There is no low maximal d.c.e. degree – Corrigendum

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Abstract

We give a corrected proof of an extension of the Robinson Splitting Theorem for the d.c.e. degrees.

The purpose of this short paper is to clarify and correct the main result and proof contained in [1]. There we gave a simple proof that there exists no low maximal d.c.e. degree. This was obtained as an immediate corollary of the following strengthening of the Robinson Splitting Theorem (Theorem 1.7 of [1]):

For any c.e. set A, any Δ^0_2 low set L, if $L <_T A$, then there is a c.e. splitting $A_0 \oplus A_1 = A$ such that $A_i \oplus L <_T A$.

Denis Hirschfeldt (private communication) was the first to notice a problem with the particular application of the recursion theorem in the proof of this result, one which does not occur in the original Robinson proof [6].

We present below a reformulation of the use of the recursion theorem sufficient to correct the proof of our main result (the non-existence of a low maximal d.c.e. degree),[1] and to give a modified degree-theoretic extension of the Robinson Splitting Theorem, which (necessarily, it turns out[2]) replaces our old Theorem 1.7:

THEOREM. For any d.c.e. degree $\mathbf{l}$, any c.e. degree $\mathbf{a}$, if $\mathbf{l}$ is low and $\mathbf{l} < \mathbf{a}$, then there are d.c.e. degrees $\mathbf{a_0}, \mathbf{a_1}$ such that $\mathbf{l} < \mathbf{a_0}, \mathbf{a_1} < \mathbf{a}$ and $\mathbf{a_0} \cup \mathbf{a_1} = \mathbf{a}$.

Proof. Let L be a d.c.e. set of low degree. Let $L = L_0 - L_1$ for some c.e. sets L_0, L_1 such that $L_0 \supset L_1$. Let f be a $1-1$ computable function such that $L_0 = \{f(x) \mid x \in \omega\}$, and $M = f^{-1}(L_1)$. Then M is c.e. and $M \leq_T L$. (M is called *Lachlan's set* for L.)

Given c.e. set A, and d.c.e. set L, assume that $L <_{\mathrm{T}} A$ and L has low degree. First we construct ω-c.e. sets A_0, A_1 to satisfy the following requirements:

$\mathcal{R}:\quad A_0, A_1 \leq_{\mathrm{T}} A \wedge A \leq_{\mathrm{T}} A_0 \oplus A_1$

$\mathcal{S}_{e,i}:\quad A \neq \Phi_e(A_i \oplus L)$

where $i = 0, 1$, $e \in \omega$, and $\{\Phi_e : e \in \omega\}$ is a standard enumeration of all *partial computable* (p.c.) functionals Φ.

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[1]Downey and Yu (see [4]) have independently proved this.

[2]This is because Downey has proved (private communication) that in the old result of Cooper and Seetapun, and independently Li, that there exists a properly Δ^0_2 degree $\mathbf{a}$ which cups all the non-zero c.e. degrees, the degree $\mathbf{a}$ can be made low. In fact, Lewis has conjectured (private communication) that his single minimal complement below $\mathbf{0}'$ for the intermediate c.e. degrees (see [5]) can be made to be low.

Let $\mathbf{a} = \deg_{\mathrm{T}}(A)$, $\mathbf{l} = \deg_{\mathrm{T}}(L)$ and $\mathbf{a}_i = \deg_{\mathrm{T}}(A_i \oplus L)$ for $i = 0, 1$. Then $\mathbf{a}_i$ is an ω-c.e. degree, each $i = 0, 1$. By the $\mathcal{R}$-requirement, $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{a}$, and by the $\mathcal{S}$-requirements, $\mathbf{l} \leq \mathbf{a}_i < \mathbf{a}$, $i \in \{0, 1\}$. Then by the $\mathcal{S}$- and $\mathcal{R}$- requirements we will have $\mathbf{l} < \mathbf{a_i}$.

We will construct ω-c.e. sets so that they are c.e. in Lachlan's set M for L. The theorem follows from the following result from [2].

PROPOSITION 1 (Arslanov, LaForte, Slaman, 1998). Let A and C be sets such that C is c.e., A is c.e. in C, $C \leq_T A$, and A is ω-c.e. Then $deg(A)$ is d.c.e.

As usual, we adopt the notational conventions, found, for instance, in [3] or [7]. In particular, we write $[s]$ after functionals and formulas to indicate that every functional or parameter therein is evaluated at stage s. In particular, for an oracle X and p.c. functional Φ, $\Phi(X; y, s)$ means simply that at most s many steps are allowed for the computation from oracle X to converge; whereas $\Phi(X; y)[s]$ means further that the approximation X_s is used as the oracle. When using a c.e. or d.c.e. oracle, we adopt the common practice of taking the use function to be nondecreasing, and to be bounded by the current stage.

