \le s$. We look at the proposition for t. By definition, (ii) is obvious. We only need to prove (i). We examine the following cases.

Case 1. For any β, if $\tau \subseteq \beta \subseteq \alpha$, and $\vec{r}(\beta)[t-1] \downarrow$, then $\vec{r}(\beta)[t]$ is defined, then by definition 7.7, and by Proposition 8.3 (iii), we have that $\varphi_\tau^*(d_\tau^\alpha(y))[t] \le \varphi_\tau^*(d_\tau^\alpha(y))[t-1]$.

Case 2. Otherwise, then there is an $\mathcal{S}$-strategy β such that $\tau \subseteq \beta \subseteq \alpha$, $\vec{r}(\beta)[t-1]$ is defined, and $\vec{r}(\beta)[t]$ is undefined. By Proposition 8.2 (iv) and (v), there are three subcases.

Subcase 2a. α is initialised at stage t.

This means that $d_\tau^\alpha(y)$ is not defined at the end of stage t. A contradiction.

Subcase 2b. There is an element $b \le p(\beta)[t-1]$ which enters B at stage t.

Let $p_0 = \varphi_\tau^*(d_\tau^\alpha(y))[t-1]$. If $b \le p_0$, then $d_\tau^\alpha(y)$ requires attention at stage t, unless $\vec{r}(\alpha)$ has been redefined to be (p_1, q_1) for some $p_1 < b$ during stage t under step 3. If $p_0 < b \le p(\beta)[t-1]$, then the cancellation of $\vec{r}(\beta)$ during stage t does not increases the valid use $\varphi_\tau^*(d_\tau^\alpha(y))$. So in either case, we have that

$$\varphi_\tau^*(d_\tau^\alpha(y))[t] \le \varphi_\tau^*(d_\tau^\alpha(y))[t-1].$$

Subcase 2c. Otherwise. This means that $\beta\hat{\ }\langle b\rangle \subseteq \alpha$, and $\beta\hat{\ }\langle b\rangle$ is visited at stage t.

Let s_1 be the stage at which $\vec{r}(\beta)[t-1]$ was created via (v, k) for some v, k. If s_1 is odd, then by Proposition 8.3, t is not α-expansionary, so that if α is visited at stage t, so is $\alpha\hat{\ }\langle 2\rangle$. A contradiction. Let s_1 be even, and let $\vec{r}(\beta)[t-1] = (p, q)$ for some p, q. If $\gamma_\tau(y)[t] \le q$, then $\gamma_\tau(y)[t]$ had been defined before stage v. By the construction at stage v, we have that $\varphi_\tau^+(d_\tau^\alpha(y)) \le b^\beta[v]$. By Proposition 8.3, $B_v \upharpoonright (b^\beta[v]+1) = B_t \upharpoonright (b^\beta[v]+1)$. Therefore the cancellation of $\vec{r}(\beta)$ at stage t does not increase the valid use $\varphi_\tau^*(d_\tau^\alpha(y))$, giving $\varphi_\tau^*(d_\tau^\alpha(y))[t] \le \varphi_\tau^*(d_\tau^\alpha(y))[t-1]$.

This completes the inductive proof of (ii). The Proposition follows. □

We now check the τ-permitting markers. We have:

8.5 PROPOSITION. (τ-Permitting Marker Proposition) Given an $\mathcal{R}$-strategy τ, and a y, if $\Gamma_\tau(y)$ is defined, then let $v \le s$ be such that $\gamma_\tau(y)$ was created at stage $v < s$, for $x = \gamma_\tau(y)[v]$, we have

$$m_\tau(x)[v] \ge m_\tau(x)[v+1] \ge \cdots \ge m_\tau(x)[s].$$

Proof. For each $j \in [v, s]$, let Q_j be the set of all nodes in $Q_\tau^y[j]$, and for each $\alpha \in Q_j$, let $a_j^\alpha = \varphi_\tau^*(d_\tau^\alpha(y))[j]$. Then by definition of Q's and by Proposition 8.4, we have:

(a) $Q_v \supseteq Q_{v+1} \supseteq \cdots \supseteq Q_s$,

(b) For each $j \in [v, s]$, and each $\alpha \in Q_j$, we have

$$a_v^\alpha \geq a_{v+1}^\alpha \geq \cdots \geq a_s^\alpha.$$

By the definition of $m_\tau(x)$, for every $j \in [v, s]$, $m_\tau(x)[j] = \max\{a_v^\alpha \mid \alpha \in Q_j\}$. By the Minimax Lemma, Lemma 4.3, we have

$$m_\tau(x)[v] \geq m_\tau(x)[v+1] \geq \cdots \geq m_\tau(x)[s].$$

Proposition 8.5 is established. □

8.6 PROPOSITION (Absolute Restraint Proposition) Let s be a stage, and τ be an $\mathcal{R}$-strategy. Then:

(i) If p^τ is defined during stage s, then τ is not visited at stage s. If τ is visited at stage s, then for any $\mathcal{S}$-strategy α, if $\alpha \subset \tau$, and $\alpha\hat{\ }\langle 2\rangle \not\subseteq \tau$, then $\vec{r}(\alpha)$ is not defined at the end of stage s.

(ii) If both $p^\tau[s]$ and $p^\tau[s+1]$ are defined, then

$$p^\tau[s] \geq p^\tau[s+1].$$

(iii) If $p^\tau[s]$ is defined, and $p^\tau[s+1]$ is not defined, then either (3a) or (3b) below occurs,

(3a) τ is initialised at stage $s+1$.

(3b) $B_s \upharpoonright (p^\tau[s]+1) \neq B_{s+1} \upharpoonright (p^\tau[s]+1)$.

Proof. For (i), for every $\mathcal{S}$-strategy α, if $\alpha \subset \tau$, and $\alpha\hat{\ }\langle 2\rangle \not\subseteq \tau$, then either $\alpha\hat{\ }\langle b\rangle \subseteq \tau$, or $\alpha\hat{\ }\langle \omega\rangle \subseteq \tau$. In the former case, step 2 of α occurs at stage s, so $\vec{r}(\alpha)$ is not defined at the end of stage s, and in the latter case, if $\vec{r}(\alpha)$ is defined at the beginning of stage s, then either step 16 or step 17 of α occurs, and if $\vec{r}(\alpha)$ is created at stage s, then step 17 of α occurs at stage s, so if $\vec{r}(\alpha)$ is defined at the end of stage s, then no node below α is further visited during stage s. A contradiction. (i) follows.

For (ii), let $p^\tau[s] \downarrow= p$. We consider the following cases.

Case 1. For any $\mathcal{S}$-strategy α, if $\alpha\hat{\ }\langle \omega\rangle \subseteq \tau$, and $\vec{r}(\alpha)[s]$ is defined, then so is $\vec{r}(\alpha)[s+1]$; and if $\alpha\hat{\ }\langle b\rangle \subseteq \tau$, and $\vec{r}(\alpha)[s]$ is defined and it was created at an odd stage, then $\vec{r}(\alpha)[s+1]$ is defined.

In this case, by Definition 7.7 and by Proposition 8.3 (iii), we have that

$$p^\tau[s+1] \leq p^\tau[s].$$

Case 2. Otherwise, and τ is not initialised at stage $s+1$. We consider the following subcases.

Subcase 2a. There is an $\mathcal{S}$-strategy α such that $\alpha\hat{\ }\langle b\rangle \subseteq \tau$, $\vec{r}(\alpha)[s] \downarrow= (p, q)$, which was created at an odd stage, is defined, and $\vec{r}(\alpha)[s+1]$ is undefined.

By Proposition 8.3 (iv), either α is initialised at stage $s+1$ or $B_s \upharpoonright (p+1) \neq B_{s+1} \upharpoonright (p+1)$. If α is initialised at stage $s+1$, so is τ. A contradiction. In the latter case, let b be the element which is enumerated into B at stage $s+1$. Then $b \leq p$. If $p^\tau[s] < b \leq p$, then it is obvious that $p^\tau[s+1] \leq p^\tau[s]$. If $b \leq p^\tau[s]$, then there is an $\mathcal{S}$-strategy $\alpha \subset \tau$ such that $\alpha\hat{\ }\langle 2\rangle \subseteq \tau$, and α receives special attention at stage

$s+1$ by redefining its conditional restraint $\vec{r}(\alpha) = (p', q')$ for some $p' < b < p^\tau[s]$. In either case, we have that $p^\tau[s+1] \leq p^\tau[s]$.

Subcase 2b. There is an $\mathcal{S}$-strategy α such that $\alpha\hat{}\langle\omega\rangle \subseteq \tau$, $\vec{r}(\alpha)[s] \downarrow = (p, q)$, and $\vec{r}(\alpha)[s+1]$ is undefined.

If $\vec{r}(\alpha)[s]$ was created at an odd stage, then by the same argument as that in subcase 2a, we have that $p^\tau[s+1] \leq p^\tau[s]$. Let s_1 be an even stage at which $\vec{r}(\alpha)[s]$ was created. By Proposition 8.3 (v) either α is initialised at stage $s+1$, or $B_s \upharpoonright (p+1) \neq B_{s+1} \upharpoonright (p+1)$, and α does not receive special attention at stage $s+1$, or step 16 of α occurs at stage $s+1$. Since τ is not initialised at stage $s+1$, we have that $B_s \upharpoonright (p+1) \neq B_{s+1} \upharpoonright (p+1)$. Then by the same argument as that in subcase 2a, we have that $p^\tau[s+1] \leq p^\tau[s]$.

(ii) follows.

For (iii), let $p^\tau[s] = p(\alpha)[s]$ for some $\mathcal{S}$-strategy $\alpha \subset \tau$. If $\vec{r}(\alpha)[s]$ was created at an odd stage, then by Proposition 8.3 (iv), either α is initialised at stage $s+1$, or B changes below p, so either τ is initialised at stage $s+1$ or B changes below $p^\tau[s]$ at stage $s+1$. Suppose that $\vec{r}(\alpha)[s]$ was created at an even stage. In this case $\alpha\hat{}\langle\omega\rangle \subseteq \tau$. By Proposition 8.3 (v), either α is initialised at stage $s+1$ or B changes below p at stage $s+1$. In either case, (iii) follows.

Proposition 8.6 follows. $\square$

8.7 PROPOSITION. (Permitting Marker Proposition) Let τ be an $\mathcal{R}$-strategy, and s, and y be such that $\Gamma_\tau(X_\tau, A; y)[s]$ is defined, and that $\gamma_\tau(X_\tau, A; y)$ was created at a stage $v \leq s$. Then for $x = \gamma_\tau(y)$, the following property holds,

$$m(x)[v] \geq m(x)[v+1] \geq \cdots \geq m(x)[s].$$

Proof. For every $j \in [v, s)$, by Definition 7.7, and by Propositions 8.5 and 8.6, we have

$$m(x)[j+1] \leq p^\tau[j+1],\ m_\tau(x)[j+1] \leq p^\tau[j],\ m_\tau(x)[j] \text{ respectively}$$

So $m(x)[j+1] \leq m(x)[j]$. Proposition 8.7 follows. $\square$

From Proposition 8.7 we know that for every γ-marker x, we define a non-increasing sequence of permitting markers $m(x)[s]$ in the construction. We enumerate x into A if and only if B changes below the most recent permitting marker $m(x)$ (without $m(x)$ immediately being redefined to be less than its previous value), in which case all lost permission occurred previously for x gets repaid. This guarantees that the permitting argument succeeds.

In the next propositions, we investigate the $\mathcal{R}$- and $\mathcal{S}$-principles listed in Definitions 4.8 and 4.9.

8.8 PROPOSITION. ($\mathcal{R}$-Principle Proposition) Given an $\mathcal{R}$-strategy τ, an $\mathcal{S}$-strategy α, and a stage s:

(i) If τ is visited at stage s, then every $\xi \subseteq \tau$ is visited at stage s.

(ii) If $p^\tau[s-1]\uparrow$, $p^\tau[s]\downarrow$, then τ is not visited during stage s.

(iii) If $p^\tau[s]\downarrow$, then τ is not visited at stage s, and τ does not receive attention at stage s.

(iv) If a restraint vector $\vec{r}(\alpha)=(p,q)$ is created at stage s, and b is the unique element which is enumerated into B at stage s, then $p<b$.

(v) Let $s^- \leq s$. Suppose that τ has not been initialised at any stage $v \in [s^-, s]$, and that $p^\tau[v]\downarrow$ for all $v\in[s^-,s]$. Then for any s_1, s_2, if $s^- \leq s_1 < s_2 \leq s$, then $p^\tau[s_1] \geq p^\tau[s_2]$.

(vi) If $p^\tau[s-1]\uparrow$, and $p^\tau[s]\downarrow = p$, then for any $k > b(\tau)$, both (a) and (b) below hold:

(a) if $\gamma_\tau(k)$ is dishonest at the end of stage s, then $m_\tau(\gamma_\tau(k)) > p$,

(b) if $\gamma_\tau(k)$ is incorrect during stage s, then $m_\tau(\gamma_\tau(k)) > p$.

[*Remark.* $\gamma_\tau(k)$ is *dishonest* if one agitator of the form $d^\alpha_\tau(k)$ is dishonest.]

(vii) If $p^\tau[s]\downarrow$, then for any $k > b(\tau)$, both (a) and (b) below hold:

(a) if $\gamma_\tau(k)$ is dishonest during stage s, then $m_\tau(\gamma_\tau(k)) > p^\tau$,

(b) if $\gamma_\tau(k)$ is incorrect during stage s, then $m_\tau(\gamma_\tau(k)) > p^\tau$.

(viii) If $p^\tau[s-1]\downarrow$ and $p^\tau[s]\uparrow$, then for any $k > b(\tau)$, both (a) and (b) below hold:

(a) if $\gamma_\tau(k)[s]\downarrow$, then $\Gamma_\tau(X_\tau, A; k)[s]\downarrow = B_s(k)$,

(b) if $\gamma_\tau(k)[s]\downarrow$, and $d^\tau_\tau(k)\downarrow$, then $d^\tau_\tau(k)$ is honest at the end of stage s.

(ix) If $p^\tau[s]\uparrow$, then for any $k > b(\tau)$, both (a) and (b) below hold:

(a) if $\gamma_\tau(k)[s]\downarrow$, then $\Gamma_\tau(X_\tau, A; k)[s]\downarrow = B_s(k)$,

(b) if $d^\tau_\tau(k)\downarrow$, then $d^\tau_\tau(k)$ is honest at the end of stage s.

(x) If τ is visited at stage s, then for any $k > b(\tau)$, both (a) and (b) below hold:

(a) if $\gamma_\tau(k)[s]\downarrow$, then $\Gamma_\tau(X_\tau, A; k)[s]\downarrow = B_s(k)$,

(b) if $d^\tau_\tau(k)[s]\downarrow$, then $d^\tau_\tau(k)$ is honest at the end of stage s.

Proof. The proposition is proved by induction on stages.

For (i): This follows from Proposition 8.1 (i).

For (ii): We consider two cases.

Case 1. $s = 2n+2$ for some n.

By the construction, there is an $\mathcal{S}$-strategy α such that $\alpha\subset\tau$, and $\alpha\hat{\ }\langle 2\rangle \not\subseteq \tau$, and such that $\vec{r}(\alpha)$ is defined at the end of stage s.

By Proposition 8.1 (iii), there is no node ξ with $\alpha\hat{\ }\langle 2\rangle \not\subseteq \xi$ can be visited during stage s. (ii) follows in this case.

Case 2. $s = 2n+1$ for some n.

