The Turing Definability of the Relation of “Computably Enumerable In”

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1. The ‘big picture’

Turing definability/invariance

Mathematical information content

Empirically observable phenomena

Scientific theory

Scientific questions e.g. in quantum theory

Mathematical foundations

Humanities v. Sciences - epistemological relativism

Computation theory

Church’s thesis
• Corresponding to the $i$th Turing machine, $\Phi_i$ denotes the $i$th partial computable (p.c.) functional $2^\omega \rightarrow 2^\omega$.

• $A$ is Turing reducible to a $B$ ($A \leq_T B$) iff $A = \Phi_i^B$, some $i \in \omega$ — and $A, B$ are Turing equivalent ($A \equiv_T B$) iff $A \leq_T B$ and $B \leq_T A$.

• The degree of unsolvability or Turing degree of $A$ is defined by

$$\text{deg} (A) = \{X \in 2^\omega \mid A \equiv_T X\}.$$ 

• $\leq$ is the induced partial ordering on $D$ (= the set of all degrees), $0$ = the least degree (consisting of all computable sets of numbers), and $\mathcal{D}$ is the structure $\langle \mathcal{D}, \leq \rangle$.

• $W_i^A = \text{dom} \ \Phi_i^A$ denotes the $i$th computably enumerable in $A$ ($A$-c.e.) set ($W_i = W_i^\emptyset$ being the $i$th c.e. set).

• The jump — or $n+1$th jump — of a set $A$ is defined by $A' = A^{(1)} = \{x \mid x \in W_x^A\}$ — or, $A^{(n+1)} = (A^{(n)})'$, respectively.
• The *jump operator* on degrees is defined by 
\[ a' = \deg(A'), \quad A \in a, \] where \( a < a' \), and \( a' \) is the l.u.b. of the degrees of sets c.e. in \( A \in a \).

• And write \( a^{(n+1)} = \deg(A^{(n+1)}) = (a^{(n)})' \).

• Define the standard \( \omega \)-*jump* of \( a \) by
\[ a^{(\omega)} = \deg(\bigoplus_{n \in \omega} A^{(n)}), \quad A \in a. \]

• Write \( D' \) for the structure \( \langle D, \leq, ' \rangle \).

• A relation on \( D \) is *Turing definable* iff it is describable in the first order theory of \( D \).

• Will assume standard computable sequences \( \{\Phi_{i,s}\}_{s \geq 0}, \{W_{i,s}^A\}_{s \geq 0} \) of finite approximations to the p.c. functionals and c.e. sets, respectively.

• Denote by \( A[s] \), or \( A^s \), the corresponding approximation to an expression \( A \) at a stage \( s \).

• The *restriction* \( \psi \upharpoonright x \) of a function \( \psi \) is taken to be its restriction to arguments \( \leq x \). And if \( \Phi \) is some functional and \( \Phi^A(x) \downarrow \), the *use* \( \varphi^A(x) \) of \( \Phi^A(x) \) will be taken to be \( \mu z[\Phi^A[z(x) \downarrow]] \).
Naturally arising information content in the Turing universe
3. Pseudo-jump operators

**Definition 3.1:** Say $J^n$ is an $n$-CEA operator iff there exist $j_0, j_1, \ldots, j_{n-1} \in \omega$ such that $J^k(A) \leq_T W_{j_k}^{J^k(A)}$, each $k < n$, $A \subseteq \omega$, and $J^n$ is inductively defined by

$$J^0(A) = A, \quad J^{k+1}(A) = W_{j_k}^{J^k(A)}, \quad (k < n).$$

- If $D = W_i - W_j$, some $i, j \geq 0$, say $D$ is a d-c.e. set (a difference of two c.e. sets).

