Post’s Programme: an update
Overview:

1. Post’s programme, and its extension to the Ershov hierarchy

2. Computational structure and new kinds of information content

3. The computability theoretic structure of the real world
The Turing landscape

- 1939 - Turing’s oracle Turing machines appear
- Provides a model of algorithmic content of structures which are based on the reals
The Turing landscape

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- Provides a model of algorithmic content of structures which are based on the reals

- 1944 - Post defines the degrees of unsolvability as a classification of reals in terms of their relative computability

- ... giving a landscape one can zoom in or out of
Mapping the landscape

- Look more closely at distinctive features of the Turing landscape ...

- And extract the underlying information content
Mapping the landscape

- Look more closely at distinctive features of the Turing landscape...

- And extract the underlying information content

- Intuition: Particular types of information are associated with distinctive causal structure
1944: Post’s Problem

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- **QUESTION**: Is there a non-recursive r.e. set of strictly lower degree of unsolvability than $K$ with respect to arbitrary recursive reducibility (now called Turing reducibility)?
The approach? - Look for a natural example

- Or, as we would ask for the property now: Is there a non-computable c.e. set of strictly lower Turing degree than $0'$?

- Of course - (Friedberg-Muchnik Theorem) - "YES"
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- For many-one reducibility - simple sets have the property

- For truth-table reducibility - hypersimple sets have it
Post’s Programme

- **Post’s Programme** (narrow version): Find a definable property of c.e. sets which delivers a non-computable c.e. set of strictly lower Turing degree than $0'$

- Gave rise to a range of immunity properties which one can use to say “the complement of $A \in \mathcal{E}$ avoids the c.e. sets”
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[Diagram showing $A$ and $W$ c.e. sets with $A \subseteq W$]
Remember - the Turing jump $A'$ is what one gets from the halting problem for a Turing machine with oracle $A$ ...

... and $A$ is high if $A' = 0''$ (and low if $A' = 0'$)
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- Martin’s Theorem (1966): Not only are there maximal sets in $O'$ (Yates, 1965) - but the degrees of maximal sets are exactly those which are high - so high is lattice invariant

Post’s Programme (broader and deeper version): Discover new relationships between natural information and computability-theoretic structure
What came later ...

Post’s remark in this paper that Hilbert’s tenth problem “begs for an unsolvability proof” had a major influence on my own work.

Martin Davis

Some negative and positive results ...

- (Martin, Soare, et al) Development of automorphism techniques to break link with $E$ ...

- **Martin:** Hypersimple is not definable in $E$ - can choose $\Phi$ so that $\Phi(h\text{-simple}) \subseteq h\text{-simple}$
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- **Harrington-Soare**: Make $\Phi(A) \in \text{high}$, given incomputable $A$
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- ... but use obstacle to building $\Phi(A) \in O'$ to get a dynamic property $Q$ giving a solution to Post's problem
More results and a question ...

- Harrington and Soare, 1996 - Non-low is not lattice invariant - but ...
- Harrington-Cholak, 1999 - This is an exception
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**OPEN QUESTION:** Find a natural lattice theoretic property characterising the high\(_2\) degrees
The need to find data non-monotonically

"... if a machine is expected to be infallible, it cannot also be intelligent. There are several theorems which say almost exactly that."

A.M. Turing, talk to the London Mathematical Society, February 20, 1947, quoted by Andrew Hodges in "Alan Turing - the enigma", p.361
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☐ By Arslanov’s Completeness Criterion - Every complete extension of Peano arithmetic of c.e. degree is of degree $0'$

☐ But - by the Low Basis Theorem - some are even low
The Ershov difference hierarchy

- **Idea**: Iterate boolean operations on c.e. sets to get a hierarchy for the $\Delta_2$ sets

- At bottom level get c.e. sets ... next level differences of c.e. sets -

- $A$ is 2-c.e. or d.c.e. iff $A = B - C$ for some c.e. sets $B, C$
The Ershov difference hierarchy