In this case, there is an $\mathcal{S}$-strategy $\alpha\subset\tau$ with $\alpha\hat{\ }\langle 2\rangle \not\subseteq \tau$ such that α receives special attention at stage s. By the choice of s, $p^\tau[s]\downarrow$, by definition 7.7, and there is no y such that $\gamma_\tau(y)$ requires attention at stage s, so that τ does not require attention at stage s. (ii) follows in this case.

(ii) follows.

For (iii): By definition of p^τ, let $p^\tau[s] = p(\alpha)[s]$ for some $\mathcal{S}$-strategy α such that $\alpha \subset \tau$ and $\alpha\hat{}\langle 2\rangle \not\subseteq \tau$. We consider two cases:

Case 1. $s = 2n + 1$ for some $n \geq 0$.

By definition of permitting marker, for every y, if $\gamma_\tau(y) \downarrow= x$, then $m(x) \leq p^\tau[s]$. Therefore τ does not require attention at stage s, so that τ does not receive attention at stage s.

Case 2. Otherwise. There are then two subcases:

Subcase 2a. α is visited at stage s.

If s is not α-expansionary, then $\alpha\hat{}\langle 2\rangle$ is visited at stage s. In this case, τ is not visited at stage s, because $\alpha\hat{}\langle 2\rangle \not\subseteq \tau$.

Suppose that s is α-expansionary, then either step 2 or step 12 of program α occurs at stage s. If step 2 occurs, then $\alpha\hat{}\langle b\rangle \subseteq \tau$, and $\vec{r}(\alpha)$ and $p_*(\alpha)$ are cancelled during stage s, if they are defined. This contradicts the assumption of $\vec{r}(\alpha)[s]$ being defined. If step 12 of program α occurs at stage s, then by the construction, either step 16 or step 17 of program α occurs at stage s. In either case, there is no ξ below α which is further visited during stage s.

Therefore τ is not visited at stage s.

Subcase 2b. Otherwise.

By Proposition 8.8 (i), there is no node below α which is visited at stage s. So τ is not visited at stage s.

Therefore, in any case, τ is not visited at stage s, and (iii) follows.

For (iv): Suppose that a restraint vector $\vec{r}(\alpha) = (p, q)$ is created at an odd stage s. Then α receives special attention at stage s. By step 2 of the construction in stage s, $p(\alpha) < b$. Therefore $p < b$ holds, and (iv) follows.

For (v): This follows from Proposition 8.6. (v) follows.

For (vi)–(x): Suppose by induction that (vi)–(x) hold for all $s' < s$.

For (vi) at s: By the choice of s, and by (ix) at the end of stage $s-1$, if $\gamma_\tau(k)$ is either dishonest or incorrect at the end of stage s, then $\gamma_\tau(k)$ becomes either dishonest or incorrect because of the enumeration of $b \in B_s - B_{s-1}$ for some b. Let $p^\tau[s] = p(\alpha)[s]$ for some s. Then $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$ is created at stage s. By proposition 8.8 (iv), $p(\alpha) < b$. If $\gamma_\tau(k)$ becomes incorrect at stage s, then $k = b$, so $m_\tau(\gamma_\tau(k)) \geq b > p^\tau[s]$. If $\gamma_\tau(k)$ becomes dishonest at stage s, then there is a node α such that $d^\alpha_\tau(k) \downarrow$, where $\varphi_\tau(d^\alpha_\tau(k)) < \gamma_\tau(k)$ holds at the beginning of stage s, but fails to hold during stage s. Therefore $b \leq \varphi_\tau(d^\alpha_\tau(k)) < \gamma_\tau(k)$. But $p^\tau[s] < b$. Therefore for any $k > b(\tau)$, if $\gamma_\tau(k)$ is either dishonest or incorrect at the end of stage s, then $m_\tau(\gamma_\tau(k))[s] > p^\tau[s]$. (vi) at s follows.

For (vii) at s: Assume that (vi) at s holds.

Let $s^- \leq s$ be minimal such that $p^\tau[s^- - 1] \uparrow$, and that for every $v \in [s^-, s]$, $p^\tau[v] \downarrow$. By (ii) and (iii), for every $v \in [s^-, s]$, τ is not visited at stage v. Therefore if there is a $k > b(\tau)$ such that $\gamma_\tau(k)$ becomes either dishonest or incorrect at a stage $v \in [s^-, s]$, then the only reason is that there is an element $b < m_\tau(\gamma_\tau(k))$ which is enumerated into B at stage v.

By the choice of s^-, and by (vi) at $s' \leq s$, if $s^- = s$, then (vii) at s holds. Suppose that $s^- < s$. By (vi) at s^-, (vii) at s^- holds. Given $v \in (s^-, s]$, suppose by induction that (vii) at s' holds for every $s' \in [s^-, v)$. By (v), $p^\tau[v-1] \geq p^\tau[v]$, so that if $\gamma_\tau(k)$ is either dishonest or incorrect at the end of stage $v-1$, then $m_\tau(\gamma_\tau(k)) > p^\tau[v-1] \geq p^\tau[v]$. If $\gamma_\tau(k)$ becomes either dishonest or incorrect at stage v, then $m_\tau(\gamma_\tau(k)) \leq p^\tau[v-1]$. Now there are two cases:

Case 1. $p^\tau[v] = p^\tau[v-1]$.

Let b_v be the element which is enumerated into B at stage v. By the construction, $p^\tau[v-1] < b_v \leq m_\tau(\gamma_\tau(k)) \leq p^\tau[v-1]$, a contradiction.

Case 2. Otherwise.

Then $p^\tau[v] < p^\tau[v-1]$, so that $p^\tau[v] < b_v < m_\tau(\gamma_\tau(k)) \leq p^\tau[v-1]$. Therefore if $\gamma_\tau(k)$ becomes either dishonest or incorrect at stage v, then $m_\tau(\gamma_\tau(k)) > p^\tau[v]$.
(vii) at v follows.

This completes the inductive proof that for every $v \in [s^-, s]$, (vii) holds at v.

For (viii) at s: Suppose that (vi)–(vii) hold at all $s' \leq s$.
Let $p^\tau[s-1] \downarrow = p$. By Proposition 8.6 (iii), either τ is initialised at stage s or $B_s \upharpoonright (p+1) \neq B_{s-1} \upharpoonright (p+1)$. Clearly if τ is initialised during stage s, then for any k, $\gamma_\tau(k)$ is not defined at the end of stage s. So (viii) holds at s.
Suppose that there is an element b such that $b \leq p$ and $b \in B_s - B_{s-1}$. By (vii) at $s-1$, for any k, if either $\gamma_\tau(k)$ is incorrect or $d^\tau_\tau(k)$ is dishonest at the end of stage s, then $m_\tau(\gamma_\tau(k)) > p^\tau = p \geq b$. By the construction, $\gamma_\tau(k)$ is enumerated into A during stage s. If either $\gamma_\tau(k)$ becomes incorrect at stage s, then $k = b$ and $m_\tau(\gamma_\tau(k)) \geq b$, and if $d^\tau_\tau(k)$ becomes dishonest at stage s, then $\varphi_\tau(d^\tau_\tau(k)) < \gamma_\tau(k)$ holds at the end of stage $s-1$, but $b \leq \varphi_\tau(d^\alpha_\tau(k))[s-1]$. By the construction, $\gamma_\tau(k)$ is also enumerated into A during stage s. Therefore for any $k > b(\tau)$, if $\gamma_\tau(k)[s] \downarrow$, then $\gamma_\tau(k)$ is correct at the end of stage s, and $d^\tau_\tau(k)$ is honest at the end of stage s. (viii) at s follows.

For (ix) at s: Suppose that (vi)–(viii) hold at all $s' \leq s$. We consider two cases:

Case 1. There is an $s' < s$ such that $p^\tau[s'] \downarrow$.

Then let s^- be the stage $s' \leq s$ such that $p^\tau[s'-1] \downarrow$ and for every $v \in [s', s]$, $p^\tau[v] \uparrow$. By the choice of s^-, and by (viii) at s^-, (ix) holds at s^-. For a v with $s^- < v \leq s$, suppose by induction that (ix) holds at s' for all s' with $s^- \leq s' < v$. If either $\gamma_\tau(k)$ becomes incorrect or $d^\tau_\tau(k)$ becomes dishonest at stage v, then there are two subcases:

Subcase 1a. v is an even stage.

In this subcase, $d^\tau_\tau(k)[v-1] \downarrow$, such that $\varphi_\tau(d^\tau_\tau(k)) < \gamma_\tau(k)$ holds at the end of stage $v-1$, and such that there is an $x \leq \varphi_\tau(d^\tau_\tau(k))[v-1]$ which is enumerated into X_τ at stage v. Then by the construction, $\Gamma_\tau(X_\tau, A; k)$ is set to be undefined during stage v.

Subcase 1b. Otherwise.

Let b_v be the $x \in B_v - B_{v-1}$. Then either $k = b_v$, in which case $m_\tau(\gamma_\tau(k)) \geq b_v$, so $\gamma_\tau(k)$ is enumerated into A during stage v, or $d_\tau^\tau(k)[v-1] \downarrow$, $\varphi_\tau(d_\tau^\tau(k)) < \gamma_\tau(k)$ holds at the end of stage $v - 1$ and $b_v \leq \varphi_\tau(d_\tau^\tau(k))[v-1]$. By the construction, $\gamma_\tau(k)[v-1]$ is enumerated into A during stage v.

Therefore (ix) at v holds in case 1.

Case 2. Otherwise.

Clearly (ix) holds at $s = 0$. Suppose by induction, for a stage v with $0 < v \leq s$, that (ix) holds at s' for all s' with $0 \leq s' < v$. If either $\gamma_\tau(k)$ becomes incorrect or $d_\tau^\tau(k)$ becomes dishonest at stage v, then there are two subcases:

subcase 2a. v is an even stage.

This is the same as subcase 1a above.

Subcase 2b. Otherwise.

This is the same as subcase 1b above.

So (ix) holds at v in case 2, from which it follows that (ix) at v holds in all cases. This completes the induction proof that (ix) holds at all $s' \leq s$.

For (x) at s: Suppose that (vi)–(ix) hold at all $s' \leq s$.

Suppose that τ is visited at stage s. Then by proposition 8.8 (iii), $p^\tau[s] \uparrow$. So (x) follows at s from (ix) at s.

This completes the inductive proof of (vi)–(x).

Proposition 8.8 follows. □

Given an $\mathcal{S}$-strategy α, suppose that $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_l$ lists all $\mathcal{R}$-strategies $\tau \subset \alpha$ which are active at α, and that $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_m$ lists all $\mathcal{S}$-strategies β such that $\beta \subset \alpha$ and $\beta\hat{\ }\langle 2\rangle \not\subseteq \alpha$. We now prove some properties of the restraint function of α.

Before verifying the $\mathcal{S}$-Principle, we introduce some notations. Given τ, α, and y, if $d_\tau^\alpha(y)$ is defined, then we define

$pd_\tau^\alpha(y) = \min\{p(\beta),\ p(\gamma) \mid \tau \subset \beta\hat{\ }\langle b\rangle \subseteq \alpha,\ \vec{r}(\beta) \downarrow= (p(\beta), q(\beta)),\ \gamma_\tau(y) \leq q(\beta),\ \vec{r}(\beta)$ was created at an odd stage, $\tau \subset \gamma \subset \gamma\hat{\ }\langle\omega\rangle \subseteq \alpha,\ \vec{r}(\gamma) \downarrow= (p(\gamma), q(\gamma)),\ \gamma_\tau(y) \leq q(\gamma)\}$.

By definition 7.7, $\varphi_\tau^*(d_\tau^\alpha(y)) = \min\{pd_\tau^\alpha(y),\ \varphi_\tau^+(d_\tau^\alpha(y))\}$.

We then have that

(1) $d_\tau^\alpha(y)$ is honest if and only if $\varphi_\tau^*(d_\tau^\alpha(y)) = \varphi_\tau^+(d_\tau^\alpha(y))$.

(2) For $x = \gamma_\tau(y)$, there is an element $b \leq m(x)$ which enters B if and only if there is an α such that $d_\tau^\alpha(y)$ is defined, and there is an element $b \leq \min\{p^\tau, pd_\tau^\alpha(y), \varphi_\tau^+(d_\tau^\alpha(y))\}$ which enters B. This means that $d_\tau^\alpha(y)$ requires attention.

(3) If both p^τ, and $pd_\tau^\alpha(y)$ are undefined, then by (1), we have that $d_\tau^\alpha(y)$ is honest.

We first prove a basic property of agitators.

8.9 PROPOSITION. Given an $\mathcal{S}$-strategy α, an $\mathcal{R}$-strategy τ, and a y such that $d_\tau^\alpha(y)$ is defined, and a stage s:

(i) If α is visited at stage s, then:

(1a) For any $\mathcal{R}$-strategy $\tau \subset \alpha$, p^{τ} is not defined at the end of stage s,

(1b) For any $\mathcal{S}$-strategy β, if $\beta \subset \alpha$, and $\beta\hat{\ }\langle 2\rangle \not\subseteq \alpha$, then $\vec{r}(\beta)[s]$ is not defined.

(ii) If both $pd^{\alpha}_{\tau}(y)[s]$ and $pd^{\alpha}_{\tau}(y)[s+1]$ are defined, then $pd^{\alpha}_{\tau}(y)[s] \geq pd^{\alpha}_{\tau}(y)[s+1]$.

(iii) If $pd^{\alpha}_{\tau}(y)[s] \downarrow = p$ for some p, and $pd^{\alpha}_{\tau}(y)$ becomes undefined at stage $s+1$, then either (3a) or (3b) below occurs,

(3a) α is initialised.

(3b) $B_s \upharpoonright (p+1) \neq B_{s+1} \upharpoonright (p+1)$.

Proof. By the proof of Proposition 8.6. The Proposition follows. □

Now we are ready to prove that the $\mathcal{S}$-principle listed in Definition 4.9 will be satisfied.

8.10 PROPOSITION (The $\mathcal{S}$-Principle Proposition) Given an $\mathcal{S}$-strategy α and a stage s:

(i) For any τ, and any y, if $d^{\alpha}_{\tau}(y)$ is defined, $pd^{\alpha}_{\tau}(y)[s-1] \uparrow$ and $pd^{\alpha}_{\tau}(y)[s] \downarrow$, then if $d^{\alpha}_{\tau}(y)$ is dishonest at the end of stage s, then $\varphi^{+}_{\tau}(d^{\alpha}_{\tau}(y)) > pd^{\alpha}_{\tau}(y)[s]$.

(ii) For any τ, and y, if $d^{\alpha}_{\tau}(y)$ is defined, $pd^{\alpha}_{\tau}(y)[s] \downarrow$, then, if $d^{\alpha}_{\tau}(y)$ is dishonest at the end of stage s, then $\varphi^{+}_{\tau}(d^{\alpha}_{\tau}(y)) > pd^{\alpha}_{\tau}(y)[s]$.

(iii) For any τ, any y, if $d^{\alpha}_{\tau}(y)$ is defined, $pd^{\alpha}_{\tau}(y)[s-1] \downarrow$ and $pd^{\alpha}_{\tau}(y)[s] \uparrow$, then, both (a) and (b) below hold:

(a) $\gamma_{\tau}(y)$ is correct at the end of stage s (if it is defined),

(b) $d^{\alpha}_{\tau}(y)$ is honest at the end of stage s.

(iv) For any τ, any y, if $d^{\alpha}_{\tau}(y)$ is defined, $pd^{\alpha}_{\tau}(y)[s] \uparrow$, then, both (a) and (b) below hold:

(a) $\gamma_{\tau}(y)$ is correct at the end of stage s (if it is defined),

(b) $d^{\alpha}_{\tau}(y)$ is honest at the end of stage s.