We satisfy the $\mathcal{R}$-requirement as follows. For each k, if k enters A at a stage s then we enumerate k into either A_0 or A_1. We may subsequently move k from A_i to A_{1-i}, $i = 0, 1$, finitely many times. Obviously, we will have $k \in A$ iff $k \in A_0 \cup A_1$. During the course of the construction, we may define an A_i-*restraint* for an $\mathcal{S}_{e,i}$-strategy, denoted by $r_{e,i}$. Given k, if $k \in A$, we define the *location function* loc (k) *of* k to be the index of the highest priority strategy $\mathcal{S}_{e,i}$ such that $r_{e,i} \geq k$. Then if $k \in A$, and loc $(k) = \langle e, i \rangle$, we enumerate k into A_{1-i}. We ensure that loc $(k) < k$. The membership of k relative to $A_0 \cup A_1$ may change due to a change of $r_{e,i}$ for some e, i, in which case we may move a number k from A_i to A_{1-i} during the course of the construction for some i. So the set splitting above does not guarantee $A_i \leq_{\mathrm{T}} A$ automatically. To satisfy $A_i \leq_{\mathrm{T}} A$, we build a Turing functional Ξ_i say, such that $A_i = \Xi_i(A, L)$. Before a number k enters A, we define $\xi_i(k) = k$ whenever we define $\Xi_i(A, L; k)$, while, after k enters A, we may redefine $\xi_i(k)$ for both $i = 0, 1$. During the course of the construction, whenever we move k from A_i to A_{1-i}, we have that for both $i = 0$ and 1, there is an element $l \leq \xi_i(k)$ which leaves L. On the other hand, of course, we ensure that for every k, and every $i \leq 1$, $\xi_i(k)[s]$ remains bounded during the construction. Hence we always have that $A_i \leq_{\mathrm{T}} A \oplus L \leq_{\mathrm{T}} A$ holds for each $i \leq 1$.

Suppose that loc $(k) = \langle e, i \rangle$. It is crucial that we simultaneously define a *permitting marker* $p_{e',i'}(k)$ for each $\langle e', i' \rangle < \langle e, i \rangle$. The intuition is as follows. It is possible that there are k', j', m, and n, such that $W_{j'}$ is the *test* set of cycle k' of the $\mathcal{S}_{e',i'}$-strategy, $\langle m, n \rangle \in W_{j'}$ is both valid and incorrect, and $k \leq \phi_{e'}(A_{i'} \oplus L; k')[v']$, where v' is the stage at which $\langle m, n \rangle$ was enumerated into $W_{j'}$, and then we define $p_{e',i'}(k)$ to be the maximum of all such n. And we define $p(k)$ to be the maximum of all $p_{e',i'}(k)$ for all $\langle e', i' \rangle < \langle e, i \rangle =$ loc (k) to be the *permitting marker of* k. Then we define $\xi_i(k) = \max\{k, p(k)\}$ whenever we define $\Xi_i(A \oplus L; k)$ after k enters A. In so doing, if at a later stage we find that there are e', i', k', and j' such that $\langle e', i' \rangle < \langle e, i \rangle$, $W_{j'}$ is the test set of cycle k' of the $\mathcal{S}_{e',i'}$-strategy, $\langle m, n \rangle \in W_{j'}$ becomes correct, in the sense that if $D_m = L\lceil n$ occurs, then we are able to restore a computation $\Phi_{e'}(A_{i'} \oplus L; k')[v']$ by redefining the membership of k to A_0 or A_1, where v' is the stage at which $\langle m, n \rangle$ was enumerated into $W_{j'}$.

Let s be a stage. We say that an element $\langle m, n \rangle \in W_j$ is *invalid at stage* s, if there is an $l \leq n$ such that $l \in D_m - L_s$, and *valid at stage* s, otherwise. We also say that $\langle m, n \rangle \in W_j$ is *correct at stage* s, if $D_m = L_s\lceil n$, and *incorrect at stage* s, otherwise.

An $\mathcal{S}_{e,i}$-strategy may define some auxiliary computably enumerable sets W_j, which are called *test sets* of the $\mathcal{S}_{e,i}$-strategy.

DEFINITION 1. Let $x \in A$. We define the *permitting marker of* x as follows.

(i) For $\langle e, i \rangle <$ loc (x), define:

$p_{e,i}(x) = \max\{\phi_{e,i}(k)[t] \mid x \leq \phi_e(A_i \oplus L; k)[t],\ (\exists m, n)[\langle m, n \rangle \in W_{j,\ \mathrm{att}}$ is both valid and incorrect at stage $s]\}$, where s is the current stage, and W_j is the test set of cycle k of the $\mathcal{S}_{e,i}$-strategy for some k.

(ii) Define $p(x) = \max\{p_{e,i}(x) \mid \langle e, i\rangle \leq \text{loc}\ (x)\}$.

DEFINITION 2. We say that s is a *progressive stage*, if for every j, if W_j is the test set of some cycle k of some $\mathcal{S}_{e,i}$-strategy for some e, i, then both (i) and (ii) below hold.