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- $A$ is **2-c.e.** or **d.c.e.** iff $A = B - C$ for some c.e. sets $B, C$

- **Dynamically**: $A$ has a sequence of finite approximations $A^s$ such that $|\{s: A^{s+1}(x) \neq A^s(x)\}| \leq 2$ (for $\leq n$ get n-c.e.)
Post's Problem: Immunity and Priority

Given the intersections of computably enumerable sets and their complements. Show that if $A$ is 1-c.e., then $A$ is c.e., and 2-c.e. if and only if $A$ is d.c.e.

(ii) Show that if $A$ is n-c.e., then $A$ can be expressed as a combination of n-c.e. sets and their complements.

[Hint: Show $A = (\ldots (A_1 \cap A_2) \cup A_3) \cap \ldots)$ where $x \in A \iff \text{defn} | \{s | A_s(x) \neq A_{s+1}(x)\} | = n$.]

This is the local framework we get from these two basic hierarchies: $0^\prime \mathrel{\leq_T} D \mathrel{\leq_T} D_n = \text{the n-c.e. degrees}$

Please do not be misled by my diagram into thinking that either of the hierarchies eventually includes everything below 0'. Far from it. This can...
QUESTION: How does Post's programme extend to other finite levels of the Ershov hierarchy?

- Is there an easy analogue of Martin's Theorem?

- Are the high d.c.e. degrees exactly those which contain a cohesive d.c.e. set?

- Are the high n-c.e. degrees exactly those which contain a cohesive n-c.e. or co-n-c.e. set?
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$0^\prime \quad 0^\prime \quad 0^\prime$

Low

High

$D_n =$ the n-c.e. degrees

Zooming in on the Turing landscape
“Give them something to take home” - Gian-Carlo Rota

Notice - $A = B - C$

is c.e. in $C$ -

So either $A$ is c.e. 
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- Now - exchange \( C \) for \( E = \{(x,s) : x \in B^s - A\} \) - \( E \) still c.e.
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- Now exchange $C$ for $E = \{(x,s) : x \in B^s - A\}$ - $E$ still c.e.

- And $A$ still c.e. in $E$ and $E \leq_T A$, in fact $\leq_m A$ - so:
  - Every incomputable d-c.e. $A$ has an incomputable c.e. $E \leq_T A$
A promising start ...

- **Defn** - X is **immune** iff it has no infinite c.e. subset - equivalently, iff it has no infinite **computable** subset

- **Defn** - X is **weak truth-table reducible** to Y, i.e. \( X \leq_{wtt} Y \) iff X is computable from Y with a computable bound on the size of each oracle question asked
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**Theorem** (Afshari, Barmpalías, C, Stephan) *Every d.c.e. wtt-degree $\neq 0$ contains an immune d.c.e. set*
Let $\#_E(x) = \text{the number of members of } E \leq x$

$B = \{(x, \#_E(x)) : x \in A\}$ is a d.c.e. set

Lachlan's c.e. set below d.c.e. A
\[ \#_E(x) = \text{the number of members of } E \leq x \]

\[ B = \{(x, \#_E(x)) : x \in A\} \quad \text{a d.c.e. set} \]

\[ B \leq_{wtt} A \text{ since } E \leq_m A \text{ and so } E \leq_{wtt} A \]

\[ \text{And } A \leq_{wtt} B \text{ since } x \in A \text{ iff } (x, y) \in B, \text{ some } y \leq x + 1 \]
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\( B \leq_{wtt} A \) since \( E \leq_m A \) and so \( E \leq_{wtt} A \)

\( A \leq_{wtt} B \) since \( x \in A \) iff \( (x,y) \in B \), some \( y \leq x + 1 \)

And \( B \) is immune - since assuming \( F \subseteq B \) infinite ...