(v) If s is α-expansionary, then for any τ, y, if $d^{\alpha}_{\tau}(y)$ is defined, then $d^{\alpha}_{\tau}(y)$ is honest at stage s.

Proof.

For (i)–(iv): Given τ, and y with $d^{\alpha}_{\tau}(y) \downarrow$, suppose by induction that (i)–(iv) hold for all $s' < s$.

For (i) at s: By (iv) at $s-1$, if either $\gamma_{\tau}(y)$ is incorrect or $d^{\alpha}_{\tau}(y)$ is dishonest at the end of stage s, then either $\gamma_{\tau_j}(k)$ becomes incorrect during stage s or $d^{\alpha}_{\tau}(k)$ becomes dishonest during stage s.

But then, by the choice of s, s is an odd stage. Let b be the element which is enumerated into B at stage s.

If $d^{\alpha}_{\tau}(y)$ becomes dishonest during stage s, then $\varphi^{+}_{\tau}(d^{\alpha}_{\tau}(y)) < \gamma_{\tau}(y)$ holds at the end of stage $s-1$, and $b \leq \varphi_{\tau}(d^{\alpha}_{\tau}(y))[s-1]$.

Let $pd^{\alpha}_{\tau}(y)[s] = p(\beta)[s]$ for some β. Then $\vec{r}(\beta)$ is created at stage s. By proposition 8.8 (iv), we have $p(\beta)[s] < b$.

Therefore, if $d^{\alpha}_{\tau}(y)$ becomes dishonest during stage s, then $\varphi^{+}_{\tau}(d^{\alpha}_{\tau}(y)) \geq b > pd^{\alpha}_{\tau}(y)[s]$. So (i) holds at stage s.

For (ii) at s: Assume that (i) holds at s.

Let $s^- \leq s$ be minimal such that $pd^\alpha_\tau(y)[s^- - 1] \uparrow$, and such that for any $v \in [s^-, s]$, $pd^\alpha_\tau(y)[v] \downarrow$.

By the γ-rules, if $d^\alpha_\tau(y)$ becomes dishonest for the first time since stage s^- at stage v, then there is an element $b < \varphi^+_\tau(d^\alpha_\tau(y))$ which is enumerated into B at stage v.

By the choice of s^-, and by (i) at $s' \leq s$, if $s^- = s$, then (ii) holds at s. Suppose that $s^- < s$.

By (i) at s^-, (ii) holds at s^-. Given $v \in (s^-, s]$, suppose by induction that (ii) holds at s' for all s' with $s^- \leq s' < v$. By Proposition 8.9, $pd^\alpha_\tau(y)[v-1] \geq pd^\alpha_\tau(y)[v]$, so if either $\gamma_\tau(y)$ is incorrect or $d^\alpha_\tau(y)$ is dishonest at the end of stage $v-1$, then $\gamma_\tau(y) > pd^\alpha_\tau(y)[v-1] \geq pd^\alpha_\tau(y)[v]$. If $d^\alpha_\tau(y)$ becomes dishonest at stage v, then $\varphi^+_\tau(d^\alpha_\tau(y)) \leq pd^\alpha_\tau(y)[v-1]$. There are two cases:

Case 1. $pd^\alpha_\tau(y)[v] = pd^\alpha_\tau(y)[v-1]$.

Let b_v be the element which is enumerated into B at stage v. By the construction and by the definition of $pd^\alpha_\tau(y)$, we must have $pd^\alpha_\tau(y)[v-1] < b_v < m_\tau(\gamma_\tau(y))) \leq pd^\alpha_\tau(y)[v-1]$, a contradiction.

Case 2. Otherwise, so that $pd^\alpha_\tau(y)[v] < pd^\alpha_\tau(y)[v-1]$.

Then $pd^\alpha_\tau(y)[v] < b_v < m_\tau(\gamma_\tau(y)) \leq pd^\alpha_\tau(y)[v-1]$. Therefore, if either $\gamma_\tau(y)$ becomes incorrect at stage v, or $d^\alpha_\tau(y)$ becomes dishonest at stage v, then $m_\tau(\gamma_\tau(y)) > pd^\alpha_\tau(y)[v]$.

(ii) at v follows.

Therefore for every $v \in [s^-, s]$, (ii) holds at v.

For (iii) at s: Assume that (i) and (ii) hold at all $s' \leq s$.

Let $pd^\alpha_\tau(y)[s-1] \downarrow = p$. By Proposition 8.9, either α is initialised during stage s or $B_s \upharpoonright (p+1) \neq B_{s-1} \upharpoonright (p+1)$. If α is initialised during stage s, then $d^\alpha_\tau(y)$ is cancelled at the end of stage s. So (iii) holds at s.

Suppose that there is an element $b \leq p$ such that $b \in B_s - B_{s-1}$. By (ii) at $s-1$, if either $\gamma_\tau(y)$ is incorrect at the end of stage $s-1$, or $d^\alpha_\tau(y)$ is dishonest at the end of stage $s-1$, then $m_\tau(\gamma_\tau(y)) > p$. By the construction, if either $\gamma_\tau(y)$ is incorrect at the end of stage $s-1$, or $d^\alpha_\tau(y)$ is dishonest at the end of stage $s-1$, then $\gamma_\tau(y)$ is enumerated into A during stage s. And if either $\gamma_\tau(y)$ becomes incorrect at stage s or $d^\alpha_\tau(y)$ becomes dishonest at stage s, then it is easy to see that $m_\tau(\gamma_\tau(y)) > b$, so $\gamma_\tau(y)$ is also enumerated into A during stage s.

(iii) at s follows.

For (iv) at s: Suppose that (i)–(iii) hold at all $s' \leq s$.

We consider two cases:

Case 1. There is an $s' < s$ such that $pd^\alpha_\tau(y)[s'] \downarrow$.

Let s^- be $\leq s$ such that $pd^\alpha_\tau(y)[s^- - 1] \downarrow$, and $pd^\alpha_\tau(y)[v] \uparrow$ for all $v \in [s^-, s]$.

If $s^- = s$, then by (iii) at s, (iv) at s follows. Suppose that $s^- < s$. By (iii) at s^-, (iv) at s^- holds. Suppose by induction for a v with $s^- < v \leq s$, that (iv) holds at s' for all s' with $s^- \leq s' < v$.

Suppose that either $\gamma_\tau(y)$ becomes incorrect at stage v or $d^\alpha_\tau(y)$ becomes dishonest at stage v. Then there are two subcases:

Subcase 1a. v is an even stage.

In this subcase, $\varphi_\tau(d^\alpha_\tau(y)) < \gamma_\tau(y)$ holds at the end of stage $v-1$, but there is an $x \leq \varphi_\tau(d^\alpha_\tau(y))[v-1]$ which is enumerated into X_τ during stage v. So by the γ-rules, $\Gamma_\tau(X_\tau, A; y)[v]$ is undefined.

Subcase 1b. Otherwise.

Let b_v be the element $x \in B_v - B_{v-1}$. Then either $k = b_v$, in which case $\gamma_\tau(y) > b_v$, or $\varphi_\tau(d^\alpha_\tau(y))[v-1] < \gamma_\tau(y)$ and $b_v \leq \varphi_\tau(d^\alpha_\tau(y))[v-1]$. In either case, $\gamma_\tau(y)$ is enumerated into A during stage v.

Therefore (iv) holds at s' in case 1 for all $s' \in [s^-, s]$.

Case 2. Otherwise.

Then set $s^- = 0$. Clearly (v) at $s^- = 0$ holds. We prove it by induction on $v \in [0, s]$. The inductive argument is the same as that in case 1.

So in any case, (iv) holds at s.

For (v): Suppose that $d^\alpha_\tau(y)$ is defined. If $pd^\alpha_\tau(y)$ is undefined, then by (iv), we have that $d^\alpha_\tau(y)$ is honest at stage s. If $pd^\alpha_\tau(y)$ is defined at stage s, then we have that $pd^\alpha_\tau(y) = p(\alpha) < \varphi^+_\tau(d^\alpha_\tau(y))$. Since s is α-expansionary, $\vec{r}(\alpha)$ was created at an even stage v for some $v \leq s$. By the construction, $B_v \upharpoonright (p(\alpha)[v]+1) = B_s \upharpoonright (p(\alpha)[v]+1)$, otherwise, $\vec{r}(\alpha)$ has been either cancelled or redefined since stage v. By the definition of $p(\alpha)[v]$, there is no agitator $d^\alpha_\tau(y)$ of α for τ for some y which has become dishonest during stages $[v, s]$. On the other hand, for any τ and y, if $d^\alpha_\tau(y)$ was defined at stage v, then it was honest at stage v. Therefore for any τ, any y, if $d^\alpha_\tau(y)$ is defined at stage s, then it is honest at stage s. (v) follows.

Proposition 8.10 follows. □

Given an $\mathcal{S}$-strategy α, assume that the notations described in Definition 7.2 are associated with α. We prove a property of agitations of α.

8.11 PROPOSITION. (Agitation Proposition) Suppose that step 17 of α occurs at stage s, that we are currently rectifying inequality $\Delta_\alpha(B; k) \neq K(k)$, and that v is the stage at which the current $\delta_\alpha(k)$ was created. If $d^\alpha_{\tau_i}(m_i)$ is enumerated into D at stage s, and $g(\alpha)$ is (re)defined at stage s, then:

(i) For any j, any y, if $d^\alpha_{\tau_j}(y) \downarrow$, and $\gamma_{\tau_j}(y) \leq \delta(k)[v]$ holds at stage s, then

$$g^\alpha_j(y)[v] \leq g(\alpha)[s].$$

(ii) For any j, any y, if $d^\alpha_{\tau_j}(y)$ is defined and $\gamma_{\tau_j}(y) \leq \delta(k)[v]$ holds at stage s, then

$$\varphi^*_{\tau_j}(d^{\leq\alpha}_{\tau_j}(\leq y))[v] \leq \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))[v].$$

(iii) If $m_i \leq k$, then for any j, and y, if $d^\alpha_{\tau_j}(y)$ is defined and $\gamma_{\tau_j}(y) \leq \delta(k)[v]$ holds at stage s, then

$$\varphi^*_{\tau_j}(d^{\leq\alpha}_{\tau_j}(\leq y))[v] \leq \varphi_{\tau_l}(d^{\alpha}_{\tau_l}(k))[v].$$

(iv) If $m_i \leq k$, then from stage s, whenever we enumerate an agitator to rectify the current inequality $\Delta_\alpha(B;k) \neq K(k)$, we have that $g(\alpha)[s]$ is defined.

Proof. For (i), we consider three cases:

Case 1. $j > i$.

If there is a $j > i$ such that m_j is defined at the current stage s, then let j_1 be the least such j. By the choice of j_1, for every j, and each y, if $i < j \leq j_1$, $d^{\alpha}_{\tau_j}(y)$ is defined and $\gamma_{\tau_j}(y) \leq \delta(k)[v]$ holds at stage s, then

$$y \leq f^{\alpha}_{j+1} \cdots f^{\alpha}_{j_1-1} h^{\alpha}_{j_1}(m_{j_1})[v]$$

By the choice of m_i, we have that

$$m_i > f^{\alpha}_{i+1} \cdots f^{\alpha}_{j_1-1} h^{\alpha}_{j_1}(m_{j_1})[v]$$

$$\geq f^{\alpha}_{i+1} \cdots f^{\alpha}_{j}(\leq y)[v].$$

By using this, we have the following:

$$g^{\alpha}_i(m_i)[v] = f^{\alpha}_1 \cdots f^{\alpha}_{i-1} h^{\alpha}_i(m_i)[v]$$

$$\geq f^{\alpha}_1 \cdots f^{\alpha}_{i-1} h^{\alpha}_i(f^{\alpha}_{i+1} \cdots f^{\alpha}_j(y) + 1)[v]$$

$$\geq f^{\alpha}_1 \cdots f^{\alpha}_{i-1} f^{\alpha}_i \cdots f^{\alpha}_j(\leq y)[v]$$

$$\geq f^{\alpha}_1 \cdots f^{\alpha}_{j-1} h^{\alpha}_j(y)[v] = g^{\alpha}_j(y)[v]$$

In particular, we have

$$g^{\alpha}_i(m_i)[v] \geq g^{\alpha}_{j_1}(m_{j_1})[v].$$

Suppose by induction that j_n is defined at stage s such that for $j_0 = i$, and for all j, and y, if $j_{n-1} < j \leq j_n$, $d^{\alpha}_{\tau_j}(y)$ is defined, and $\gamma_{\tau_j}(y) \leq \delta(k)[v]$ holds at stage s, then

(1) $g^{\alpha}_j(y)[v] \leq g^{\alpha}_{j_{n-1}}(m_{j_{n-1}})[v]$, and in particular,

(2) $g^{\alpha}_{j_n}(m_{j_n})[v] \leq g^{\alpha}_{j_{n-1}}(m_{j_{n-1}})[v]$.

If there are $j > j_n$ such that m_j is defined during stage s, then by the same argument as that for $j = 1$, we can prove both (1) and (2) for $n + 1$.

This establishes inductively that for all j, all y, if $j > i$, $d^{\alpha}_{\tau_j}(y)$ is defined, and $\gamma_{\tau_j}(y) \leq \delta(k)[v]$ holds at stage s, then

$$g^{\alpha}_j(y)[v] \leq g^{\alpha}_i(m_i)[v].$$

Case 2. $j = i$.

By the choice of m_i, for every y, if $d^{\alpha}_{\tau_i}(y)$ is defined, and $\gamma_{\tau_i}(y) \leq \delta(k)[v]$ holds at stage s, then by the definition of g, we have

$$g^{\alpha}_i(y)[v] \leq g^{\alpha}_i(m_i)[v].$$

Case 3. $j < i$.

By the choice of m_i, there is no $j < i$ such that m_j is defined at stage s, so for any $j < i$, and any y, if $d^{\alpha}_{\tau_j}(y)$ is defined, and $\gamma_{\tau_j}(y) \leq \delta(k)[v]$ holds at stage s, then

$$y \leq f^{\alpha}_{j+1} \cdots f^{\alpha}_{i-1} h^{\alpha}_i(m_i)[v]$$

By using this and the definition of g, we have

$$g^{\alpha}_j(y)[v] = f^{\alpha}_1 \cdots f^{\alpha}_{j-1} h^{\alpha}_j(y)[v]$$

$$\leq f^{\alpha}_1 \cdots f^{\alpha}_{j-1} h^{\alpha}_j f^{\alpha}_{j+1} \cdots f^{\alpha}_{i-1} h^{\alpha}_i(m_i)[v]$$

$$\leq f^{\alpha}_1 \cdots f^{\alpha}_{j-1} f^{\alpha}_j \cdots f^{\alpha}_{i-1} h^{\alpha}_i(m_i)[v]$$

$$= g^{\alpha}_i(m_i)[v].$$

(i) follows.

For (ii), since $g(\alpha)$ is defined at stage s, we have that

$$p(\alpha)[s] = \max\{b^{\alpha}[v],\ \varphi^{*}_{\tau_j}(d^{\leq\alpha}_{\tau_j}(\leq y)[v],\ p_{*}(\alpha)[v] \mid \gamma_{\tau_j}(y) \leq \delta(k)[v], p_{*}(\alpha)[v]\downarrow\}$$

$$\leq \varphi_{\tau_l}(d^{\alpha}_{\tau_l}(k))[v]$$

(ii) follows.

For (iii), let j, y be such that $d^{\alpha}_{\tau_j}(y)$ is defined, and $\gamma_{\tau_j}(y) \leq \delta(k)[v]$ holds at stage s. We consider three cases.

Case 1. $j > i$.