(i) If $g(j, s) = 1$, then there are m, n such that $\langle m, n\rangle \in W_j$, and such that $D_m = L_s\lceil n$,

(ii) If $g(j, s) = 0$, then for any m, n, if $\langle m, n\rangle \in W_j$, then $D_m \neq L_s\lceil n$,

where $g(j, s)$ is the limit value of $\lim_t g(j, t)$ which is observed at stage s.

At a nonprogressive stage, we will not enumerate W_j for any j.

To satisfy an $\mathcal{S}_{e,i}$-requirement, we modify the Robinson Splitting Theorem for the c.e. case (see [7], p.224). First, we need the following

LEMMA 1. If L is a low set (not necessarily d.c.e.) then
$X =_{\text{dfn}} \{j : \ (\exists m, n)[\langle m, n\rangle \in W_j \text{ and } D_m = L\lceil n]\} \leq_{\text{T}} \emptyset'$.

It follows from this Lemma, that there is a computable function
$g : \omega^2 \to \{0, 1\}$ such that for all j, $X(j) = \lim_v g(j, v)$.

To satisfy an $\mathcal{S}_{e,i}$-requirement $A \neq \Phi_e(A_i \oplus L)$ we build a p.c. functional $\Gamma_{e,i}$ to show that if $A = \Phi_e(A_i \oplus L)$, then $\Gamma_{e,i}(L)$ is total and $\Gamma_{e,i}(L) = A$, contradicting the hypothesis of the theorem.

To define $\Gamma_{e,i}(L; k)$, we build during the construction an auxiliary c.e. set V. By the Recursion Theorem, we may assume that we have in advance an index j such that $V = W_j$.

At a stage s the $\mathcal{S}_{e,i}$- strategy for some integer k may restrain some numbers $\leq n$. We will denote this number n by $r_{e,i}(k)[s]$.

Now the $\mathcal{S}_{e,i}$-strategy proceeds via an ω-sequence of cycles $k \geq 0$ as follows.

For a given $k \geq 0$:

(1) Wait for a stage s at which

(1a) $l(A, \Phi_e(A_i \oplus L)) > k$, where $l(A, \Phi_e(A_i \oplus L))$ is the usual length function;

(1b) For every j', if $W_{j'}$ is the test set of cycle k' of the $\mathcal{S}_{e,i}$-strategy for some $k' < k$, then $g(j', s) = 1$.

Then proceed to step 2 below.

(2) (*Open cycle* k). Then:

– Let n_0 be the maximal n' for which there are m', and j' such that $\langle m', n'\rangle \in W_{j'}$ is valid at stage s, and $W_{j'}$ is the test set of cycle k' of the $\mathcal{S}_{e,i}$-strategy for some k',

– Let $n = \max\{\varphi_e(k),\ n_0,\ p(x) \mid x \leq \phi_e(A_i \oplus L; k),\ \text{loc}\ (x) > \langle e, i\rangle\} + 1$, – Let $D_m = L_s\lceil n$ and enumerate $\langle m, n\rangle$ into W_j.

[**Remarks**. 1. At the current stage, there is no $\mathcal{S}$-strategy which *requires special attention* as described in step 3 below.

2. The definition of n ensures that if there is an $x \leq \phi_e(A_i \oplus L; k)$ such that loc $(x) > \langle e, i\rangle$ and the membership of x to A_0 and A_1 is changed, then $\langle m, n\rangle \in W_j$ becomes invalid automatically, so that we will never restore the computation $\Phi_e(A_i \oplus L; k)$ built at the current stage.

3. The definition of W_j above ensures that if s is a progressive stage, and $g(j, s) = 1$, then if m, n is such that $\langle m, n\rangle \in W_j$ is the first element enumerated into W_j and $D_m = L_s\lceil n$, then for the stage v at which $\langle m, n\rangle$ was enumerated into W_j, we have that $A_{i,v}\lceil(\phi_e(A_i \oplus L; k)[v] + 1) \subseteq A_{i,s}$.]

(3) (*Special Attention*) Wait for a stage $v \geq s$ at which $g(j, v) = 1$ and $D_{m'}\lceil n' = L_v\lceil n'$, for some $\langle m', n'\rangle \in W_j$. Choose $\langle m_0, n_0\rangle$ to be the $\langle m', n'\rangle$ with n' minimal.

Let s_0 be the stage at which $\langle m_0, n_0\rangle$ was enumerated into W_j.