Then \( F \) is not computable, since can compute \( E \) from \( F \):

Look for a \( (y, \#_E(y)) \in F \) with \( x \leq y \), and enumerate \( E \) to see if \( x \) is in \( E \) or not
moreover ...

- **Theorem** (Afshari, Barmpalias, C, Stephan) Every d.c.e. Turing degree $\neq 0$ contains a hyperimmune d.c.e. set

- And - these results can be extended to every finite level of the n-c.e. hierarchy -
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But cohesive d.c.e. sets are all co-c.e.

- Friedberg splitting theorem for incomputable c.e. A

\[ \text{incomputable c.e.} \]

\[ B \quad \text{incomputable c.e.} \quad \text{C} \]
But cohesive d.c.e. sets are all co-c.e.

- Friedberg splitting theorem for incomputable c.e. $A$

- Owings splitting theorem for d.c.e. $A-D$ which is not co-c.e.
Using weaker notions than cohesive?

- Weaker properties of sets characterising the high $\Delta^2_2$ degrees include hyperhyperimmune, r-cohesive and dense immune - but

(ABCS) None of these work for n-c.e. with $n > 1$
Using weaker notions than cohesive?

- Weaker properties of sets characterising the high $\Delta_2$ degrees include hyperhyperimmune, r-cohesive and dense immune - but

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  Theorem: If $A$ is n-c.e. and hh-immune then $A$ is co-c.e.
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- For the other notions the situation is more interesting
Re-enter Lachlan’s c.e. predecessor ...

- Let $A$ be n-c.e. Then:
  - $A$ can be r-cohesive \iff \exists an r-cohesive c.e. $E \leq_T A$

- And:
  - can be dense immune \iff \exists dense immune c.e. $E \leq_T A$

- Corollary ...
High c.e. degrees ⊊ degrees of r-cohesive d.c.e. sets ⊊ degrees of dense immune d.c.e. sets ⊊ high d.c.e. degrees
Since high c.e. degrees contain r-cohesive/dense immune co-c.e. sets

And - by C, Lempp, Watson - there are properly d.c.e. degrees above any incomplete c.e. degree
Since the equality holds over the degrees of co-c.e. sets
Since all r-cohesive/dense immune $\Delta_2$ sets are high.

And - Wu, Yang (2003) - there exist high d.c.e. degrees all of whose c.e. predecessors are low.
So - a number of questions to answer

What sort of natural information content does extend to other levels of the Ershov and high/low hierarchies??

- Can one extend automorphism techniques from the computably enumerable context to break the link between computability theoretic structure and natural information content?

- Or will Harrington-Nies type codings work here?

- What about the non-low\_2 n-c.e. degrees?
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New kinds of information content

- **Intuition:** A string $\sigma \in 2^{<\omega}$ is **random** if it has no shorter description than itself.

- Given TM $U$ define the **Kolmogorov complexity** relative to $U$ to be $C_\sigma = |\tau|$ for shortest $\tau$ with $U(\tau) \downarrow = \sigma$.
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- Want to say a real \( A \) is random if the Kolmogorov complexity of each beginning \( A \upharpoonright n \) is \( n \)-c.
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- Want to say a real $A$ is random if the Kolmogorov complexity of each beginning $A|n$ is $n$-c.

**But** - Martin-Löf - no such reals (we ask too much!)
New kinds of information content

- A TM $M$ is **prefix-free** if $M$ is defined on no pair $\sigma \preceq \rho$
- Can define a sensible notion of **universal** prefix-free $U$
- The **prefix-free complexity** $K(\sigma) = K_U(\sigma) = |\tau|$ for shortest $\tau$ with $U(\tau) \downarrow = \sigma$
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- 2 books available online -
New kinds of information content

- **Defn** (Levin, Schnorr, Chaitin): A is \textit{K-random} if
  \[ K(A^n) \geq n - c \text{ for all } n \]
New kinds of information content

Defn (Levin, Schnorr, Chaitin): A is K-random if
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Schnorr: K-random \iff Martin-Löf random