If there are $j > i$ such that m_j is defined at stage s. Then let i' be the least such j. Then, by the choice of m_i,

$$m_i > f^{\alpha}_{i+1} \cdots f^{\alpha}_{i'-1} h^{\alpha}_{i'}(m_{i'})[v]$$

And for all j, y, if $i < j \leq i'$, $d^{\alpha}_{\tau_j}(y)$ is defined, and $\gamma_{\tau_j}(y) \leq \delta(k)[v]$ holds during stage s, then

$$y \leq f^{\alpha}_{j+1} \cdots f^{\alpha}_{i'-1} h^{\alpha}_{i'}(m_{i'})[v]$$

$$\leq f^{\alpha}_{i+1} \cdots f^{\alpha}_{i'-1} h^{\alpha}_{i'}(m_{i'})[v] < m_i \leq k.$$

Notice that $y \leq k$, so by the well ordering at stage v, we have

$$\varphi_{\tau_l}(d^{\alpha}_{\tau_l}(k))[v] > f^{\alpha}_1 \cdots f^{\alpha}_{l-1} h^{\alpha}_l(k)[v]$$

$$\geq f^{\alpha}_j \cdots f^{\alpha}_{l-1} h^{\alpha}_l(k)[v] \geq f^{\alpha}_j(y)[v] \geq \varphi^*_{\tau_j}(d^{\leq\alpha}_{\tau_j}(\leq y))[v].$$

If there is a $j > i'$ such that m_j is defined at stage s, then notice that $m_i > m_{i'}$, we repeat the same argument by replacing i by i' finitely many times. (iii) in Case 1 follows.

Case 2. $j = i$.

For any y, if $d^{\alpha}_{\tau_i}(y)$ is defined, and $\gamma_{\tau_i}(y)[s] \leq \delta(k)[v]$, then $y \leq m_i \leq k$.

By the well ordering at stage v, and by the argument in Case 1, we have that

$$\varphi^*_{\tau_i}(d^{\leq\alpha}_{\tau_i}(\leq y))[v] \leq \varphi_{\tau_l}(d^{\alpha}_{\tau}(k))[v].$$

Case 3. $j < i$.

For any $j < i$, any y, $d^{\alpha}_{\tau_j}(y)$, $\gamma_{\tau_j}(y)[s] \leq \delta(k)[v]$ holds at stage s, since m_j is not defined at stage s, we have that

$$(1) \quad y \leq f^{\alpha}_{j+1} \cdots f^{\alpha}_{i-1} h^{\alpha}_i(m_i)[v]$$

$$\leq f^{\alpha}_{j+1} \cdots f^{\alpha}_{l-1} h^{\alpha}_l(m_i)[v]$$

$$\leq f^{\alpha}_{j+1} \cdots f^{\alpha}_{l-1} h^{\alpha}_l(k)[v]$$

The first inequality by the construction at stage s, the second by definition, and the third by definition and by the assumption of $m_i \leq k$.

By the well ordering at stage v, we have that

$$(2) \quad \varphi_{\tau_l}(d^{\alpha}_{\tau_l}(k))[v] > f^{\alpha}_1 \cdots f^{\alpha}_j(f^{\alpha}_{j+1} \cdots f^{\alpha}_{l-1} h^{\alpha}_l(k)[v])[v]$$

$$f^{\alpha}_1 \cdots f^{\alpha}_j(y)[v] \geq f^{\alpha}_j(y)[v] \geq \varphi^*_{\tau_j}(d^{\leq\alpha}_{\tau_j}(\leq y))[v]$$

The first inequality by the well ordering at stage v, the second by (1), and the last two by definition of f.

So in any case, if $m_i \leq k$, then for any j, any y, if $d^{\alpha}_{\tau_j}(y)$ is defined, and $\gamma_{\tau_j}(y)[s] \leq \delta(k)[v]$ holds at stage s, then

$$\varphi^*_{\tau_j}(d^{\leq\alpha}_{\tau_j}(\leq y))[v] \leq \varphi_{\tau_l}(d^{\alpha}_{\tau_l}(k))[v]$$

(iii) follows.

For (iv), this follows from (iii) and from the construction.

Proposition 8. 11 follows. □

8.12 PROPOSITION. ($\mathcal{P}$-Satisfaction Proposition) $A \leq_{\mathrm{T}} B$.

Proof. By the construction, if x is enumerated into $A_s - A_{s-1}$, then $x = \gamma_\tau(m)$ for some τ, m and some odd stage s, and $m(\gamma_\tau(m)) \geq b$, where $b \in B_s - B_{s-1}$. By

Proposition 8.7, $m(\gamma_\tau(m)) < \gamma_\tau(m)$ holds at all stages after $\gamma_\tau(m)$ is specified and before it is enumerated into A. We now prove $A \leq_T B$ as follows.

Given an x, let $s(x)$ be the least odd stage v such that $B_v \upharpoonright (x+1) = B \upharpoonright (x+1)$. Then $x \in A$ if and only if $x \in A_{s(x)}$, giving $A \leq_T B$.

Proposition 8.12 follows. □

We now verify that the construction satisfies the requirements. Firstly we define the *true Path TP of the construction* as follows:

(i) For a stage $s > 0$, define δ_s to be the longest strategy which is visited at stage s.

(ii) Define *the true path TP of the construction* by

$$TP = \liminf_{s=2n+2} \delta_s.$$

Given an $\mathcal{S}$-strategy α, suppose that $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_l$ comprise all $\mathcal{R}$-strategies $\tau \subset \alpha$ which are active at α. We first prove some basic properties of the $\mathcal{S}$-strategy α.

8.13 PROPOSITION. (Basic Properties of an $\mathcal{S}$-Strategy) Let $\alpha \in TP$ be an $\mathcal{S}$-strategy. Suppose that there are infinitely many α-expansionary stages. Then:

(i) If $k > n_1 = n(\tau_1, \alpha)$ and $\Delta_\alpha(B;k)\downarrow \neq K(k)$ occurs at a stage v, then there is a stage $s > v$ at which either

(a) $B_v \upharpoonright (\delta^*_\alpha(k)[s]+1) \neq B_s \upharpoonright (\delta^*_\alpha(k)[s]+1)$, i.e., $\Delta(B;k)[s]$ becomes undefined, or

(b) Δ_α is set to be totally undefined at stage s.

(ii) For a fixed k, there are only finitely many stages at which repair(α) is defined to be (i, τ_i, m_i, k) for some $i \in \{1, 2, \cdots, l\}$, and some m_i.

(iii) For a fixed pair (i, m_i), $i \in \{1, 2, \cdots, l\}$, $m_i > n_i = n(\tau_i, \alpha)$, there is a stage s_0 such that for any $s > s_0$, any k, if repair$(\alpha)[s]\downarrow = (i, \tau_i, m_i, k)$, then $m_i \leq k$.

Proof. For (i): If there is a stage $s > v$ at which α is initialised, then Δ_α is set to be totally undefined at the end of stage s, in which case (i) holds. On the other hand, suppose that α is initialised at no stage $s > v$, and, since there are only finitely many (i, m_i) such that $\gamma_i(m_i) \leq \delta_\alpha(k)[v]$.

If $\Delta_\alpha(B;k)[v]$ is kept at every stage greater than v, i.e., for any $s > v$, B will not change below $\delta^*_\alpha(k)[s]$. By the definition of $\delta^*_\alpha(k)$, before we start the procedure of rectification of $\Delta_\alpha(B;k) \neq K(k)$, $\delta^*_\alpha(k) = \delta_\alpha(k)[v]$, and once the cycle of rectification of $\Delta_\alpha(B;k) \neq K(k)$ starts, and we enumerate some agitator of α into D, we always define $\delta^*_\alpha(k)$ to be $p(\alpha)$ which is the maximal φ-uses of agitators of α with corresponding γ-marker less than or equal to $\delta_\alpha(k)[v]$. Therefore if the inequality $\Delta_\alpha(B;k) \neq K(k)$ is kept forever, then since there are infinitely α-expansionary stages, we will reach a stage $s > v$ at which step 16 occurs, in which case, Δ_α is reset.

(i) follows.

For (ii): By the construction, if $\Delta_\alpha(B;k)\downarrow= 1$, then we have that repair$(\alpha) \neq (i,\tau_i,m_i,k)$ for any i,m_i. By (i), for a fixed $k > n_1$, if $k \in K$, then there is a stage s_0 such that for any $s > s_0$, if $\Delta_\alpha(B;k)[s]$ is defined, then $\Delta_\alpha(B;k)[s] = 1$. Clearly if $k \notin K$, then repair$(\alpha) \neq (i,\tau_i,m_i,k)$ for any i,m_i at any stage. Therefore for a fixed k, there are only finitely many stages at which repair$(\alpha) = (i,\tau_i,m_i,k)$ for some i,m_i.

(ii) follows.

(iii) follows from (ii).

This completes the proof of Proposition 8.13. □

We say that a stage s is a ξ-stage, if s is even and ξ is visited at stage s.

8.14 PROPOSITION. (True Path TP Proposition) Suppose that $B \not\leq_{\mathrm{T}} \emptyset$ and $K \not\leq_{\mathrm{T}} B$. Given a strategy $\alpha \in TP$:

(i) If $\alpha = \tau$ is an $\mathcal{R}$-strategy, and $\lim_s p^\tau[s]$ does not exist, then there are infinitely many τ-stages s for which $p^\tau[s]$ is undefined.

(ii) There is a possible outcome a of α such that $\alpha\hat{\ }\langle a\rangle \in TP$.

(iii) $\alpha\hat{\ }\langle a\rangle \in TP$ is initialised only finitely often.

(iv) If $\alpha\hat{\ }\langle a\rangle \in TP$, then there are infinitely many $\alpha\hat{\ }\langle a\rangle$-stages, the stages at which $\alpha\hat{\ }\langle a\rangle$ is visited.

(v) If $\alpha\hat{\ }\langle a\rangle \in TP$, and there is some φ-node $\xi \subseteq \alpha\hat{\ }\langle a\rangle$, then $u(\alpha\hat{\ }\langle a\rangle)[s]$ becomes unbounded during the course of the construction.

(vi) If either $\alpha\hat{\ }\langle 1\rangle \in TP$ or $\alpha\hat{\ }\langle 2\rangle \in TP$, then α acts only finitely often.

Given an $\mathcal{S}$-strategy α, suppose that $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_l$ comprise all $\mathcal{R}$-strategies $\tau \subset \alpha$ which are active at α.

(vii) If:

(A) α is an $\mathcal{S}$-strategy,

(B) a restraint vector $\vec{r}(\alpha) = (p,q)$ is created at stage s, and

(C) b is the unique element which is enumerated into B at stage s,

then:

(a) $p < b \leq q$,

(b) $\Theta_\alpha(A;b)[s]\downarrow= 0 \neq 1 = B_s(b)$ with $\theta_\alpha(b)[s] \leq q$, and

(c) if α is not initialised after stage s and $B_s \upharpoonright (p+1) = B \upharpoonright (p+1)$, then $A_s \upharpoonright (q+1) = A \upharpoonright (q+1)$.

(viii) If:

(A) α is an $\mathcal{S}$-strategy,

(B) $\lim_s \vec{r}(\alpha)[s]$ does not exist,

(C) $\lim_s n_j[s]\downarrow= n_j < \omega$, and

(D) there is a $j \in \{1,2,\cdots,l\}$ and a $k \leq n_j = n(\tau_j,\alpha)$ such that $\varphi^*_{\tau_j}(d^{<\alpha}_{\tau_j}(k))[s]$ becomes unbounded during the course of the construction,

then letting i be the greatest such j, and k the least corresponding k, we have

– $\alpha\hat{\ }\langle b\rangle\hat{\ }\langle(\varphi_{\tau_i}(d^{<\alpha}_{\tau_i}(k)))\rangle \in TP$.

(ix) If:

(A) α is an $\mathcal{S}$-strategy,

(B) $\lim_s \vec{r}(\alpha)[s]$ does not exist,

(C) $\lim_s n_1[s] \downarrow= n_1 < \omega$ exists,

(D) $b^\alpha_0[s]$ is bounded during the course of the construction, and

(E) f_α is built infinitely often,

then:

– f_α is total and $f_\alpha =^* B$.

(x) If:

(A) (A)–(D) of (ix) hold,

(B) f_α is built only finitely often, and

(C) $\Delta_\alpha(B)$ is total,

then:

– $\Delta_\alpha(B) =^* K$.

Also, recalling the definitions of the auxiliary functions h^α_l, h^α_i and g^α_i, for i with $1 \leq i \leq l$, defined in the construction by:

$$\begin{aligned} h^\alpha_i(m) =& \max\{\varphi^*_{\tau_i}(d^\beta_{\tau_i}(\leq m)),\ \varphi^*_{\tau_i}(d^\alpha_{\tau_i}(< m)), m \ \mid \beta < \alpha\} \\ g^\alpha_i(m) =& f^\alpha_1 f^\alpha_2 \cdots f^\alpha_{i-1} h^\alpha_i(m) \end{aligned}$$

for every $m \in \omega$, we have:

(xi) If:

(A)–(D) (A)–(D) of (ix) respectively, and

(E) f_α is built only finitely often, and $\lim_s g^\alpha_i(m)[s] \downarrow= g^\alpha_i(m) < \omega$,

then $\lim_s d^\alpha_{\tau_i}(m)[s] \downarrow= d^\alpha_{\tau_i}(m) < \omega$.

(xii) If $\alpha\hat{\ }\langle(\varphi, e, \tau, p, k)\rangle \in TP$ for some e, τ, k, then

(a) $\gamma_\tau(k)[s]$ becomes unbounded over the course of the construction,

(b) for every $k' < k$, $\gamma_\tau(k')[s]$ is bounded over the course of the construction.

Proof. We prove the proposition by induction on the length of α.

For the root node λ, we have that λ is an $\mathcal{R}_0$-strategy. Clearly $\lim_s b(\lambda)[s] \downarrow= b(\lambda) < \omega$. By definition of p^λ, $p^\lambda[s]$ is undefined for all $s \geq 0$. By the construction, for every even stage $s > 0$, λ is visited at stage s, and $p^\lambda[s]$ is undefined. So (i) holds for λ. (vii)–(xii) are empty for λ. We prove (ii)–(vi) by cases.

Case 1. There are infinitely many λ-expansionary stages.

By the construction, after $\lim_s b(\lambda)[s] \downarrow= b(\lambda)$ is defined, $\lambda\hat{\ }\langle 0\rangle$ will not be initialised. If there are infinitely many $\lambda\hat{\ }\langle 0\rangle$-stages, then (ii)–(vi) hold for $\lambda\hat{\ }\langle 0\rangle$. Therefore it suffices to prove that there are infinitely many $\lambda\hat{\ }\langle 0\rangle$-stages.

Let s_0 be the stage at which $\lim_s b(\lambda)[s] \downarrow= b(\lambda)$ is defined. Suppose to the contrary that s_1 is a stage $> s_0$ after which there is no $\lambda\hat{}\langle 0\rangle$-stage. By the construction, for any even stage $s > s_1$, λ is visited at stage s, but step 5 of program λ does not occur at stage s.

By the construction, Γ_λ is built only at $\lambda\hat{}\langle 0\rangle$-stages, so Γ_λ is a finite set. By the construction, if there is an $\mathcal{S}$-strategy α and a stage $s > s_1$ at which $d^\alpha_\lambda(k)$ receives X_λ-honestification for some k, then there is a k such that $\Gamma_\lambda(k)[s-1] \downarrow$ but $\Gamma_\lambda(k)[s] \uparrow$.

So we can choose a stage $s_2 > s_1$ after which step 2 of program λ will never occur. Again since Γ_λ is finite, we can choose a stage $s_3 > s_2$ after which step 6 of program λ will not occur.