(4) Define $\Gamma_{e,i}(L; k) = A_v(k)$ with $\gamma_{e,i}(k) = n_0$, and $r_{e,i}(k) = \max\{\text{old } r_{e,i}(k), n_0\}$. Define $A_{i,s+1}\lceil n_0 = A_{i,s_0}\lceil n_0$ and $A_{1-i,s+1}\lceil n_0 = A_s\lceil n_0 - A_{i,s+1}\lceil n_0$. We say that $\Gamma_{e,i}(L; k)$ is defined via $\langle m_0, n_0\rangle \in W_j$. And we say that the $\mathcal{S}_{e,i}$-strategy *receives special attention* at the current stage, and every lower priority $\mathcal{S}$-strategy is initialised.

[**Remark**. We ensure that $\Phi_e(A_i \oplus L; k)[s_0]$ is *recoverable* at the current stage, in the sense that $A_{i,s_0}\lceil(\phi_e(A_i \oplus L; k)[s_0] + 1) \subseteq A_i$ holds currently.]

(5) We define the A_i-restraint $r_{e,i}$ for the $\mathcal{S}_{e,i}$-strategy to be

$$r_{e,i} = \max\{r_{e,i}(k) \mid k \in \omega\}.$$

The possible outcomes for the $\mathcal{S}$-strategy are:

A. There are only finitely many cycles. This means that some cycle k eventually waits at step (1). In this case the $\mathcal{S}_{e,i}$ is satisfied via cycle k.

B. There is a cycle k which enumerates its test set W_j infinitely many times. This means that $\Phi(A_i, L)$ is partial, and so $\mathcal{S}_{e,i}$ is satisfied.

In this case, there are only finitely many k such that cycle k of the $\mathcal{S}_{e,i}$-strategy acts, and there is a unique k such that cycle k of the $\mathcal{S}_{e,i}$-strategy acts infinitely many times. Let j be such that W_j is the test set of cycle k of the $\mathcal{S}_{e,i}$-strategy. Then W_j is an infinite set, and for almost every pair after $\langle m, n\rangle$ is enumerated into W_j, $g(j, s) = 1$ will never occur, because we have, in fact, that $\lim_s g(j, s) \downarrow= 0$. Therefore the $\mathcal{S}_{e,i}$-strategy receives special attention only finitely many times.

In either of cases of A and B above, $r_{e,i}[s]$ will be bounded during the course of the construction.

We also notice that the A_i-restraint $r_{e,i}[s]$ will be increasing in stages, unless the $\mathcal{S}_{e,i}$-strategy is initialised.

C. There are infinitely many cycles, and each cycle acts only finitely often. In this case, we have that for each k, $\Gamma_{e,i}(L; k)$ is eventually defined and $\Gamma_{e,i}(L) =^* A$, which is a contradiction. In this case, all the $\mathcal{S}$-requirements are satisfied.

The construction.

We define the priority ranking of the requirements $\mathcal{S}$ as follows: for all e, i, e', i', $\mathcal{S}_{e,i} < \mathcal{S}_{e',i'}$, if $\langle e, i\rangle < \langle e', i'\rangle$, where $\langle ., .\rangle$ is the usual pairing function.

The $\mathcal{R}$-requirement indirectly participates in this priority ordering in the following way. If at a stage s we enumerate some $k \in A$ into A_0 or A_1, then we initialise all $\mathcal{S}_{e,i}$-strategies for all e, i such that $k < \langle e, i\rangle$. If the $\mathcal{S}_{e,i}$-strategy at a stage s is initialised, then at this stage $\Gamma_{e,i}(L)$ is set to be totally undefined, and for each k, the test set W_j of cycle k of the $\mathcal{S}_{e,i}$-strategy is cancelled.

For the $\mathcal{S}_{e,i}$-requirement $A \neq \Phi_e(A_i, L)$, we define the length function $l(e, i, s))$ as usual. We say that s is $\mathcal{S}_{e,i}$-*expansionary*, if $s = 0$ or $l(e, i, s) > l(e, i, v)$ for all $v < s$. At an $\mathcal{S}_{e,i}$-expansionary stage, s say, we set $k \leq l(e, i, s)$ to be the least (if any) x such that $\Gamma_{e,i}(L; x)$ is undefined, and proceed as above to define $\Gamma_{e,i}(L; k)$. In this case, we say that the $\mathcal{S}_{e,i}$-strategy *requires attention via* k. We say that the $\mathcal{S}_{e,i}$-strategy *requires attention* if it requires attention via k for some k.

DEFINITION 3. Let s be a stage. We say that the $\mathcal{S}_{e,i}$-strategy *requires special attention at stage* s, if there are k, j, m, and n, such that

(i) W_j is the test set of cycle k of the $\mathcal{S}_{e,i}$-strategy,
(ii) $\langle m, n\rangle \in W_j$,
(iii) $g(j, s) \downarrow= 1$,
(iv) $D_m = L_s \lceil n$.