**Lemma 3.2:** If $D = W_i - W_j$ is a d-c.e. set then $A \oplus (W_i^A - W_j^A)$ is a 2-CEA operator.

**Proof** (Lachlan; Jockusch and Shore [1984]).

- Make a special choice of the indices $j_0, j_1$ in definition 3.1 in relation to $i, j$ —
• Choose $j_0, j_1$ so that for each set $X$ of numbers

\[
\begin{align*}
W_{j_0}^X & = X \oplus \{ \langle x, s \rangle \mid x \in (W_{i,s+1}^X - W_{i,s}^X) \} \\
& \quad \cup \{ \langle x, s + 1 \rangle \mid x \in (W_{i,s+1}^X - W_{i,s}^X) \cap W_j^X \}, \\
W_{j_1}^{X \oplus Y} & = X \oplus \{ x \mid \exists s [\langle x, s \rangle \in Y \& \langle x, s \pm 1 \rangle \notin Y] \},
\end{align*}
\]

and define the operator $J^2$ by $J^2(A) = W_{j_1}^{J^1(A)}$, where $J^1(A) = W_{j_0}^A$.

• Need to verify:

(a) that $J^2(A) = A \oplus (W_i^A - W_j^A)$ and
(b) that $J^2$ is a 2-CEA operator.

• For (a): Notice that

\[
(J^2(A))_0 = (W_{j_1}^{J^1(A)})_0 = (J^1(A))_0 = (W_{j_0}^A)_0 = A
\]

and —
\[ x \in (J^2(A))_1 \iff x \in (W_{j_1}^{J^1(A)})_1 \]
\[ \iff \exists s [ \langle x, s \rangle \in (W_{j_0}^A)_1 \& \langle x, s + 1 \rangle \notin (W_{j_0}^A)_1 ] \]
\[ \iff x \in (W_i^A - W_j^A), \]
giving \( J^2(A) = A \oplus (W_i^A - W_j^A) \).

- And for (b): It follows straight from the definition that \( X \leq_T W_{j_0}^X \), so \( J^1 \) is a 1-CEA operator.

- To show that \( J^1(A) \leq_T W_{j_1}^{J^1(A)} \) need to check that \( W_{j_0}^A \leq_T J^2(A) = A \oplus (W_i^A - W_j^A) \).

- But \( x \in (W_{j_0}^A)_0 \iff x \in A \) and
\[ \langle x, s \rangle \in (W_{j_0}^A)_1 \]
\[ \iff x \in (W_{i,s+1}^A - W_{i,s}^A) \]
\[ \vee x \in (W_{i,s}^A - W_{i,s-1}^A) \cap W_j^A \]
\[ \iff x \in (W_{i,s+1}^A - W_{i,s}^A) \]
\[ \vee [ x \in (W_{i,s}^A - W_{i,s-1}^A) \& x \notin W_i^A - W_j^A ], \]
so \( W_{j_0}^A \leq_T A \oplus (W_i^A - W_j^A) \), and \( J^2 \) is a 2-CEA operator, as required. \( \square \)
4. Jump inversion

String notation:

- σ, τ etc. denote finite binary strings (i.e. 0-1 valued functions with finite ordinal domains).
- |σ| = the length of σ.
- σ ^ τ denotes the concatenation of σ, τ (= σ followed by τ).
- Write σ ⊆ τ iff τ is an extension of σ, σ ⊂ A iff σ is a beginning of (the characteristic function of) A.
- σ, τ are compatible (σ ≈ τ) iff σ ⊆ τ or τ ⊆ σ – otherwise write σ | τ.
- Write ∅ = the empty string, and S = the set of all strings.
- T : S → S is a tree iff ∀τ ⊂ σ ∈ S:
  
  (i) T(σ) ↓ ⇒ T(τ) ↓ ⊊ T(σ), and, for i ≤ 1,  
  (ii) T(τ ^ i) ↓ ⇒ T(τ ^ (1 − i)) ↓ | T(τ ^ i).
JUMP INVERSION THEOREM FOR 2-CEA OPERATORS: If $J^2$ is a 2-CEA operator, then for each $C \succeq_T \emptyset''$ there is a set $A$ such that $C \equiv_T J^2(A)$.