Let s_4 be minimal $> s_3$ such that for any x, $\gamma_\lambda(x) = \gamma_\lambda(x)[s_4]$, and let k be the least $x > b(\lambda)$ such that $\Gamma_\lambda(X_\lambda, A; x)[s_4] \uparrow$.

Let $Q^k_\lambda = \{\lambda, \alpha \mid n(\lambda, \alpha) \downarrow< k\}$. Then Q^k_λ is a finite set. First we prove that, for every $\beta \in Q^k_\lambda$, we have $\lim_s d^\beta_\lambda(k)[s] \downarrow= d^\beta_\lambda(k) < \omega$. Clearly, if $\beta = \lambda$, then $\lim_s d^\beta_\lambda(k)[s] \downarrow= d^\beta_\lambda(k) < \omega$. If $\beta \in Q^k_\lambda$ and $\beta \neq \lambda$, then $\lambda\hat{}\langle 0\rangle \subseteq \beta$ and λ is active at β. By the choice of s_1, Δ_β is built only finitely often.

Suppose that α is an $\mathcal{S}$-strategy and $\alpha \in Q^k_\lambda$. If s is a stage $> s_4$ at which repair(α) is defined to be $(0, \lambda, m, k)$ for some m, k at stage s, then by the assumption of case 1, there is a stage $v > s$ at which either $\Gamma_\lambda(m)$ is set to be undefined, or $\Delta_\alpha(B; k)$ is set to be undefined.

However, for a fixed $\alpha \in Q^k_\lambda$, Δ_α is a finite set, therefore there are only finitely many stages at which repair(α) is defined to be $(0, \lambda, m, k)$ for some m, k during the course of the construction. This ensures that $\lim_s d^\alpha_\lambda(k)[s] \downarrow= d^\alpha_\lambda(k) < \omega$.

Let $d^k_\lambda = \max\{d^\alpha_\lambda(k) \mid \alpha \in Q^k_\lambda\}$. Then $\lim_s d^k_\lambda[s] \downarrow= d^k_\lambda < \omega$.

Therefore we can choose a stage $s_5 > s_4$ such that

(a) for every $\alpha \in Q^k_\lambda$, $\lim_s d^\alpha_\lambda(k)[s] \downarrow= d^\alpha_\lambda(k)[s_5]$,

(b) $\lim_s d^k_\lambda[s] \downarrow= d^k_\lambda[s_5]$.

By the assumption of this case, there are infinitely many λ-expansionary stages. So let s_6 be minimal $> s_5$ such that

(i) s_6 is a λ-stage,

(ii) s_6 is λ-expansionary, and

(iii) $l(\lambda)[s_6] > d^k_\lambda[s_5]$.

By the choice of s_6, step 5 of program λ occurs at stage s_6, so s_6 is a $\lambda\hat{}\langle 0\rangle$-stage, contradicting the choice of s_1.

Therefore if there are infinitely many λ-expansionary stages, then $\lambda\hat{}\langle 0\rangle \in TP$, $\lambda\hat{}\langle 0\rangle$ is initialised only finitely often, and there are infinitely many $\lambda\hat{}\langle 0\rangle$-stages. Note that (v) and (vi) are empty for λ in case 1.

Case 2. Otherwise.

Let s_0 be the stage at which $\lim_s b(\lambda)[s] \downarrow= b(\lambda) < \omega$, and let s_1 be the least stage $> s_0$ such that there is no $\lambda\hat{}\langle 0\rangle$-stage $> s_1$.

Then Γ_λ remains a finite set at the end of the construction.

First we prove that there are only finitely many odd stages at which some strategy $\supseteq \lambda\hat{}\langle 0\rangle$ requires attention.

By the choice of s_1, we do not build Γ_τ for $\tau \supseteq \lambda\hat{}\langle 0\rangle$ at even stages $> s_1$. By the construction, for no τ do we build Γ_τ at odd stages. Therefore the collection of all Γ_τ for $\tau \supseteq \lambda\hat{}\langle 0\rangle$ is a finite set. By the same argument, we have that the collection of all Δ_α for $\alpha \supseteq \lambda\hat{}\langle 0\rangle$ is finite.

Suppose that for some $\mathcal{S}$-strategy $\alpha \supseteq \lambda\hat{}\langle 0\rangle$, and some stage $s > s_1$, repair(α) is defined to be $(e(\tau), \tau, m, k)$ for some τ, m, k at stage s. Then either $l(D, \Phi_\tau(X_\tau, B))[v] \not> d^\alpha_\tau(m)[s]$ holds for all $v > s$, or there is a stage $v > s$ at which either $\Gamma_\tau(X_\tau, A; m)$ or $\Delta_\alpha(B; k)$ is set to be undefined.

Therefore there are only finitely many stages at which repair(α) is created for some $\mathcal{S}$-strategy $\alpha \supseteq \lambda\hat{}\langle 0\rangle$. So we can choose a stage $s_2 > s_1$ such that no $\mathcal{S}$-strategy $\alpha \supseteq \lambda\hat{}\langle 0\rangle$ receives attention at any odd stage $> s_2$.

Now let s_3 be a stage $> s_2$ such that no $\mathcal{R}$-strategy $\tau \supseteq \lambda\hat{}\langle 0\rangle$ receives honestification or rectification at any stage $> s_3$.

By the choice of s_3, $\lambda\hat{}\langle 1\rangle$ will be initialised at no odd stage $> s_3$.

By the proof in case 1, let s_4 be a stage $> s_3$ such that neither step 2 nor step 4 of program λ occurs at any even stage $> s_4$. By the construction, for any even stage $s > s_4$, $\lambda\hat{}\langle 1\rangle$ is visited at stage s, and λ will never define any parameter after stage s_4. Therefore $\lambda\hat{}\langle 1\rangle \in TP$, $\lambda\hat{}\langle 1\rangle$ is initialised only finitely often, there are infinitely many $\lambda\hat{}\langle 1\rangle$-stages, and λ acts only finitely often. And note that $u(\lambda\hat{}\langle 1\rangle)$ is never defined during the course of the construction. So (ii)–(vi) hold for λ in case 2.

It follows that in each case, proposition 8.14 holds for the root node λ.

Suppose by induction that the proposition holds for all $\alpha' \subset \alpha$ and that $\alpha \in TP$. Let s_0 be the least stage such that:

(i) no $\alpha' \subseteq \alpha$ is initialised at any stage $> s_0$,

(ii) if $\alpha'\hat{}\langle 1\rangle \subseteq \alpha$ or $\alpha'\hat{}\langle 2\rangle \subseteq \alpha$, then α' will act at no stage $> s_0$.

We consider the following cases:

Case 1. $\alpha = \tau$ is an $\mathcal{R}$-strategy.

For (i): By proposition 8.8 (iii), if s is a τ-stage, then $p^\tau[s]$ is not defined. By the inductive hypothesis, there are infinitely many τ-stages. Therefore there are infinitely many τ-stages s for which $p^\tau[s]$ is not defined, and (i) follows. We prove (ii)–(vi) according to the following subcases:

Subcase 1a. There are infinitely many τ-expansionary stages.

Let s_1 be minimal $> s_0$ such that $\lim_s b(\tau)[s] \downarrow= b(\tau)[s_1]$. By the construction, $\tau\hat{}\langle 0\rangle$ will never be initialised after stage s_1. If there are infinitely many $\tau\hat{}\langle 0\rangle$-stages, then (ii)–(iv) hold, and (v) follows from the inductive hypothesis, and (vi) is empty in this subcase. So we need only prove that there are infinitely many $\tau\hat{}\langle 0\rangle$-stages.

8.15 LEMMA. If there are infinitely many τ-expansionary stages, then there are infinitely many $\tau\hat{}\langle 0\rangle$-stages.

Proof. Suppose to the contrary that s_2 is a stage $> s_1$ after which there is no $\tau\hat{}\langle 0\rangle$-stage.

By the construction, for any $\mathcal{R}$-strategy $\tau' \supseteq \tau\hat{}\langle 0\rangle$, $\Gamma_{\tau'}$ will not be built at any stage $s > s_2$, and for any $\mathcal{S}$-strategy $\alpha \supseteq \tau\hat{}\langle 0\rangle$, Δ_α will not be built at any stage $s > s_2$. Therefore by proposition 8.13 (i), (ii), there is a stage $s_3 > s_2$ such that for any $\mathcal{S}$-strategy $\alpha \supseteq \tau\hat{}\langle 0\rangle$, any odd stage $s > s_3$, α does not receive attention at stage s.

And we can choose a stage $s_4 > s_3$ such that for any $\mathcal{R}$-strategy $\tau' \supseteq \tau\hat{}\langle 0\rangle$, any odd stage $s > s_4$, τ' does not receive attention at stage s.

By the choice of s_4, for any odd stage $s > s_4$, and any $\beta \supseteq \tau\hat{}\langle 0\rangle$, β does not receive attention at stage s.

By the choice of s_1, Γ_τ is a finite set. By proposition 8.13 (i), (ii), let s_5 be minimal $> s_4$ after which neither step 2 nor step 4 of program τ occurs.

Let k be the least x such that $\Gamma_\tau(X_\tau, A; x)$ is undefined eventually. Then $Q_\tau^k = \{\tau, \alpha \mid n(\tau,\alpha)\downarrow < k\}$ is a finite set.

By the choice of s_1, for every $\alpha \in Q_\tau^k$, with $\alpha \neq \tau$, Δ_α is a finite set. If $\alpha <_{\mathrm{L}} TP$, then $\lim_s d_\tau^\alpha(k)[s] \downarrow = d_\tau^\alpha(k) < \omega$. While if $\alpha \in TP$, then by proposition 8.13 (i), (ii), $\lim_s d_\tau^\alpha(k)[s] \downarrow = d_\tau^\alpha(k) < \omega$. And if $TP <_{\mathrm{L}} \alpha$, then $\lim_s n(\tau,\alpha)[s]$ does not exist. Therefore for every α, if $\alpha \in Q_\tau^k$ holds eventually and permanently, then $\lim_s d_\tau^\alpha(k)[s] \downarrow = d_\tau^\alpha(k) < \omega$.

By the assumption of this subcase, there are infinitely many τ-expansionary stages. By program τ, we can choose a stage $s_6 > s_5$ such that for every $\alpha \in Q_\tau^k$, $\lim_s d_\tau^\alpha(k)[s] \downarrow = d_\tau^\alpha(k)[s_6] \notin D$.

Let s_7 be a τ-expansionary stage $> s_6$ at which $l(D, \Phi_\tau(B, X_\tau)) > d_\tau^\alpha(k)$ for all $\alpha \in Q_\tau^k$. Then by the choice of s_7, step 5 of program τ occurs. This contradicts the choice of s_7.

Lemma 8.15 follows. □

Then by Lemma 8.15, Proposition 8.14 holds for τ in subcase 1a.

Subcase 1b. Otherwise.

Let s_1 be minimal $> s_0$ such that $\lim_s b(\tau)[s] \downarrow = b(\tau)[s_1]$, and let s_2 be minimal $> s_1$ such that there is no τ-expansionary stage $> s_2$. We first prove:

8.16 LEMMA. $\tau\hat{}\langle 1\rangle$ will be initialised only finitely often .

Proof. By the construction, for any $\mathcal{R}$-strategy $\tau' \supseteq \tau\hat{}\langle 0\rangle$, $\Gamma_{\tau'}$ will be built at no stage $s > s_2$, and for any $\mathcal{S}$-strategy $\alpha \supseteq \tau\hat{}\langle 0\rangle$, any odd stage $s > s_3$, α does not receive attention at stage s. And we can choose a stage $s_4 > s_3$ such that for any $\mathcal{R}$-strategy $\tau' \supseteq \tau\hat{}\langle 0\rangle$, and odd stage $s > s_4$, τ' does not receive attention at stage s.

By the choice of s_4, for any odd stage $s > s_4$, any $\beta \supseteq \tau\hat{}\langle 0\rangle$, β is not visited at stage s.

By the choice of s_2, Γ_τ is a finite set. By Proposition 8.13 (i), (ii), let s_5 be minimal $> s_4$ after which neither step 2 nor step 4 of program τ ever occurs.

By the choice of s_5 and by the construction, $\tau\hat{}\langle 1\rangle$ will be initialised at no even stage $> s_5$. By the choice of s_4, s_5, $\tau\hat{}\langle 1\rangle$ will be initialised at no stage $> s_5$.

Lemma 8.16 follows. □

Let s_5 be the stage chosen in the proof of Lemma 8.16. By the construction, for any even stage $s > s_5$, if τ is visited at stage s, then $\tau\hat{}\langle 1\rangle$ is visited at stage s. Therefore there are infinitely many $\tau\hat{}\langle 1\rangle$-stages. Clearly τ will never act after stage s_5. So (ii)–(iv) and (vi) hold, and (v) follows from the inductive hypothesis.

Hence Proposition 8.14 holds for τ in subcase 1b.

Case 2. α is an $\mathcal{S}$-strategy.

Suppose that $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_l$ lists all $\mathcal{R}$-strategies $\tau \subset \alpha$ which are active at α.

We first look at the case $l = 0$. We now need to prove (vi) and (ix). Notice that $l = 0$, so that $\vec{r}(\alpha)$ will never be defined.

By the inductive hypothesis, let s_1 be the least stage $> s_0$ such that for any ξ, if $\xi = \xi^-\hat{}\langle(\varphi, e, \tau, p, k)\rangle \subseteq \alpha$ for some e, τ, k, then for all $k' < k$, $\lim_s \gamma_\tau(k')[s] \downarrow= \gamma_\tau(k')[s_1]$.

By the choice of s_0, s_1, if $f_\alpha(x)$ is defined at a stage $s > s_1$, then $\Theta_\alpha(A; x)[s] \downarrow= B_s(x)$, and $A_s \upharpoonright (\theta_\alpha(x)[s] + 1) = A \upharpoonright (\theta_\alpha(x)[s] + 1)$.

Therefore if f_α is built infinitely often, then f_α is total, and $f_\alpha =^* B$. So (ix) holds.

Otherwise, there are only finitely many α-expansionary stages. Let x be the least y such that $f_\alpha(y)$ is undefined eventually. By the construction, either $l(B, \Theta_\alpha(A)) \not> x$ holds at almost every stage, or $u(\alpha) \leq \theta_\alpha(x)$ occurs infinitely often. In the former case, $B \neq \Theta_{e(\alpha)}(A)$, and α acts only finitely often . In the latter case, by the inductive hypothesis, $u(\alpha)[s]$ becomes unbounded, but $u(\alpha) \downarrow\leq \theta_\alpha(x)$ occurs infinitely often, and $\theta_\alpha(x)[s]$ becomes unbounded during the course of the construction. So $B \neq \Theta_{e(\alpha)}(A)$ holds, and α acts only finitely often . (vi) follows.

Hence Proposition 8.14 holds for α if $l = 0$.

Suppose that $l > 0$. We first prove Proposition 8.14 (vii).

8.17 Lemma. If $l > 0$, then Proposition 8.14 (vii) holds for α.

Proof. Suppose that α receives special attention at stage s. Let k be the x such that $\Delta_\alpha(B; x) \neq K(x)$ holds at the beginning of stage s, and $q(\alpha) = \delta_\alpha(k)$ or $q_*(\alpha) = \delta_\alpha(k)$ holds at the beginning of stage s.

Let v be the stage at which $\delta_\alpha(k)$ was created. Suppose that $v_1 < v_2 < \cdots < v_n$ are all stages v' such that $v < v' < s$, and step 12 of program α occurred via $\Delta_\alpha(B; k)[v]$. Let $v = v_0$. We prove by induction that for each $j = 0, 1, \cdots, n-1$,

$$A_v \upharpoonright (\delta_\alpha(k)[v] + 1) = A_{v_j} \upharpoonright (\delta_\alpha(k)[v] + 1).$$

Suppose by induction that the property holds for $j = i$.