- algorithmic
- measure theoretic
New kinds of information content

- **Defn (Levin, Schnorr, Chaitin):** $A$ is **K-random** if $K(A \upharpoonright n) \geq n - c$ for all $n$

- **Schnorr:** K-random $\iff$ Martin-Löf random

- **Defn:** $A \leq_K B \iff K(A \upharpoonright n) \leq K(B \upharpoonright n) + c$ for all $n$

- $A$ is **K-trivial** $\iff A \leq_K \mathbb{N}$
New kinds of information content

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- **Defn:** \( A \leq_K B \iff K(A \upharpoonright n) \leq K(B \upharpoonright n) + c \) for all \( n \)

- **A is K-trivial** \( \iff A \leq_K N \)

- **Solovay**: There exist incomputable K-trivals
More robustness of notions ...

- **Defn:** $A \leq_{LR} B \iff$ every $B$-random real is $A$-random

- (Kučera-Terwijn) Say $A$ is **low for random** $\iff A \leq_{LR} \mathbb{N}$
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- A low for random \( \iff \) A is K-trivial

- **Notice:** \( \leq_{LR} \) is an extension of Turing reducibility, giving rise to the structure of the **LR-degrees** (see Nies, Barmpalias, Lewis, Miller ... )
A novel solution to Post’s problem

- **Nies**: K-trivials \( \leq \) low - in fact \( \leq \) superlow ...

... giving a very different solution to Post’s problem
A novel solution to Post’s problem

- **Nies**: $\text{K-trivials} \not\subseteq \text{low} - \text{in fact} \not\subseteq \text{superlow} \ldots$

  ... giving a very different solution to Post’s problem

- **Nies**: There is a $\text{low}_2$ c.e. degree $\geq$ all K-trivials!
A novel solution to Post’s problem

- **Nies**: $K$-trivials $\not\subseteq$ low - in fact $\not\subseteq$ superlow ...
  
  ... giving a very different solution to Post’s problem

- **Nies**: There is a low$_2$ c.e. degree $\geq$ all $K$-trivials!

**Open question**: Can this degree be low? (cannot be c.e.)
Real world randomness

- **Quantum randomness** is a familiar experimental and theoretical phenomenon.
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  It passes all reasonable statistical properties of randomness.

- **Cris Calude:** It is Turing incomputable.
Real world randomness

- **Quantum randomness** is a familiar experimental and theoretical phenomenon

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- **Cris Calude**: It is Turing incomputable

**Open question**: How random is quantum randomness?
Zooming out of the Turing landscape ...

- Can describe global relations in terms of local structure ...

- ... so capturing the emergence of large-scale structure
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- So capturing the emergence of large-scale structure

- Mathematically - formalise as definability over Turing degree structure
Zooming out of the Turing landscape ...

- Can describe global relations in terms of local structure ...
- ... so capturing the emergence of large-scale structure
- Mathematically - formalise as definability over Turing degree structure
- More generally - as invariance under automorphisms
Zooming out of the Turing landscape ...

**Fundamental problem:** Characterise the Turing invariant relations
Zooming out of the Turing landscape ...

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**Intuition:** These are key to pinning down how basic laws and entities emerge as mathematical constraints on causal structure
Zooming out of the Turing landscape ...

**Fundamental problem:** Characterise the Turing invariant relations

- **Intuition:** These are key to pinning down how basic laws and entities emerge as mathematical constraints on causal structure

- **Notice:** The richness of Turing structure discovered so far becomes the raw material for a multitude of non-trivially definable relations
But - the demise of computability theory?

**Bi-interpretability Conjecture**  
(Harrington): The Turing definable relations are exactly those with information content describable in second-order arithmetic
But - the demise of computability theory?

Bi-interpretability Conjecture (Harrington): The Turing definable relations are exactly those with information content describable in second-order arithmetic

- Notice: Conjecture not consistent with the existence of non-trivial Turing automorphisms ...