Note: Since pseudo-jumps are not necessarily degree theoretic, cannot just iterate the jump inversion theorem for 1-CEA operators.

PROOF.

• Let $J^2(X) = W_i^{W_j^X} = W_i(W_j^X)$ define the 2-CEA operator $J^2$ from indices $i, j$.

• Need to construct a set $A$ such that

$$W_i(W_j^A) \equiv_T A \oplus \emptyset'' \equiv_T C.$$ 

• Define an increasing sequence $\{\sigma_n\}_{n \geq 0}$ of strings chosen off a tree $T$ — and take (the characteristic function of) $A = \bigcup_{n \geq 0} \sigma_n$. 
• **Aim:**
  (i) Construct $T$ so that, for each string $τ$ with $|τ| > n$, $T(τ)$ decides whether $n ∈ W_σ$ for any $σ ⊃ T(τ)$, and then
  (ii) Choose the $σ$’s $⊂ A$ on $T$ to code $C$ into $A$ with help from $∅''$ and to decide whether $n ∈ J^2(A)$.

**Definition of $T$:**

- Define $T(∅) = ∅$.
- Assume $T(τ) ↓$ with $|τ| = n ≥ 0$ (that is, with $T(τ)$ at level $n$ on $T$).
- Ask if $∃ σ' ⊇ T(τ)$ with $n ∈ W_σ'$.
- Then define $T(τ^0), (τ^1) = σ^0, σ^1$, respectively, where $σ$ is the first such $σ'$ (in some standard listing of strings) if such a $σ'$ exists — and otherwise $σ$ is the first $σ' ⊃ T(τ)$.
- Notice that $T ≤_T ∅'$.
The construction of $\{\sigma_n\}_{n \geq 0}$

**Stage 0.** Define $\sigma_0 = T(\emptyset)$.

**Stage $2n+1$.** If there exists a string $\sigma \supset \sigma_{2n}$ at some level $x + 1$, say, on $T$ with $n \in W_i(W_j^\sigma \mid x)[|\sigma|]$, let $\sigma_{2n+1}$ be the first such $\sigma$. Otherwise let $\sigma_{2n+1} = \sigma_{2n}$.

**Stage $2n + 2$.** Define $\sigma_{2n+2} = \text{the first } T(\tau^\wedge C(n)) \supset \sigma_{2n+1}$.

Now observe the following sequence of facts —

(1) $\{\sigma_n\}_{n \geq 0} \leq_T C$.

- This holds because stages $2n + 1, n \geq 0$, can be carried out computably in $T' \leq_T \emptyset'' \leq_T C$, and stages $2n, n \geq 0$, can be carried out computably in $T, C \leq_T C$.

- Hence:

(2) $A \oplus \emptyset'' \leq_T C$; and —
Since, for each \( n \geq 0 \),
\[
n \in W_i(W_j^A) \iff \sigma_{2n+1} \supset \sigma_{2n}.
\]

Also:

\[
C \leq_T \{\sigma_n\}_{n \geq 0}
\]

Since, if one writes \( \sigma_n = T(\tau_n) \), each \( n \in \omega \), one has
\[
C(n) = \tau_{2n+2}(|\tau_{2n+2}| - 1)
= \sigma_{2n+2}(|\sigma_{2n+2}| - 1),
\]
each \( n \geq 0 \),

\[
\{\sigma_n\}_{n \geq 0} \leq_T A \oplus \emptyset''
\]

Since stage \( 2n + 1 \) can be carried out computably in \( \emptyset'' \), and stage \( 2n + 2 \) can be carried out computably in \( T \leq_T \emptyset' \) and \( A \), and

\[
\{\sigma_n\}_{n \geq 0} \leq_T W_i(W_j^A).
\]