Then either step 16 or step 17 of program α occurs at stage v_i.

If step 16 occurs at stage v_i, then $g_*(\alpha)$, $p_*(\alpha)$, and $q_*(\alpha)$ are created at stage v_i. By definition of $g_*(\alpha)$, $p_*(\alpha)$, and $q_*(\alpha)$ at stage v_i, there is no element $b \leq q_*(\alpha)$ which will be enumerated into A, unless there are elements $\leq \max\{p_*(\alpha)[v_i], g_*(\alpha)[v_i]\}$, which are enumerated into B.

If step 17 occurs at stage v_i, then if by the definition of $p(\alpha)[v_i]$, by the conditional restraint $\vec{r}(\alpha) = (p(\alpha), \delta(k)[v])$, and by the construction, there is no element

$b \le p(\alpha)[v_i]$ which has been enumerated into B during stages $[v_i, v_{i+1}]$, and so the restraint vector $\vec{r}(\alpha)[v_i]$ has ensured that there are no elements $\le \delta_\alpha(k)[v]$ which have been enumerated into A during stages $[v_i, v_{i+1}]$.

Therefore $A_v \upharpoonright (\delta_\alpha(k)[v]+1) = A_{v_n} \upharpoonright (\delta_\alpha(k)[v]+1)$.

We now prove the lemma by cases.

Case 1. α receives special attention at stage s via case 2a.

Let s^- be the stage at which the current $g_*(\alpha)$ and $p_*(\alpha)$ was created. Then $s^- = v_n$, and $\Theta_\alpha(A)[s^-] \upharpoonright (p_*(\alpha)+1) = B_{s^-} \upharpoonright (p_*(\alpha)+1)$, with $\theta_\alpha(p_*(\alpha))[s^-] \le \delta_\alpha(k)[v]$. By the choice of s, if b is the element which is enumerated into B at stage s, then $g_*(\alpha) < b \le p_*(\alpha)$. Therefore $\Theta_\alpha(A;b) \downarrow \ne B(b)$ holds during stage s. By the construction at stage s, a restraint vector $\vec{r}(\alpha) = (g_*(\alpha), \delta_\alpha(k)[v])$ is created. Therefore if $B_s \upharpoonright (g_*(\alpha)+1) = B \upharpoonright (g_*(\alpha)+1)$ holds, then $\Theta_\alpha(A;b)[s] \downarrow \ne B(b)$ will be preserved forever.

Case 2. α receives special attention at stage s via case 2b of the construction at stage s.

Suppose that $s^- = v_i$ is the stage at which the current $g(\alpha)$ and $p(\alpha)$ were created. By the proof above and by the construction at stage s^-. $\Theta_\alpha(A)[s^-] \upharpoonright (p(\alpha)+1) = B_{s^-} \upharpoonright (p(\alpha)+1)$. with $\theta_\alpha(p(\alpha)) \le q(\alpha) = \delta_\alpha(k)[v]$. Let b be the element which is enumerated into B at stage s. Then by the choice of s, $\Theta_\alpha(A;b) \downarrow \ne B(b)$ holds during stage s. By the assumption that $b > g(\alpha)$, the honestification of agitators of strategies $< \alpha$ will not enumerate elements $\le \delta_\alpha(k)[v]$ into A, while, the honestification of agitators of strategies $\beta \supseteq \alpha$ will not enumerate elements $\le \delta_\alpha(k)[v]$ into A unless $B_s \upharpoonright (g(\alpha)+1) \ne B \upharpoonright (g(\alpha)+1)$.

Lemma 8.17 holds in either case.

Lemma 8.17 follows. □

8.18 LEMMA. If $l > 0$, then Proposition 8.14 (ix) holds.

Proof. By the assumption of proposition 8.14 (ix), let s_1 be the least stage $> s_0$ after which f_α is never set to be totally undefined.

By the inductive hypothesis, let $s_2 > s_1$ be a stage such that for any $\xi = \xi^{-\frown}\langle(\varphi, e, \tau, p, k)\rangle \subseteq \alpha$, some e, τ, k, we have $\lim_s \gamma_\tau(k')[s] = \gamma_\tau(k')[s_2]$ for every $k' < k$.

By the choice of s_1, s_2, and by the definition of $p_*(\alpha)$, the parameter $p_*(\alpha)$ will be increasing in stages, so if we have $f_\alpha(x)$ defined at a stage $s > s_2$, then $\Theta_\alpha(A;x)[s] \downarrow = 0 = B_s(x)$, and $A_s \upharpoonright (\theta_\alpha(x)[s]+1) = A \upharpoonright (\theta_\alpha(x)[s]+1)$. Therefore $f_\alpha(x) = B(x)$ unless $l(B, \Theta_\alpha(A))[s]$ is bounded over the course of the construction.

By Proposition 8.13, if f_α is built infinitely often, then f_α is total, and we have $f_\alpha =^* B$.

Lemma 8.18 follows. □

8.19 LEMMA. If $l > 0$, then Proposition 8.14 (x) holds.

Proof. By the construction, if (x) (A) holds, then f_α is built infinitely often if and only if Δ_α is set to be totally undefined infinitely often.

By (B), we can choose a stage $s_1 > s_0$ after which Δ_α will not be set to be totally undefined.

Suppose to the contrary that $\Delta_\alpha(B) =^* K$. Then there are infinitely many stages s at which a permanent inequality $\Delta_\alpha(B;k) \downarrow\neq K(k)$ occurs for some k. By Proposition 8.14, there is a stage $v > s$ at which Δ_α is set to be totally undefined. But this means f_α is built infinitely often, a contradiction.

So Lemma 8.19 holds. □

8.20 LEMMA. If $l > 0$, then Proposition 8.14 (xi) holds.

Proof. Let s_1 be a stage $> s_0$ such that:

(a) for every $j \in \{1, 2, \cdots, l\}$, $n_j = n_j[s_1]$,

(b) $\lim_s b_0^\alpha[s] \downarrow= b_0^\alpha < \omega$ exists, and

(c) f_α is not built at any stage $> s_1$.

We prove the lemma by induction on i.

For $i = 1$, we need to prove the following:

8.21 LEMMA. For every $m > n_1 = n(\tau_1, \alpha)$, if $\lim_s b_1^\alpha[s] \downarrow= b_1^\alpha < \omega$ exists, and $\lim_s g_1^\alpha(m)[s]$ exists $= g_1^\alpha(m) < \omega$ exists, then $\lim_s d_{\tau_1}^\alpha(m)[s]$ exists $= d_{\tau_1}^\alpha(m) < \omega$.

Proof. We prove this by induction on m.

For $m = n_1 + 1$: By definition, $g_1^\alpha(m) = h_1^\alpha(m) = \max\{\varphi_{\tau_1}^*(d_{\tau_1}^\beta(m)),\ \varphi^*(d_{\tau_1}^\alpha(< m)) \mid \beta < \alpha\}$. Let s_2 be minimal $> s_1$ such that $\lim_s b_1^\alpha[s] = b_1^\alpha[s_1]$, and $\lim_s g_1^\alpha(m)[s] \downarrow= g_1^\alpha(m)[s_1]$, and let s_3 be minimal $> s_2$ such that $B_{s_3} \upharpoonright (g_1^\alpha(m)[s_2] + 1) = B \upharpoonright (g_1^\alpha(m)[s_2] + 1)$.

By the choice of s_1, $d_{\tau_1}^\alpha(m)$ will never be enumerated into D by the well ordering in step 10 of program α.

Suppose to the contrary that $d_{\tau_1}^\alpha(m)[s]$ becomes unbounded over the course of the construction. This means there are infinitely many stages at which repair(α) is defined to be $(e(\tau_1), \tau_1, m, k)$ for some k. By Proposition 8.13, let s_4 be minimal $> s_3$ such that for any $s > s_4$, any k, if repair(α) is defined to be $(e(\tau_1), \tau_1, m, k)$ at stage s for some k, then $m \leq k$.

Let s_5 be the least stage $> s_4$ at which repair(α) is defined to be $(e(\tau_1), \tau_1, m, k)$ for some k. Let v be the stage at which the current $\delta_\alpha(k)$ was created. And let s_6 be the stage at which either α receives special attention, or α is visited and step 12 occurs. By the choice of s_3, s_4, either α receives special attention or step 12 of program α occurs at stage s_6.

If α receives special attention, then by the choice of s_3, $\vec{r}(\alpha)[s_6]$ is created and is kept forever. So by Proposition 8.14 (vii), α acts only finitely often during the course of the construction, so that $\lim_s d_{\tau_1}^\alpha(m)[s] \downarrow= d_{\tau_1}^\alpha(m) < \omega$, a contradiction.

On the other hand, assume α is visited and step 12 occurs at stage s_6. In this case, $\Gamma_{\tau_1}(X_{\tau_1}, A; m)$ is set to be undefined at stage s_6, but $\Delta_\alpha(B;k) \downarrow\neq K(k)$ still holds.

If step 16 of α occurs at stage s_6, then f_α is built at stage s_6, contradicting the choice of s_1. Otherwise, then by the choice of s_5, and by Proposition 8.11, we have:

(1) step 17 of α occurs at stage s_6.

Let $d^{\alpha}_{\tau_i}(m_i)$ be the agitator which is chosen at stage s_6. Then:

(2) $p(\alpha)[s_6] \leq \varphi_{\tau_l}(d^{\alpha}_{\tau_l}(k))[v]$,

(3) $B_{s_6} \upharpoonright (g^{\alpha}_i(m_i)[v]+1) = B \upharpoonright (g^{\alpha}_i(m_i)[v]+1)$.

(3) follows from the choice of s_3 and from Proposition 8.11.

By (1), (2), and (3), if we will get a B-change after the agitation at stage s_6, the α will receive a special attention and preserve a permanent inequality between $\Theta_\alpha(A)$ and B.

Otherwise, let s_7 be the least α-expansionary stage $> s_6$, then $\Gamma_{\tau_i}(m_i)$ has been lifted by an X_{τ_i}-change, and step 12 of α occurs. If step 16 occurs then f_α is built. A contradiction. Otherwise, then by Propositions 8.10, 8.11, all agitators of α are honest, if there is an agitator $d^{\alpha}_{\tau_i}(y)$ is defined, and $\gamma_{\tau_i}(y) \leq \delta(k)[v]$, then step 17 occurs at stage s_7, so that we can repeat the same argument as that at stage s_6. Each time, when we enumerate some agitator $d^{\alpha}_{\tau_i}(y)$ into D, the corresponding γ-marker has been lifted. Since there are only finitely many $\gamma_{\tau_i}(y)$ threatening the $\delta_\alpha(k)[v]$-use, the above procedure of rectification is finite so that we will either reach step 16 of α or we create and preserve a permanent inequality between $\Theta_\alpha(A)$ and B. In either case, we get a contradiction.

So Lemma 8.21 holds for $m = n_1 + 1$.

Suppose now by induction that for all m', if $n_1 < m' < m$, we will have $\lim_s d^{\alpha}_{\tau_1}(m')[s] \downarrow = d^{\alpha}_{\tau_1}(m') < \omega$, that $\lim_s g^{\alpha}_1(m)[s] \downarrow = g^{\alpha}_1(m) < \omega$, and that s_1 is a stage $> s_0$ such that both (a) and (b) below hold:

(a) for any $s \geq s_1$, any m', if $n_1 < m' < m$, then $d^{\alpha}_{\tau_1}(m')[s] = d^{\alpha}_{\tau_1}(m')[s_1]$,

(b) for any $s \geq s_1$, $g^{\alpha}_1(m)[s] = g^{\alpha}_1(m)[s_1]$.

Let s_2 be a stage $> s_1$ such that $B_{s_2} \upharpoonright (g^{\alpha}_1(m)[s_1]+1) = B \upharpoonright (g^{\alpha}_1(m)[s_1]+1)$. Let s_3 be a stage $> s_2$ such that for any $s \geq s_3$, any k, if repair(α) is defined to be $(e(\tau_1), \tau_1, m, k)$ at stage s for some k, then $m \leq k$.

Suppose to the contrary that $d^{\alpha}_{\tau_1}(m)[s]$ becomes unbounded during the course of the construction. Then by the same argument as that for $m = n_1 + 1$, we will get a contradiction.

This completes the inductive proof of Lemma 8.21. □

By Lemma 8.21, Lemma 8.20 holds for $i = 1$. Suppose by induction that Lemma 8.20 holds for all j' with $1 \leq j' < j$. We prove:

8.22 LEMMA. Proposition 8.14 (xi) holds for $i = j$.

Proof. We prove it by induction on m.

For $m = n_j + 1$: Let s_1 be the least stage $> s_0$ for which:

(a) for any $s \geq s_1$, $b^{\alpha}_j[s] = b^{\alpha}_j[s_1]$,

(b) for any $s \geq s_1$, $g^{\alpha}_j(m)[s] = g^{\alpha}_j(m)[s_1]$, and

(c) $d^{\alpha}_{\tau_j}(m)$ is enumerated by step 10 of program α at no stage $s \geq s_1$.

Let s_2 be minimal $> s_1$ such that $B_{s_2} \upharpoonright (g^{\alpha}_j(m)[s_1]+1) = B \upharpoonright (g^{\alpha}_j(m)[s_1]+1)$. By Proposition 8.13, let s_3 be minimal $> s_2$ such that for any $s \geq s_3$, and any k, if repair(α) is defined to be $(e(\tau_j), \tau_j, m, k)$ at stage s, then $m \leq k$.

Suppose to the contrary that $d^{\alpha}_{\tau_j}(m)$ is enumerated into D infinitely often. Let s_4 be minimal $> s_3$ at which repair(α) is defined to be $(e(\tau_j), \tau_j, m, k)$ for some k. By the choice of s_4, for every i, if $1 \leq i < j$, then $m_i[s_4]$ is undefined.

Let $m_j[s_4] = m = n_j + 1$. Then $g^{\alpha}_j(m_j) = f^{\alpha}_1 \cdots f^{\alpha}_{j-1} h^{\alpha}_j(m_j)$.

Notice that for each i, if $1 \leq i < j$, then $m_i[s_4] \uparrow$, so for any y, if $\gamma_{\tau_i}(y) \leq \delta_\alpha(k)$, then

$$y \leq f^{\alpha}_{i+1} \cdots f^{\alpha}_{j-1} h^{\alpha}_{j-1}(m_j).$$

Using this, we have that, for any i, any y, if $1 \leq i < j$ and $\gamma_{\tau_i}(y) \leq \delta_\alpha(k)$, then

$$g^{\alpha}_i(y) \leq f^{\alpha}_1 \cdots f^{\alpha}_{i+1} \cdots f^{\alpha}_{j-1} h^{\alpha}_j(m_j) = g^{\alpha}_j(m_j).$$

And for any i, and any y, if $j < i \leq l$, and $\gamma_{\tau_i}(y) \leq \delta_\alpha(k)$, then

$$m_j > f^{\alpha}_{j+1} \cdots f^{\alpha}_{i-1} h^{\alpha}_i(y).$$

Therefore

$$g^{\alpha}_j(m_j) = f^{\alpha}_1 \cdots f^{\alpha}_{j-1} h^{\alpha}_j(m_j) \geq f^{\alpha}_1 \cdots f^{\alpha}_{i-1} h^{\alpha}_i(y) = g^{\alpha}_i(y).$$

Let s_5 be the least stage $> s_4$ at which repair(α) $= (e(\tau_j), \tau_j, n_j + 1, k)$ is cancelled. By the choice of s_2, s_3, either α receives special attention at stage s_5 or α is visited at stage s_5, and step 12 of program α occurs at stage s_5. If α receives special attention at stage s_5, then by the choice of s_2, a permanent restraint vector $\vec{r}(\alpha)$ is created at stage s_5. By Proposition 8.14 (vii), α acts only finitely often , a contradiction. Therefore, α is visited at stage s_5, in which case, $\Gamma_{\tau_j}(X_{\tau_j}, A; n_j + 1)$ is set to be undefined, but $\Delta_\alpha(B; k) \downarrow\neq K(k)$ is retained. There are two cases:

Case 1. There exist no i, y such that $j < i \leq l$, $y > n_i$ and $\gamma_{\tau_i}(y) \leq \delta_\alpha(k)$.