In this case, let k_0 be the least k, and j_0 be the test set of cycle k_0 of the $\mathcal{S}_{e,i}$-strategy, s_0 be the least stage at which some $\langle m, n\rangle$ with the corresponding k_0 was enumerated into W_{j_0}, and let $\langle m_0, n_0\rangle$ be the $\langle m, n\rangle$ which was enumerated by cycle k_0 of the $\mathcal{S}_{e,i}$-strategy at stage s_0. We say that the $\mathcal{S}_{e,i}$-strategy *requires special attention* at stage s via $(k_0, j_0, m_0, n_0, s_0)$.

Assume A enumerated so that for every $s + 1$, $\mid A_{s+1} - A_s \mid= 1$, and $A_0 = \emptyset$.

The construction will proceed as follows.

DEFINITION 4. We describe the construction by stages.

Stage $s = 0$. Set $A_0 = A_1 = \emptyset$, and initialise all $\mathcal{S}$-strategies.
Stage $s > 0$. There are five cases.

Case 1. s is not progressive. Then go to the *Action Phase of Stage s*.

Case 2. (Special Attention) Otherwise, and there is an $\mathcal{S}$-strategy which requires special attention at stage s. Then let $\mathcal{S}_{e,i}$ be the highest priority strategy which requires special attention at stage s, and let the $\mathcal{S}_{e,i}$-strategy receives special attention at stage s as follows.

Suppose that the $\mathcal{S}_{e,i}$-strategy receives special attention at stage s via (k, j, m, n, v). Then:
– for each x, if $x \leq \phi_e(A_i\oplus; k)[v]$, loc $(x) > \langle e, i\rangle$, $x \in A_i - A_{i,v}$, then extract x from A_i,
– define $\Gamma_{e,i}(L; k) \downarrow= A(k)$ with $\gamma_{e,i}(k) = n$,
– define $r_{e,i} = \max\{\text{old } r_{e,i},\ \phi_e(A_i \oplus L; k)[v]\}$,
– initialise all $\mathcal{S}$-strategies with priority lower than the $\mathcal{S}_{e,i}$-strategy, and go to Case 4 of stage s below.

Case 3. (*Enumerating* W) Otherwise, and Case 4 has occurred since Case 3 was last implemented, then in decreasing order of the priority of strategies, for each $\mathcal{S}_{e,i}$-strategy with $\langle e, i\rangle < s$, implement the following:

Program. (Enumerating W) Let k be minimal such that

(1a) For any j', if $W_{j'}$ is the test set of cycle k' of the $\mathcal{S}_{e,i}$-strategy for some $k' < k$, then $g(j', s) \downarrow= 1$,
(1b) $\Gamma_{e,i}(L; k)$ is undefined,
(1c) $l(A, \Phi_e(A_i \oplus L)) > k$. Then:
– open cycle k,
– let j be such that W_j is the test set of cycle k of the $\mathcal{S}_{e,i}$-strategy,
– let n' be the maximal n such that there are m and j', such that $W_{j'}$ is the test set of cycle k' of the $\mathcal{S}_{e,i}$-strategy for some k', $\langle m, n\rangle \in W_{j'}$ is both valid and incorrect at stage s,
– let n_0 be the maximal n such that there are m, j', k' such that $\langle e', i'\rangle > \langle e, i\rangle$, and , $\langle m, n\rangle$ was enumerated into $W_{j'}$ at stage v for some v, where j' is the index such that $W_{j'}$ is the test set of cycle k' of the $\mathcal{S}_{e',i'}$-strategy, and $\langle m, n\rangle \in W_{j'}$ is both valid and incorrect at stage s, and there is an $x \leq \phi_e(A_i \oplus L; k), \phi_{e'}(A_{i'} \oplus L; k')[v]$ such that x-splitting has occurred after $\langle m, n\rangle$ was enumerated into $W_{j'}$.
– let $n = \max\{n', n_0, \phi_e(A_i \oplus L; k)[s]\} + 1$, let $D_m = L_s\lceil n$, and
– enumerate $\langle m, n\rangle$ into W_j.

Case 4. (*Splitting* A) In increasing ordering of x, for each $x \in A$, if $x \notin A_0 \cup A_1$, then execute the following:

1. (*Updating* loc (x)) Then if there are e, i such that $\langle e, i\rangle \leq \min\{\text{old loc}\ (x),\ x\}$, if old loc $(x)\downarrow$, such that $x \leq r_{e,i}$, then let $\langle e_0, i_0\rangle$ be the $<$-greatest such pair $\langle e, i\rangle$, and define loc $(x) = \langle e_0, i_0\rangle$, otherwise, then if old loc $(x)\downarrow$, then define loc $(x) =$ old loc (x), and x, otherwise.

2. Let loc $(x) = \langle e, i\rangle$. Define the permitting marker of x as prescribed in definition 1; let v be the greatest stage $\leq s$ at which some $l \leq p(x)$ is enumerated into L, and if there is no such l, then let $v = -1$; and enumerate $\langle x, M_s\lceil(v+1)\rangle$ into A_{1-i}, where M is the c.e. set of all stages at which some $l \notin L$ is enumerated into L.