To verify this, one first notices that since \( J^2 \) is a 2-CEA operator one has \( A \leq_T W_j^A \) and \( W_j^A \leq_T W_i(W_j^A) \). Then —
• To carry out stage $2n + 1$ one can compute

$$\{\langle \sigma, x \rangle \mid \sigma_{2n} \subset \sigma \subset A \& W_j^A \upharpoonright x = W_j^{\sigma} \mid \sigma \mid \upharpoonright x\}$$

with the use of $A$ and $W_j^A$, and hence also

$$\{T(\tau) \subset A \mid \tau \supset \tau_{2n}\}.$$

• Can then compute $\sigma_{2n+1}$ with help from $W_i(W_j^A)$.

• Similarly, one can carry out stage $2n$ of the construction using $A$ and $W_j^A$.

• So from (5) and (6) one gets:

$$(7) \ C \leq_T W_i(W_j^A) \text{ and } A \oplus \emptyset''.$$

• Combining (2), (3) and (7) the theorem follows. \(\square\)

**Note:** Only need the analogue of Friedberg’s theorem for a 2-CEA operator derived as in Lemma 3.2 from a d-c.e. set. This is a main ingredient of —
5. A jump and join theorem

The basic jump-join theorem for 2-CEA operators derived from a d-c.e. set: If $J^2$ is a 2-CEA operator derived from a d-c.e. set, then if $C \geq_T \emptyset'' \oplus X$ and $X \not\leq_T \emptyset'$, one can find an $A$ such that

$$X \oplus A \equiv_T C \equiv_T J^2(A).$$


- Choose $i, j$ s.t. $J^2(X) = W^X_j W^X_i = W_i(W^X_j)$.
- From the proof of Lemma 3.2, can assume that $W_i, W_j$ are given by equations of the form

$$W^X_j = X \oplus \{\langle x, s \rangle \mid x \in (W^X_{i'_s}, s+1 - W^X_{i'_s}, s)\} \cup \{\langle x, s + 1 \rangle \mid x \in (W^X_{i'_s}, s+1 - W^X_{i'_s}, s) \cap W^X_{j'_s}\},$$

$$W^X_i \oplus Y = X \oplus \{x \mid \exists s [\langle x, s \rangle \in Y \& \langle x, s \pm 1 \rangle \notin Y]\},$$

where $J^2(A) = A \oplus (W^A_{i'_s} - W^A_{j'_s}).$
• Without changing the degree of $X$, can assume that $X$ is $\emptyset'$-immune (i.e., has no infinite $\emptyset'$-c.e. subsets).

• Wish to construct a set $A$ satisfying the picture:

\[
A \oplus X \equiv_T C \equiv_T J^2(A) \equiv_T W_i(W_j^A)
\]

• As before, define $A = \bigcup_{n \geq 0} \sigma_n$, where the $\sigma$'s $\subset A$ are chosen to —

• Code $C$ into $A$ with help from $X$, —
• To force certain \( \langle x, s \rangle \in W_j^{\sigma'} \), \( \sigma' \supset \sigma_n \), —

• And to ensure that, for each \( n \in \omega \),

\[
n \in J^2(A) \iff n \in W_i(W_j^{\sigma} \upharpoonright M)[|\sigma|],
\]

\( \sigma \) corresponding to \( n \), and \( M \) depending on the construction.

**Note:** One cannot choose \( \{\sigma_n\}_{n \geq 0} \) off a tree \( T \) as in the jump inversion theorem above — since then one would not be able to obtain \( T \) both from \( A \oplus X \) and from \( J^2(A) \).

• Instead, one chooses \( \sigma_n \) off a tree \( T_{n,\pi} \) specifically adapted for stage \( n + 1 \) of the construction — and used in such a way that the construction is retrievable from \( A \oplus X \) and from \( J^2(A) \).