By the choice of s_4, for every $i \in \{1, 2, \cdots, j-1\}$, if $\gamma_{\tau_i}(y) \leq \delta_\alpha(k)$, then $y \leq f^{\alpha}_{i+1} \cdots f^{\alpha}_{j-1} h^{\alpha}_j(m_j)$. Therefore we have that:

$B_{s_2} \upharpoonright (g^{\alpha}_i(y) + 1) = B \upharpoonright (g^{\alpha}_i(y) + 1)$. By Propositions 8.10 and 8.11, this ensures that for any $i < j$, if α enumerates $d^{\alpha}_{\tau_i}(y)$ into D for some y to repair $\Delta_\alpha(B; k)$, then at the stage at which a response to the enumeration of $d^{\alpha}_{\tau_i}(y)$ occurs, we either create a permanent restraint vector $\vec{r}(\alpha)$, or we simply set $\Gamma_{\tau_i}(X_{\tau_i}, A; y)$ to be undefined, while $\Delta_\alpha(B; k) \downarrow\neq K(k)$ is retained.

Clearly if a permanent restraint vector $\vec{r}(\alpha)$ is created, then α acts only finitely often , a contradiction.

Therefore we will reach step 16 of program α at a stage $> s_5$, at which f_α is built, contradicting the choice of s_0.

Case 2. Otherwise, then there is an x such that $j < x \leq l$ and $m_x[s_4] \downarrow$.

Let i be the least such x, and let $m_i = m_i[s_4]$. By the choice of s_4, we have that $n_j + 1 = m_j > f^{\alpha}_{j+1} \cdots f^{\alpha}_{i-1} h^{\alpha}_i(m_i)$.

By the choice of s_4 and by the proof of case 1 above, there is a stage $s_6 > s_5$ say at which we have that for any x, if $1 \leq x < j$, then there is no y such that

$y > f_{x+1}^{\alpha} \cdots f_j^{\alpha} \cdots f_{i-1}^{\alpha} h_i^{\alpha}(m_i)$ and $\gamma_{\tau_x}(y) \leq \delta_\alpha(k)$, and $\Delta_\alpha(B;k) \downarrow \neq K(k)$ is still kept.

By the construction, at stage s_6, we enumerate $d_{\tau_i}^{\alpha}(m_i)$ into D and we define repair$(\alpha) = (e(\tau_i), \tau_i, m_i, k)$. By the definition of $g_i^{\alpha}(m_i)$, we have

$$\begin{aligned} g_i^{\alpha}(m_i) =& f_1^{\alpha} f_2^{\alpha} \cdots\ f_{j-1}^{\alpha}\ f_j^{\alpha} \cdots f_{i-1}^{\alpha} h_i^{\alpha}(m_i) \\ \leq& f_1^{\alpha} \cdots f_{j-1}^{\alpha} h_j^{\alpha}(n_j+1) \\ =& g_j^{\alpha}(n_j+1). \end{aligned}$$

And by the choice of s_2, $B_{s_2} \upharpoonright (g_i^{\alpha}(m_i)+1) = B \upharpoonright (g_i^{\alpha}(m_i)+1)$.

Let s_7 be the stage $> s_6$ at which repair$(\alpha) = (e(\tau_i), \tau_i, m_i, k)$ is cancelled. Then by the choice of s_3, and by Propositions 8.10 and 8.11, either α receives special attention at stage s_7 or α is visited at stage s_7, and step 12 of program α occurs at stage s_7. Clearly the former case is impossible. Therefore $\Gamma_{\tau_i}(X_{\tau_i}, A; m_i)$ is set to be undefined and $\Delta_\alpha(B;k) \downarrow \neq K(k)$ is retained.

Continuing with the same argument as above, we will get a stage at which step 16 of program α occurs, contradicting the choice of s_0.

So Lemma 8.22 holds for $m = n_j + 1$.

Suppose by induction that for all m', if $n_j < m' < m$, then we have that $\lim_s d_{\tau_j}^{\alpha}(m')[s]$ exists $= d_{\tau_j}^{\alpha}(m') < \omega$, that $\lim_s b_j^{\alpha}[s]$ exists $= b_j^{\alpha} < \omega$, and that $\lim_s g_j^{\alpha}(m)[s]$ exists $= g_j^{\alpha}(m) < \omega$.

Let s_1 be minimal $> s_0$ such that:

(a) for any m', if $n_j < m' < m$, then $\lim_s d_{\tau_j}^{\alpha}(m')[s] \downarrow = d_{\tau_j}^{\alpha}(m')[s_1]$,

(b) $d_{\tau_j}^{\alpha}(m)$ is enumerated by step 10 of program α at no stage $\geq s_1$, and

(c) $\lim_s g_j^{\alpha}(m)[s] \downarrow = g_j^{\alpha}(m)[s_1]$.

Let s_2 be minimal $> s_1$ such that $B_{s_2} \upharpoonright (g_j^{\alpha}(m)+1) = B \upharpoonright (g_j^{\alpha}(m)+1)$. By Proposition 8.13, let s_3 be minimal $> s_2$ such that for any $s \geq s_3$, any i, any k, if repair(α) is defined to be $(e(\tau_i), \tau_i, m, k)$ for some m_i at stage s, then $m \leq k$.

Suppose to the contrary that $d_{\tau_j}^{\alpha}(m)$ is enumerated infinitely often during the course of the construction. Then by the same argument as that for $m = n_j + 1$, we will get a contradiction.

So Proposition 8.14 (xi) holds for j, and Lemma 8.22 follows. □

We now prove Proposition 8.14 for an $\mathcal{S}$-strategy α. We consider seven subcases:

Subcase 2a. $l = 0$.

We have proved the proposition in this subcase before Lemma 8.17.

Subcase 2b. $l > 0$ and $\lim_s \vec{r}(\alpha)[s]$ exists $= (p, q)$ for some p, q.

Let s_1 be minimal $> s_0$ such that $\lim_s \vec{r}(\alpha)[s] \downarrow = \vec{r}(\alpha)[s_1] = (p, q)$ for some p, q. By the choice of s_1, and by program α, if α is visited at an even stage $s > s_1$, then $\alpha\hat{\ }\langle 2\rangle$ is visited at stage s.

By the proof of Lemmas 8.15 and 8.16, there is a stage $s_2 > s_1$ such that no strategy β with $\alpha \subset \beta$ and $\alpha\hat{\ }\langle 2\rangle \not\subseteq \beta$ acts at any odd stage $s > s_2$. Therefore $\alpha\hat{\ }\langle 2\rangle$ will be initialised at no stage $s > s_2$.

(ii)–(iv) and (vi) hold for α in this subcase. And (v) follows from the inductive hypothesis.

So Proposition 8.14 holds for α in subcase 2b.

Subcase 2c. $l > 0$ and there are only finitely many α-expansionary stages.

This follows immediately by the proof for subcase 2b above.

Subcase 2d. $l > 0$, and neither of $\lim_s \vec{r}(\alpha)[s]$ or $\lim_s n_1[s]$ exist.

Let i be the greatest j such that $\lim_s n_j[s]$ does not exist. By the construction, for every j, if $i < j \leq l$, $d^{\alpha}_{\tau_j}(n_{i+1} + 1)[s]$ becomes unbounded during the course of the construction.

In this case, there is an $x \leq n_{i+1}$ such that $\varphi_{\tau_{i+1}}(d^{<\alpha}_{\tau_{i+1}}(x))[s]$ becomes unbounded during the course of the construction. Let k be the least such x. Then $\alpha\hat{}\langle b\rangle\hat{}\langle(\varphi_{\tau_{i+1}}(d^{<\alpha}_{\tau_{i+1}}(k)))\rangle$ will be visited at infinitely many even stages, and for any $k' < k$, $\alpha\hat{}\langle(\varphi_{\tau_{i+1}}(d^{<\alpha}_{\tau_{i+1}}(k')))\rangle$ will be visited at only finitely many even stages.

By the choice of i and by the construction, $\alpha\hat{}\langle(\varphi_{\tau_{i+1}}(d^{<\alpha}_{\tau_{i+1}}(k)))\rangle$ will be initialised at only finitely many even stages, and then by the proof of Lemmas 8.15 and 8.16, it will be initialised only finitely often during the course of the construction.

So Proposition 8.14 follows for α in this subcase.

Subcase 2e. $l > 0$, $\lim_s \vec{r}(\alpha)[s]$ does not exist, $\lim_s n_1[s] \downarrow= n_1 < \omega$ exists, and $\lim_s b^{\alpha}_0[s] \downarrow= b^{\alpha}_0 < \omega$ exists, and f_α is built infinitely often.

But in this case, by Lemma 8.18, f_α is total and $f_\alpha =^* B$, contradicting the hypothesis of Proposition 8.14.

So subcase 2e does not occur.

Subcase 2f. $l > 0$, $\lim_s \vec{r}(\alpha)[s]$ does not exist, $\lim_s n_1[s] \downarrow= n_1 < \omega$ exists, and $\lim_s b^{\alpha}_0[s] \downarrow= b^{\alpha}_0 < \omega$ exists, f_α is built only finitely often, and $\Delta_\alpha(B)$ is total.

But then, by Lemma 8.19, $\Delta_\alpha(B)$ is total and $\Delta_\alpha(B) =^* K$, giving $K \leq_{\mathrm{T}} B$. This contradicts the assumption of Proposition 8.14.

Again, subcase 2f is impossible.

Subcase 2g. Otherwise.

Then let k be the least $x > n_1$, with $n_1 = \lim_s n_1[s]$, such that $\Delta_\alpha(B; x)$ diverges.

By the construction, $\alpha\hat{}\langle(d, k)\rangle$ will be visited at infinitely many even stages, and by the choice of k, there are only finitely many even stages at which some $\beta <_{\mathrm{L}} \alpha\hat{}\langle(d, k)\rangle$ is visited.

By the proof of Lemma 8.15 and 8.16, $\alpha\hat{}\langle(d, k)\rangle$ will be initialised only finitely often , and will be visited at infinitely many even stages.

So Proposition 8.14 for α follows in this subcase.

Hence, in each case, Proposition 8.14 holds for an $\mathcal{S}$-strategy α.

We now prove Proposition 8.14 for a decision strategy.

Case 3. α is a decision strategy. Let α^- be the longest β such that $\beta \subset \alpha$. There are six subcases:

Subcase 3a. $\alpha = \alpha^-\hat{}\langle(\varphi, e, \tau, p, k)\rangle$ for some e, τ, k.

Let τ^* be the longest τ' such that $\tau \subset \tau\hat{}\langle 0\rangle \subseteq \tau' \subset \tau'\hat{}\langle 0\rangle \subseteq \alpha^-$ and τ' has not been Σ^0_3-injured at α. By the construction in case 3a, $\gamma_{\tau^*}(k)[s]$ becomes unbounded during the course of the construction.

Let k^* be the least x such that $\gamma_{\tau^*}(x)[s]$ becomes unbounded. Then by the construction, $\alpha\hat{}\langle(\varphi, e(\tau^*), \tau^*, p, k^*)\rangle$ will be visited at infinitely many even stages.

By the choice of k^* and by the proof of Lemmas 8.15 and 8.16, we have that $\alpha\hat{}\langle(\varphi, e(\tau^*_, \tau^*, p, k^*)\rangle$ can be initialised only finitely often .

By the choice of k^*, for $a = (\varphi, e(\tau^*), \tau^*, p, k^*)$, (ii)–(vi) and (xii) of Proposition 8.14 follow.

Subcase 3b. $\alpha = \alpha^-\hat{}\langle(\varphi_\tau(d^\tau_\tau(k)))\rangle$ for some $\mathcal{R}$-strategy τ and some k.

By the construction, $\lim_s d^\tau_\tau(k)[s] \downarrow= d^\tau_\tau(k) < \omega$, but $\varphi_\tau(d^\tau_\tau(k))[s]$ becomes unbounded during the course of the construction. Let y be the least $x \leq k$ such that $\gamma_\tau(x)[s]$ are unbounded during the course of the construction. Then by the construction in subcase 3b, $\alpha\hat{}\langle(\varphi, e, \tau, p, y)\rangle$ will be visited at infinitely many even stages. (ii)–(vi) and (xii) follow.

Subcase 3c. $\alpha = \alpha^-\hat{}\langle(\varphi_\tau(d^\beta_\tau(k)))\rangle$ for some $\beta \neq \tau$ and some k.

Let $e = e(\tau)$.

If there is an $x \leq k$ such that $d^\beta_\tau(x)[s]$ becomes unbounded during the course of the construction, then let y be the least such x. So $\alpha\hat{}\langle(d^\beta_\tau(y))\rangle$ is visited at infinitely many even stages, so (ii)–(vi) follow.

Otherwise, we will have that $\lim_s d^\beta_\tau(k)[s]$ exists $= d^\beta_\tau(k) < \omega$, but that $\lim_s \varphi_\tau(d^\beta_\tau(k))[s]$ does not exist. In which case let y be the least x such that $\gamma_\tau(x)[s]$ becomes unbounded during the course of the construction. Then by the construction in subcase 3c, $\alpha\hat{}\langle(\varphi, e, \tau, p, y)\rangle$ will be visited at infinitely many even stages, in which case (ii)–(vi) and (xii) follow.

Subcase 3d. $\alpha = \alpha^-\hat{}\langle(\varphi_\tau(d^{<\beta}_\tau(k)))\rangle$ for some τ, β and k.

The proof for this subcase follows immediately from the construction in subcase 3d.

Subcase 3e. $\alpha = \alpha^-\hat{}\langle(d^\beta_\tau(k))\rangle$ for some $\beta \neq \tau$ and some k.

Again, this is immediate from the construction.

Subcase 3f. $\alpha = \alpha^-\hat{}\langle(c, k)\rangle$ for some k.

Taking, $e = e(\alpha^-)$, the proof for this subcase is immediate from the construction in subcase 3f. Let $e = e(\alpha^-)$.

So Proposition 8.14 holds in case 3.

This completes the inductive proof of Proposition 8.14. □

We now prove some properties of the outcomes along the true path TP.

8.23 PROPOSITION. (Outcomes Along True Path TP Proposition) Given a strategy $\xi \in TP$:

(i) If $\xi = \tau$ is an $\mathcal{R}$-strategy, then:

(a) if $\tau\hat{}\langle 0\rangle \in TP$, then

– Γ_τ is built infinitely often, and

– if $\Gamma_\tau(X_\tau, A)$ is total, then $\Gamma_\tau(X_\tau, A) =^* B$.

(b) if $\tau\hat{}\langle 1\rangle \in TP$, then $D \neq \Phi_{e(\tau)}(B, X_{e(\tau)})$.

(ii) If $\xi = \alpha$ is an $\mathcal{S}_e$-strategy, then:

(a) if $\alpha\hat{}\langle 2\rangle \in TP$, then $B \neq \Theta_e(A)$,

(b) if $\alpha\hat{}\langle\omega\rangle\hat{}\langle(d,k)\rangle\hat{}\langle(\theta, e, \alpha, p)\rangle \in TP$ for some $k \in \omega$, then $\Theta_e(A)$ is partial and so $B \neq \Theta_e(A)$.