3. Go on to Case 5 of stage s.

Case 5 (*Building* Ξ_i) For each $x \leq s$, and every $i \leq 1$, if $\Xi_i(A \oplus L; x)$ is undefined, then define $\Xi_i(A \oplus L; x)\downarrow = A_i(x)$, and define $\xi_i(x)$ as follows.

Case 1. $x \notin A_s$, then $\xi_i(x) = x$.

Case 2. Otherwise, then define $\xi_i(x) = \max\{x, p(x)\}$, where $p(x)$ is the permitting marker of x.

Action Phase of Stage s.

For every $x \leq s$, if $x \in A$ and $x \notin A_0 \cup A_1$, then enumerate x into $A_0 \cup A_1$ in the same way as in step 2 of Case 4.

We define the sets $A_i = \cup_{s\in\omega} A_{i,s}$, $i = 0, 1$, in the usual way, so completing the description of the construction.

THE VERIFICATION

We first notice that

LEMMA 2. There are infinitely many progressive stages.

Proof. This follows from Lemma 1. ■

LEMMA 3. For any k, j, m, n, v, and s, if W_j is the test set of cycle k of the $\mathcal{S}_{e,i}$-strategy, $\langle m, n\rangle$ was enumerated into W_j at stage v, and $\langle m, n\rangle \in W_j$ is valid at stage s, then

$$A_{i,v}\lceil(\phi_e(A_i \oplus L; k)[v] + 1) \subseteq A_{i,s}.$$

Proof. Suppose to the contrary that there is a stage s' such that $v < s' \leq s$, and there is an $x \leq \phi_e(A_i \oplus L; k)[v]$, and $x \in A_{i,v} - A_{i,s'}$. Let s_0 be the least s', and let v_0 be the greatest stage $\leq v$ at which an x-splitting occurred. By the construction, $v_0 < v$ holds. If loc $(x)[v_0] < \langle e, i\rangle$, then the $\mathcal{S}_{e,i}$-strategy is initialised at stage s_0, which contradicts the choice of s. Therefore loc $(x)[v_0] > \langle e, i\rangle$. Let $\langle x, M_{v_0}\lceil(a+1)\rangle$ be the axiom with a minimal of $x \in A_a$ at the end of stage v_0. By the choice of v, $M_{v_0}\lceil(a+1) = M_v\lceil(a+1)$, and by the construction at stage v, $n \geq p(x)[v_0] = p(x)[v]$. By the choice of s_0, there is an $l \leq p(x)[v_0]$ such that $l \in L_{v_0} \cap L_v$, and $l \notin L_{s_0}$. Therefore $\langle m, n\rangle \in W_j$ is no longer valid at the end of stage s_0, contradicting the choice of s_0 and s.

Lemma 3 follows. ■

We assume that if $r_{e,i}$ is not defined explicitly in the construction, then we define it to be -1. If the $\mathcal{S}_{e,i}$-strategy is initialised, then $r_{e,i}$ is set to be -1.

LEMMA 4. (i) For each x, if x is enumerated into A at a stage v, then for any $s \geq v$, loc $(x)[s] \geq$ loc $(x)[s+1]$.

(ii) Given e, i, and $v < s$, if the $\mathcal{S}_{e,i}$-strategy has not been initialised during stages $s' \in [v, s]$, then

$$r_{e,i}[v] \leq r_{e,i}[v+1] \leq \cdots \leq r_{e,i}[s].$$

(iii) For every $x \in A$, $\lim_s p(x)[s] \downarrow= p(x) < \omega$ exists.

Proof. Both (i) and (ii) follow straightforwardly from the construction. We look at (iii). By (i), let e, i be such that $\lim_s \operatorname{loc}(x)[s] = \langle e, i\rangle$, and let v be minimal such that $\operatorname{loc}(x)[s] = \langle e, i\rangle$ holds for all $s \geq v$. By the choice of v, for any $s \geq v$, if an x-splitting occurs at stage s, then there is an axiom $\langle x, M\lceil(a+1)\rangle$ which is enumerated into A_{1-i}. By the construction, for any $\langle e', i'\rangle < \langle e, i\rangle$, any j', and any k', if cycle k' of the $\mathcal{S}_{e',i'}$-strategy enumerates some $\langle m', n'\rangle$ into $W_{j'}$ via a computation $\Phi_{e'}(A_{i'} \oplus L; k')$ at a stage $s \geq v$ with $x \leq \phi_{e'}(A_{i'} \oplus L; k')[s]$, then at the next stage at which x leaves A_{1-i}, $\langle m', n'\rangle \in W_{j'}$ becomes invalid. Therefore $p(x)[v] \geq p(x)[v+1] \geq \cdots$. (iii) follows.

Lemma 4 holds. ■

LEMMA 5. $A_0 \cap A_1 = \emptyset$ and $A_0 \cup A_1 = A$.