• \( T_{n,\pi} \) is constructed so that, for each \( T_{n,\pi}(\tau) \) at level \( s \) on \( T_{n,\pi} \), \( T_{n,\pi}(\tau) \) decides whether \( \langle n, s \rangle \in W_j^{\sigma} \) for any \( \sigma \supset T_{n,\pi}(\tau) \).
Definition of $T_{n,\pi}$:

- Let $T_{n,\pi}(\emptyset) = \sigma_{2n} \hat{\pi}$.
- Assume $T_{n,\pi}(\tau) \downarrow$ with $|\tau| = s \geq 0$ — i.e., with $T_{n,\pi}(\tau)$ at level $s$ on $T_{n,\pi}$.
- Ask if there exists a $\sigma' \supseteq T_{n,\pi}(\tau)$ with $\langle n, s + 1 \rangle \in W_{j,|\sigma'|}$.
- Then define $T_{n,\pi}(\tau^0), (\tau^1) = \sigma^0, \sigma^1$, respectively, where $\sigma$ is the first such $\sigma'$ if such a $\sigma'$ exists — and otherwise $\sigma$ is the first $\sigma' \supseteq T_{n,\pi}(\tau)$.
- Notice that $T_{n,\pi} \leq T \emptyset'$.

The construction of $\{\sigma_n\}_{n \geq 0}$:

Stage $n = 0$.
- Define $\sigma_0 = \emptyset$.
Stage $2n + 1$.

- Let $0^m$ denote a string of $m$ zeros.
- Define $T_{n,\pi}[\tau] = \text{the full subtree of } T_{n,\pi}$ above $T_{n,\pi}(\tau)$.
- Say $T_{n,\pi}[\tau]$ forces $n \in J^2(A)$ at level $s + 1$ iff $\langle n, s \rangle \in W^T_{j}^{T_{n,\pi}(\tau)}$ and $\langle n, s \pm 1 \rangle \notin W^T_{j}^{T_{n,\pi}(\tau)}$.
- $T_{n,\pi}[\tau]$ forces $n \notin J^2(A)$ iff for no $s \in \omega$, and no $\tau' \supseteq \tau$, does $T_{n,\pi}[\tau']$ force $n \in J^2(A)$ at level $s + 1$.
- At stage $2n + 1$, choose the least $m = \langle n, s \rangle$, some $s \in \omega$, such that either
  (a) $m \notin X$ and there is some $T_{n,0^m \pmb{^\hat{1}}}[\tau]$ which forces $n \in J^2(A)$ at some level $s + 1$, or
  (b) $m \in X$ and $T_{n,0^m \pmb{^\hat{1}}}[\emptyset]$ forces $n \notin J^2(A)$.
- Define $\sigma_{2n+1} = T_{n,0^m \pmb{^\hat{1}}}[\tau_{2n+1}]$ — where $\tau_{2n+1}$ = the least string $\tau$ for which (a) holds, if appropriate — and otherwise = $\emptyset$. 
Note: In case (a), $\sigma_{2n+1} \supset T_{n,0}^{m-1}(\tau_{2n+1})$ will give $n \in J^2(A)$ by virtue of $\langle n, s \rangle \in W^\sigma_{j}\sigma_{2n+1}$ and $\langle n, s \pm 1 \rangle \notin W^\sigma_{j}\sigma_{2n+1}$ — giving $\langle n, s \pm 1 \rangle \notin W^\sigma_j$, any $\sigma \supset \sigma_{2n+1}$.

But — From case (b) may get $\langle n, s \rangle \in W^\sigma_{j}\sigma_{p+1}$, some $p \geq 2n$ — but $\langle n, s \pm 1 \rangle \notin W^\sigma_{j}\sigma_{p+1}$, any $p$ — so $n \in J^2(A)$ again.