- And - its exact status is still a matter of controversy
More trouble ... this time with physics

By 1973, physicists had in place what was to become a fantastically successful theory of fundamental particles and their interactions, a theory that was soon to acquire the name of the ‘standard model’. Since that time, the overwhelming triumph of the standard model has been matched by a similarly overwhelming failure to find any way to make further progress on fundamental questions.

Introduction to Peter Woit: “Not Even Wrong - The Failure of String Theory and the Continuing Challenge to Unify the Laws of Physics”, Jonathan Cape, 2006
... I would like to state a theorem which at present can not be based upon anything more than upon a faith in the simplicity, i.e. intelligibility, of nature ... nature is so constituted that it is possible logically to lay down such strongly determined laws that within these laws only rationally completely determined constants occur (not constants, therefore, whose numerical value could be changed without destroying the theory) ...
Peter Woit: “One way of thinking about what is unsatisfactory about the standard model is that it leaves seventeen non-trivial numbers still to be explained, ...”
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String theory as a unifying explanatory theory - “the only game in town” ...?
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String theory as a unifying explanatory theory - “the only game in town” ... ?

The longstanding crisis of string theory is its complete failure to explain or predict any large distance physics. ... String theory is incapable of determining the dimension, geometry, particle spectrum and coupling constants of macroscopic spacetime. ... The reliability of string theory cannot be evaluated, much less established. String theory has no credibility as a candidate theory of physics.

Lee Smolin’s 5 Great Problems:

1. Combine general relativity and quantum theory into a single theory that can claim to be the complete theory of nature.

2. Resolve the problems in the foundations of quantum mechanics.

3. The unification of particles and forces problem: Determine whether or not the various particles and forces can be unified in a theory that explains them all as manifestations of a single, fundamental entity.

4. Explain how the values of the free constants in the standard model of physics are chosen in nature.

5. Explain dark matter and dark energy. Or, if they don’t exist, determine how and why gravity is modified on large scales.
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“The style of the string theory community ... is a continuation of the culture of elementary-particle theory. This has always been a more brash, aggressive, and competitive atmosphere, in which theorists vie to respond quickly to new developments ... and are distrustful of philosophical issues. This style supplanted the more reflective, philosophical style that characterized Einstein and the inventors of quantum theory, and it triumphed as the center of science moved to America and the intellectual focus moved from the exploration of fundamental new theories to their application.”


The state of physics today is like it was when we were mystified by radioactivity ... They were missing something absolutely fundamental. We are missing perhaps something as profound as they were back then.
“Causality is fundamental”

Early champions of the role of causality - Roger Penrose, Rafael Sorkin, Fay Dowker, and Fotini Markopoulou

It is not only the case that the spacetime geometry determines what the causal relations are. This can be turned around: Causal relations can determine determine the spacetime geometry ...

It’s easy to talk about space or spacetime emerging from something more fundamental, but those who have tried to develop the idea have found it difficult to realize in practice. ... We now believe they failed because they ignored the role that causality plays in spacetime. These days, many of us working on quantum gravity believe that causality itself is fundamental - and is thus meaningful even at a level where the notion of space has disappeared.

Lee Smolin, The Trouble With Physics, p.241
A deconstructed informational Universe

Described in terms of reals ... With natural laws based on algorithmic relations between reals

- Emergence described in terms of definability/invariance
- ... with failures of definable information content modelling quantum ambiguity
- ... which gives rise to new levels of algorithmic structure
- ... and a fragmented scientific enterprise
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## The Turing landscape revisited

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Post’s real anticipation

Early recognition of the importance of the relation between information content and computability theoretic structure

- ... leading to Hartley Rogers’ seminal focus on globally emergent relations
- ... giving rise to a Programme which remains full of interest, surprises, and open questions
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Thank you!

Solvability, Provability, Definability: The Collected Works of Emil L. Post

Martin Davis
Editor

BIRKHAUSER
Boston • Basel • Berlin