Proof. This is immediate from the construction. ■

LEMMA 6. $A_i \leq A \oplus L, i = 0, 1$.

Proof. Given an x, if $x \notin A$, then for both $i = 0, 1$, $\Xi_i(A \oplus L; x) \downarrow= 0 = A_i(x)$. Suppose that x is enumerated into A at stage v, then by definition of Ξ_i in Case 5 of the construction, both $\Xi_0(A \oplus L; x)$ and $\Xi_1(A \oplus L; x)$ becomes undefined during stage v permanently. For any $s \geq v$, if $\Xi_i(A \oplus L; x)$ is defined in Case 5 of stage s, then $\xi_i(x) = \max\{x, p(x)\}$, and x is enumerated into A_0 or A_1 relative to $M\lceil(a+1)$, where a is the greatest stage at which some element $l \leq p(x)$ is enumerated into L. Therefore at the stage at which x leaves $A_0 \cup A_1$, the axioms for both $\Xi_i(A \oplus L; x)$ and $\Xi_{1-i}(A \oplus L; x)$ become invalid. This allows us to redefine $\Xi_i(A \oplus L; x)$ for both $i = 0$ and 1.

Therefore if $\Xi_i(A \oplus L; x)\downarrow$, then $\Xi_i(A \oplus L; x) = A_i(x)$ holds, for each $i \leq 1$.

By Lemma 4 (iii), for every x, $\Xi_i(A \oplus L; x)$ is defined eventually, $i = 0, 1$. Therefore for each $i \leq 1$, $\Xi_i(A \oplus L) = A_i$.

Lemma 6 follows. ■

LEMMA 7. Degrees of sets A_0 and A_1 are d.c.e.

Proof. By Proposition 1 it is enough to prove that

1) $A_i, i = 0, 1$, is ω-c. e. , and

2) A_i is c. e. in the Lachlan's set M, which was defined at the beginning of the proof.

For 1). Given x, we change the membership of x to A_0 and A_1 only if loc (x) changes. By Lemma 4 (i), there are at most x many times we redefine loc (x) during the course of the construction. By the construction in Case 4 and the Action Phase of Stage s for all s, we have that for each $i \leq 1$, $|\{s \mid A_{i,s}(x) \neq A_{i,s+1}(x)\}| \leq x$. A_i is an ω-c.e. set. 1) follows.

For 2). Let M be the c.e. set of all stages at which some element $l \notin L$ is enumerated into L. By definition of A_i, A_i is computably enumerable in M.

Lemma 7 follows. ■

Now we prove that all requirements are satisfied.

LEMMA 8. For all e, i, $\mathcal{S}_{e,i}$ are satisfied.

Proof. Given e, i. Suppose by induction that for every pair $\langle e', i'\rangle < \langle e, i\rangle$, the followings hold:

(a) $\Gamma_{e',i'}(L)$ is partial.

(b) $r_{e',i'}[s] \downarrow= r_{e',i'} < \omega$ exists.

(c) $\mathcal{S}_{e',i'}$ is satisfied.

Let $r = \max\{r_{e',i'}\}$, and let s_0 be minimal such that for all $s \geq s_0$, all x, and all $\langle e', i'\rangle < \langle e, i\rangle$, we have that the following properties hold:

(d) $r_{e',i'}[s] = r_{e',i'}[s_0]$.

(e) If $x \leq r$, and $x \in A$, then loc $(x)[s_0]$ is the limit value of loc (x).

(f) If $x \leq r$, and $x \in A$, then $x \in A_0 \cup A_1$ holds at the end of stage s_0.

We consider the following cases.

Case 1. There is a k' such that cycle k' of the $\mathcal{S}_{e,i}$-strategy acts infinitely many times.

Let k be the least (actually, the unique) such k', and let j be such that W_j is the test set of cycle k of the $\mathcal{S}_{e,i}$-strategy. Then:

(1) W_j is an infinite set.

(2) Every $\langle m, n\rangle \in W_j$ is either invalid or incorrect permanently.

Therefore $\lim_s g(j, s) \downarrow= 0$. By the construction, there are only finitely many stages at which cycle k' of the $\mathcal{S}_{e,i}$-strategy acts for all $k' > k$, and by the choice of k, for each $k' < k$, $\lim_s g(j', s) \downarrow= 1$, where j' is the index of the test set W of cycle k' of the $\mathcal{S}_{e,i}$-strategy. Therefore there are only finitely many stages at which cycle k' of the $\mathcal{S}_{e,i}$-strategy acts for some $k' \neq k$ during the course of the construction. So the $\mathcal{S}_{e,i}$-strategy requires special attention via an axiom created by cycle $k' \neq k$ of the $\mathcal{S}_{e,i}$-strategy only finitely many times.