•• And — Cannot in case (b) treat the cases:
  
  (i) $\langle n, s \rangle \in W^\sigma_j$, some $\sigma \supset \sigma_{2n}$,
  (ii) $\langle n, s \rangle \notin W^\sigma_j$, any $\sigma \supset \sigma_{2n}$,

differently — to get

$$n \in J^2(A) \iff n \in J^2(\sigma_{2n+1}),$$

since could not then retrieve $\sigma_{2n+1}$ from $J^2(A)$.

Consequently —

Following case (b), the forcing of $n \notin J^2(A)$ is current at all stages $2p + 2 > 2n + 1$ for which there is no existing $\langle n, s \rangle \in W^\sigma_{j,p}$, with $s \leq p$. 
Stage $2n + 2$.

- Check if there is a current forcing of any $p \notin J^2(A)$, for which $\langle p, s \rangle \in W_{j,n}^{\sigma_{2n+1}}$, some $s \leq n$ — and which now ceases to be active —

- Choose the least $\pi \supset \sigma_{2n+1}$ for which some $\langle p, s \pm 1 \rangle \in W_{j,|\pi|}^{\pi}$, each such $p$ —

- And define $\sigma_{2n+2} = \pi^C(n)$.

- Notice — It now follows that in case (b) of stage $2n + 1$ — in which such a $\tau \supset \emptyset$ does not exist — one has $n \notin J^2(A)$ by virtue of $n \notin J^2(\sigma)$, all $\sigma \supseteq \sigma_{2p+2}$, some $p \geq n$.

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One needs to check that the construction can be carried out — which means verifying that $m$ exists in the definition of $\sigma_{2n+1}$.
• To see this — first notice that $S$ is $\Sigma_2$ — and so c.e. in $\emptyset'$ — where $S$ is defined by

$$S = \{m \mid (\exists \tau)(\exists s)[T_{n,0^m}^1[\tau] \text{ forces } n \in J^2(A) \text{ at level } s + 1] \}.$$ 

• If $S$ is finite then there is some $m \in X - S$ — so $m \in X$ and

$$(\forall \tau)(\forall s)[T_{n,0^m}^1[\tau] \text{ does not force } n \in J^2(A) \text{ at level } s + 1]$$

• Giving $m \in X$ and $T_{n,0^m}^1[\emptyset]$ forces $n \notin J^2(A)$ — which is (b).

• And if $S$ is infinite there is some $m \in S - X$, since $X$ is $\emptyset'$-immune.

• So for this $m$ — one has $m \notin X$ and $(\exists \tau)(\exists s)[T_{n,0^m}^1[\tau] \text{ forces } n \in J^2(A) \text{ at level } s + 1]$ — giving (a).
Now verify the following sequence of facts —

(1) \( \{\sigma_p\}_{p \geq 0} \leq_T C \).

- This is because stage 2\( n + 1 \) can be carried out computably in \( X \oplus \emptyset'' \leq_T C \) —

- And stage 2\( n + 2 \) is executed computably, apart from the coding of \( C(n) \) into \( \sigma_{2n+2} \).

- Hence:

(2) \( A \oplus X \leq_T C \); and

(3) \( W_i(W_j^A) \leq_T C \) —

- Since for each \( n \geq 0 \), \( m \) in stage 2\( n + 1 \) is retrievable from \( \sigma_{2n} \) and \( A \) and

\[
n \in W_i(W_j^A) \iff |\sigma_{2n+1}| > |\sigma_{2n}^0 m^1|.
\]

- To see this, first notice that —
\[ |\sigma_{2n+1}| > |\sigma_{2n}^{\hat{m}}1| \]
\[ \iff m \notin X \& (\exists \tau, s)[T_{n,0^m1}\tau \text{ forces } n \in J^2(A) \text{ at level } s+1] \]
\[ \iff m \notin X \& (\exists s)[\langle n, s \rangle \in W_j(T_{n,0^m1}(\tau_{2n+1})) \]
\[ \& (\forall \tau' \supset \tau_{2n+1})(\langle n, s \pm 1 \rangle \notin W_j(T_{n,0^m1}(\tau')))]] \]
\[ \iff m \notin X \& (\forall Y \supset T_{n,0^m1}(\tau_{2n+1}))(n \in W_i(W_j^Y)) \]
\[ \Rightarrow n \in W_i(W_j^A). \]