(iii) If $\xi = \xi^-\hat{}\langle(\varphi_\tau(d^\alpha_\tau(k)))\rangle$ for some τ, α, k, then there is a φ-node δ such that $\xi \subset \delta \in TP$ and $i^\delta \leq e(\tau)$.

(iv) If $\xi = \xi^-\hat{}\langle(d^\alpha_\tau(k))\rangle$ for some τ, α, k, then there is a φ-node δ such that $\xi \subset \delta \in TP$ and $i^\delta \leq e(\tau)$.

(v) If $\xi = \xi^-\hat{}\langle(\varphi_\tau(d^{<\alpha}_\tau(k)))\rangle$ for some τ, α, k, then there is a φ-node δ such that $\xi \subset \delta \in TP$ and $i^\delta \leq e(\tau)$.

Proof. For (i): By the proof of Proposition 8.14, if $\tau\hat{}\langle 0\rangle \in TP$, then step 5 of program τ occurs infinitely often, so that Γ_τ is built infinitely often. By Proposition 8.8 (x), if $\Gamma_\tau(X_\tau, A)$ is total, then $\Gamma_\tau(X_\tau, A; x) = B(x)$ for every $x > b(\tau)$, where $b(\tau) = \lim_s b(\tau)[s]$. If $\tau\hat{}\langle 1\rangle \in TP$, then by the proof of Proposition 8.14, there are only finitely many τ-expansionary stages, so $l(D, \Phi_{e(\tau)}(B, X_\tau))[s]$ is bounded during the course of the construction. So $D \neq \Phi_{e(\tau)}(B, X_{e(\tau)})$, (i) follows.

For (ii): By the construction, if $\alpha\hat{}\langle 2\rangle \in TP$, then either $\lim_s \vec{r}(\alpha)[s]$ exists $= \vec{r}(\alpha)$, or there are only finitely many α-expansionary stages. In the former case, by Proposition 8.14 (vii), there is a fixed number b such that $\Theta_e(A; b)\downarrow = 0 \neq 1 = B(b)$ is created and preserved forever. In the latter case, $l(B, \Theta_e(A))[s]$ is bounded during the course of the construction, or there is an x such that $\theta_e(x)[s]$ is unbounded during the course of the construction. So in each case, $\Theta_e(A) \neq B$.

If $\alpha\hat{}\langle\omega\rangle\hat{}\langle(d,k)\rangle\hat{}\langle(\theta, e, \alpha, p)\rangle \in TP$ for some k, then by the construction, $\lim_s \varphi_{\tau_l}(d^\alpha_{\tau_l}(k))(k)[s] \downarrow = \varphi_{\tau_l}(d^\alpha_{\tau_l}(k)) < \omega$ but $\theta_e(\varphi_{\tau_l}(d^\alpha_{\tau_l}(k)))[s]$ becomes unbounded during the course of the construction. Therefore $\Theta_e(A)$ is partial.

For (iii)–(v): We prove these by induction on $e(\tau)$.

For $e(\tau) = 0$: In this case, we have to prove the following:

8.24 LEMMA. If $e(\tau) = 0$, and either

(a) $\xi = \xi^-\hat{}\langle(d^\alpha_\tau(k))\rangle$, or

(b) $\xi = \xi^-\hat{}\langle(\varphi_\tau(d^\alpha_\tau(k))\rangle$, or

(c) $\xi = \xi^-\hat{}\langle(\varphi_\tau(d^{<\alpha}_\tau(k)))\rangle$

for some τ, α, k, then there is a φ-node δ such that $\xi \subset \delta \in TP$ and $i^\delta = 0$.

Proof. We prove the Lemma by induction on the length of α. We consider three cases:

Case 1. $\xi = \xi^-\hat{}\langle(d^\alpha_\tau(k))\rangle$ for some τ, α, k.

Suppose without loss of the generality that $\lim_s d^\tau_\tau(k)[s]$ exists $= d^\tau_\tau(k) < \omega$. Therefore α is an $\mathcal{S}$-strategy. Note that $\alpha \subset \xi$ and $\alpha\hat{}\langle 2\rangle \not\subseteq \xi$. By Proposition 8.8

(vii), $\lim_s \vec{r}(\alpha)[s]$ does not exist. By the choice of ξ, the conditions of Proposition 8.14 (xi) hold for α. Therefore by Proposition 8.14 (xi) for α, $\lim_s g_1^\alpha(k)[s]$ does not converge over the course of the construction. By definition of g_1^α, $g_1^\alpha(k) = h_0^\alpha(k) = \max\{\varphi_\tau(d_\tau^\beta(k)) \mid \beta < \alpha\}$.

If there is a $\beta \subset \alpha$ such that $d_\tau^\beta(k)[s]$ becomes unbounded during the course of the construction, then by the construction, $\xi\hat{\ }\langle(d_\tau^\beta(x))\rangle \in TP$ for some $\beta \subset \alpha$ and some $x \leq k$. Then by the inductive hypothesis, there is a φ-node δ such that $\xi \subset \delta \in TP$ and $i^\delta \leq 0 = e(\tau)$.

Otherwise, by the construction in case 3, there is an $x \leq k$ such that $\xi\hat{\ }\langle(\varphi, 0, \tau, p, x)\rangle \in TP$.

Therefore Lemma 8.24 holds in case 1.

Case 2. $\xi = \xi^-\hat{\ }\langle(\varphi_\tau(d_\tau^\alpha(k)))\rangle$ for some τ, α, k.

If $d_\tau^\alpha(k)[s]$ becomes unbounded during the course of the construction, then by the construction, there is an $x \leq k$ such that $\xi\hat{\ }\langle(d_\tau^\alpha(x))\rangle = \xi_1 \in TP$. In which case, by the proof in case 1 above, there is a φ-node δ such that $\xi\hat{\ }\langle(d_\tau^\alpha(x))\rangle \subset \delta \in TP$ and $i^\delta \leq 0 = e(\tau)$.

Otherwise, by the construction, we have some number $x \leq k$ such that $\xi\hat{\ }\langle(\varphi, 0, \tau, p, x)\rangle \in TP$. So the Lemma holds in case 2.

Case 3. $\xi = \xi^-\hat{\ }\langle(\varphi_\tau(d_\tau^{<\alpha}(k)))\rangle$ for some τ, α, k.

Suppose that $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_m$ comprise all $\beta \subset \alpha$ such that $d_\tau^\beta(k) \downarrow$.

If there is an i such that $d_\tau^{\beta_i}(k)[s]$ becomes unbounded during the course of the construction, then let j be the greatest such i. By the construction, we have $\xi_1 = \xi\hat{\ }\langle(d_\tau^{\beta_i}(k))\rangle \in TP$. By the proof in case 1, there is a φ-node δ such that $\xi_1 \subset \delta \in TP$ and $i^\delta \leq 0 = e(\tau)$.

Otherwise, by the construction, there is an $x \leq k$ such that $\xi\hat{\ }\langle(\varphi, 0, \tau, p, x)\rangle \in TP$.

This completes the inductive proof for Lemma 8.24. □

Suppose by induction that Proposition 8.23 (iii)–(v) holds for ξ with corresponding $e(\tau) < e$. We need to prove:

8.25 LEMMA. For any ξ, τ, α, k, if $e(\tau) = e$, and either

(a) $\xi = \xi^-\hat{\ }\langle(d_\tau^\alpha(k))\rangle$,

(b) $\xi = \xi^-\hat{\ }\langle(\varphi_\tau(d_\tau^\alpha(k)))\rangle$, or

(c) $\xi = \xi^-\hat{\ }\langle(\varphi_\tau(d_\tau^{<\alpha}(k)))\rangle$,

then there is a φ-node δ such that $\xi \subset \delta \in TP$ and $i^\delta \leq e$.

Proof. We prove the Lemma by induction on the length of α. Suppose by induction that the Lemma holds for all $\alpha' \subset \alpha$. We consider three cases:

Case 1. $\xi = \xi^-\hat{\ }\langle(d_\tau^\alpha(k))\rangle$.

If there is an $\mathcal{R}$-strategy τ' such that $\tau' \subset \tau$ and τ' is active at α, then let τ^* be the longest such τ'. By Proposition 6.13, $e(\tau^*) < e(\tau) = e$. If $\varphi_{\tau^*}(d_{\tau^*}^\alpha(k))[s]$ becomes unbounded during the course of the construction, then by the construction, there is an $x \leq k$ such that $\xi_1 = \xi\hat{}\langle(\varphi_{\tau^*}(d^{\alpha}_{\tau^*}(x)))\rangle \in TP$. By the inductive hypothesis of Proposition 8.23 (iii)–(v), there is a φ-node δ such that $\xi \subset \xi_1 \subset \delta \in TP$ and $i^{\delta} \leq e(\tau^*) < e(\tau) = e$. Otherwise, by Proposition 8.14 (xi) for α, we have that $g^{\alpha}_j(k)[s]$ becomes unbounded during the course of the construction, where $\tau_j = \tau$, and $g^{\alpha}_j(k) = f^{\alpha}_1 \cdots f^{\alpha}_{j-1} h^{\alpha}_j(k)$.

If there is a $\beta \subset \alpha$ such that $\varphi_{\tau}(d^{\beta}_{\tau}(k))[s]$ becomes unbounded during the course of the construction, then there is a node $\xi_1 = \xi\hat{}\langle(\varphi_{\tau}(d^{\beta}_{\tau}(k)))\rangle \in TP$, so by inductive hypothesis, there is a φ-node δ such that $\xi \subset \xi_1 \subset \delta \in TP$ and $i^{\delta} \leq e(\tau) = e$.

Otherwise, we have by the construction that there is an $i < j$ such that $\xi\hat{}\langle(\varphi_{\tau_i}(d^{<\alpha}_{\tau_i}(x)))\rangle \in TP$ for some x. By Proposition 6.13, $e(\tau_i) < e(\tau_j) = e(\tau) = e$. So by the inductive hypothesis of Proposition 8.23 (iii)–(v), there is a φ-node δ such that $\xi \subset \delta \in TP$ and such that $i^{\delta} \leq e(\tau_i) < e$.

Case 2. $\xi = \xi^{-}\hat{}\langle(\varphi_{\tau}(d^{<\alpha}_{\tau}(k)))\rangle$.

This follows easily by the proof of case 1 and of case 3 of Lemma 8.24.

Lemma 8.25 follows. □

This completes the inductive proof of Proposition 8.23 (iii)–(v).

Proposition 8.23 follows. □

8.26 PROPOSITION. ($\mathcal{S}$-Satisfaction Proposition) Every $\mathcal{S}_e$ is satisfied.

Proof. Given $e \in \omega$, by Proposition 8.14, there is an $\mathcal{S}_e$-strategy α such that either $\alpha\hat{}\langle 2\rangle \in TP$ or $\alpha\hat{}\langle(\theta, e, \alpha, p)\rangle \in TP$. By Proposition 8.23 (ii), in either case, $B \neq \Theta_e(A)$. So $\mathcal{S}_e$ is satisfied, and Proposition 8.26 follows. □

8.27 PROPOSITION. ($\mathcal{R}$-Satisfaction Proposition) For every e, $\mathcal{R}_e$ is satisfied.

Proof. Given an $e \in \omega$, by Proposition 6.14, let τ be the longest $\mathcal{R}_e$-strategy $\in TP$.

If $\tau\hat{}\langle 1\rangle \in TP$, then by Proposition 8.23 (i), $D \neq \Phi_e(B, X_e)$, and so $\mathcal{R}_e$ is satisfied.

Suppose that $\tau\hat{}\langle 0\rangle \in TP$. By Proposition 8.23 (i), if $\Gamma_{\tau}(X_{\tau}, A)$ is total, then $\Gamma_{\tau}(X_{\tau}, A) =^* B$. Again, $\mathcal{R}_e$ is satisfied.

Suppose that $\Gamma_{\tau}(X_{\tau}, A)$ is partial. Then there are two cases:

Case 1. $\Phi_e(B, X_e)$ is partial.

This means that $D \neq \Phi_e(B, X_e)$, and $\mathcal{R}_e$ is satisfied.

Case 2. $\Phi_e(B, X_e)$ is total.

Let k be the least $x > b(\tau)$ such that $\gamma_{\tau}(x)[s]$ becomes unbounded during the course of the construction. By the assumption of the totality of $\Phi_e(B, X_e)$, there is a strategy β such that $d^{\beta}_{\tau}(k)[s]$ becomes unbounded during the course of the construction. Let α be the shortest such β. If $\alpha <_{\mathrm{L}} TP$, then α acts only finitely often , so $\lim_s d^{\alpha}_{\tau}(k)[s]$ exists $= d^{\alpha}_{\tau}(k) < \omega$. If $TP <_{\mathrm{L}} \alpha$, then α will be initialised infinitely often, so for a fixed k, $\lim_s d^{\alpha}_{\tau}(k)[s]$ exists $= d^{\alpha}_{\tau}(k) < \omega$. Therefore $\alpha \in TP$. We consider three subcases:

Then by the definition of the priority tree T, there is an $\mathcal{R}_e$-strategy τ' such that $\alpha \subset \tau' \in TP$, contradicting the choice of τ.

Subcase 2a. $d^\alpha_\tau(k)$ is enumerated infinitely often by step 10 of program α.

By the construction, for $\tau = \tau_{i+1}$, we have that $\varphi_{\tau_i}(d^\alpha_{\tau_i}(k))[s]$ becomes unbounded during the course of the construction. Therefore $\alpha\hat{\ }\langle(\varphi_{\tau_i}(d^\alpha_{\tau_i}(k)))\rangle \in TP$. By Proposition 8.23 (iii), there is a φ-node δ such that $\alpha \subset \delta \in TP$ and $i^\delta \leq e(\tau_i) < e(\tau_{i+1}) = e$. By the definition of the priority tree T, there is an $\mathcal{R}_e$-strategy τ' such that $\alpha \subset \tau' \in TP$, contradicting the choice of τ.

Subcase 2b. Otherwise.

In this subcase, we have that $d^\alpha_\tau(k)$ is enumerated infinitely often by step 17 of program α. Let $\tau_i = \tau$. By Proposition 8.14 (xi), $\lim_s g^\alpha_i(k)[s]$ does not exist, where $g^\alpha_i(k)[s] = f^\alpha_1 \cdots f^\alpha_{i-1} h^\alpha_i(k)$.

By the choice of α and by the totality of $\Phi_e(B, X_e)$, we have that $\lim_s h^\alpha_i(k)[s]$ exists $= h^\alpha_i(k) < \omega$.

Let j be the greatest x such that $\lim_s f^\alpha_{x-1} f^\alpha_x \cdots f^\alpha_{i-1} h^\alpha_i(k)[s]$ does not exist. Then $j < i$. By the construction, there is a node ξ such that $\alpha \subset \xi^- \subset \xi = \xi^-\hat{\ }\langle(\varphi^{\leq\alpha}_{\tau_j}(x)))\rangle \in TP$ for some fixed $x \in \omega$. By Proposition 8.23 (v), there is a φ-node δ such that $\xi \subset \delta \in TP$ and such that $i^\delta \leq e(\tau_j) < e(\tau_i) = e$. By the definition of the priority tree T, there is an $\mathcal{R}_e$-strategy τ' such that $\alpha \subset \tau' \in TP$. This contradicts the choice of τ.

Therefore in each subcase, case 2 does not occur, and so $\mathcal{R}_e$ is satisfied.

Proposition 8.27 follows. □

This completes the proof of Theorem 1.19. □

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