- Conversely, if \( n \in W_i(W_j^A) \) there exists a \( y \) such that \( \langle n, y \rangle \in W_j^A \) and \( \langle n, y \pm 1 \rangle \notin W_j^A \).

- This means that case (b) cannot apply at stage \( 2n+1 \) — since then the forcing of \( n \notin J^2(A) \) would cease to be active at some stage \( 2p+2 > 2n+1 \) — giving \( \langle n, y \pm 1 \rangle \in W_j^A \).

- But this means that at stage \( 2n+1 \) one chooses \( \tau_{2n+1} \supset \emptyset \) so that \( T_{n,0^m1}[\tau_{2n+1}] \) forces \( n \in J^2(A) \) at level \( y + 1 \) — so \( |\sigma_{2n+1}| > |\sigma_{2n}^{\hat{m}}1| \), as required.

- Also —
(4) \( C \leq_T \{ \sigma_p \}_{p \geq 0} \)

- Since one has \( C(n) = \sigma_{2n+2}(\lvert \sigma_{2n+2} \rvert - 1) \), each \( n \geq 0 \).

(5) \( \{ \sigma_p \}_{p \geq 0} \leq_T A \oplus X \).

- To reproduce stage \( 2n + 1 \) of the construction, one can use \( \sigma_{2n} \) and \( A \) to find \( m \)

- And then check whether \( m \in X \) or not to see which of cases (a) or (b) apply at stage \( 2n + 1 \).

- If \( m \in X \), so (b) applies, one has \( \tau_{2n+1} = \emptyset \).

- So \( T_{n,0^m \cdot 1}(\tau_{2n+1}) = \sigma_{2n} \cdot 0^m \cdot 1 \).

- If \( m \notin X \) — so (a) applies — one can find \( T_{n,0^m \cdot 1}(\tau_{2n+1}) \supset \sigma_{2n} \cdot 0^m \cdot 1 \) with help from \( A \).

- And then \( \sigma_{2n+1} = T_{n,0^m \cdot 1}(\tau_{2n+1}) \).

- And to reproduce stage \( 2n + 2 \) of the construction, one can computably obtain \( \pi \)

- And get \( \sigma_{2n+1} = \pi \cdot A(\lvert \pi \rvert) \).
\[(6) \{ \sigma_p \}_{p \geq 0} \leq_T W_i(W_j^A) .\]

- To verify this, first notice that since \( J^2 \) is a 2-CEA operator one has \( A \leq_T W_j^A \) and \( W_j^A \leq_T W_i(W_j^A) .\)

- To carry out stage 2n + 1 one can find m from \( \sigma_{2n} \) and A — verify whether \( n \in W_i(W_j^A) \) so as to see if one is in case (a) or (b) — and find \( T_{n,0}^m \cdot 1(\tau_{2n+1}) \), and hence \( \sigma_{2n+1} \), with help from A and \( W_j^A .\)

- While one can carry out stage 2n + 2 computably as far as obtaining \( \pi \) — and then \( \sigma_{2n+2} = \pi \uparrow A(|\pi|) \).

- So from (5) and (6) one gets:

\[(7) \ C \leq_T W_i(W_j^A) \ \text{and} \ A \oplus X .\]

- Combining (2), (3) and (7) the theorem follows. \( \Box \)

**Note:** The basic jump-join theorem is sufficient for a natural Turing definition of the jump. But a local version is needed to provide that of the relation of ‘computably enumerable in’, which applies to 2-CEA operators derived from *special* d.c.e. sets —