We also notice that if the $\mathcal{S}_{e,i}$-strategy requires special attention via an axiom $\langle m, n\rangle \in W_j$ at a stage s, then we have that both $D_m = L_s\lceil n$ and $g(j, s) \downarrow= 1$ hold. Therefore the $\mathcal{S}_{e,i}$-strategy requires special attention via axioms created by cycle k of the $\mathcal{S}_{e,i}$-strategy only finitely often.

Then we have that $r_{e,i}[s]$ will be bounded during the course of the construction.

By Case 3 of the construction, we have that the number $n[s]$ created by cycle k of the $\mathcal{S}_{e,i}$-strategy will be unbounded during the construction. By Lemma 4 (iii), $\phi_e(A_i \oplus L; k)[s]$ will be unbounded during the course of the construction.

Case 2. Otherwise, and $\Gamma_{e,i}(L)$ is a total function.

We prove the following Lemma.

LEMMA 9. Given k, and $v + 1$, if:

(i) $v + 1 > s_0$;

(ii) $\Gamma_{e,i}(L; k)[t] \uparrow$ for all $t \leq v$, and $\Gamma_{e,i}(L; k)[v + 1] \downarrow$; and

(iii) $L_{v+1}\lceil(\gamma_{e,i}(L; k)[v + 1] + 1) = L\lceil(\gamma_{e,i}(L; k)[v + 1) + 1)$, then

$$\Gamma_{e,i}(L; k) \downarrow= A(k).$$

Proof. By the choice of s_0, any element x which enters A after stage s_0 is greater than r. Notice that $\Phi_e(A_i \oplus L; k) \downarrow= A(k)$ holds at the beginning of stage $v + 1$. By the choice of s_0 and by the assumption of (iii), for any x, if $x \in A_v$, $x \leq \phi_e(A_i \oplus L; k)[v + 1]$, and loc $(x)[v + 1] \downarrow$, then for any $s \geq v + 1$, x-splitting will not occur at stage s. Therefore it suffices to prove the following Lemma.

LEMMA 10. For any x, if $x \leq \phi_e(A_i \oplus L; k)[v + 1]$, and loc $(x)[v + 1] \uparrow$, then $x \notin A_i$.

Proof. By the construction, there is no x such that x-splitting occurs during stage $v+1$. Suppose that $\langle m, n\rangle$ is the element which is enumerated into W_j by cycle k of the $\mathcal{S}_{e,i}$-strategy, where $n = \gamma_{e,i}(k)[v + 1]$. By the assumption of (iii) of Lemma 9, there is a stage $s_1 > v + 1$ (the least) at which the $\mathcal{S}_{e,i}$-strategy receives special attention via $\langle m, n\rangle \in W_j$. By the construction for any $s' \in [v + 1, s_1]$, an element $x \leq \phi_e(A_i \oplus L; k)[v + 1]$ can be enumerated into A_i at stage s', only if there is an $l \leq n$ such that $l \in L_{s'} - L_{v+1}$, otherwise, the $\mathcal{S}_{e,i}$-strategy has received special attention via $\langle m, n\rangle \in W_j$ so that $r_{e,i} \geq \phi_e(A_i \oplus L; k)[v+1]$ has been created. This means that at the end of stage s_1, every $x \leq \phi_e(A_i \oplus L; k)[v + 1]$ which was enumerated into A_i since $v + 1$ has left A_i. By the definition of $r_{e,i}[s_1]$, there is no $x \leq \phi_e(A_i \oplus L; k)[v+1]$ which can be enumerated into A_i after stage s_1.

Lemma 10 follows. ■

So does Lemma 9. ■

By Lemma 9, if $\Gamma_{e,i}(L)$ is total, then for almost every k, $\Gamma_{e,i}(L; k) \downarrow= A(k)$. In this case, every $\mathcal{S}_{e,i}$ is satisfied.

Case 3. Otherwise.

Let k be the least k' such that $\Gamma_{e,i}(L;k')$ diverges, and let W_j be the test set of cycle k of the $\mathcal{S}_{e,i}$-strategy. We have that $\lim_s g(j,s) \downarrow= 0$. By the choice of k, and by cycle k of the $\mathcal{S}_{e,i}$-strategy, there are infinitely many stages at which $l(A, \Phi_e(A_i \oplus L)) \not> k$. Therefore $\Phi_e(A_i \oplus L) \neq A$.

In this case, $\Gamma_{e,i}(L)$ is partial, $\lim_s r_{e,i}[s] \downarrow= r_{e,i} < \omega$ exists, and $\mathcal{S}_{e,i}$ is satisfied.

We then have that one of the following holds:

(1) $\Gamma_{e,i}(L)$ is total, and $\Gamma_{e,i}(L) =^* A$.

In this case, every $\mathcal{S}_{e,i}$ is satisfied.

(2) $\Gamma_{e,i}(L)$ is partial, $\lim_s r_{e,i}[s] \downarrow= r_{e,i} < \omega$ exists and $\Phi_e(A_i \oplus L) \neq A$.

This completes the inductive proof of Lemma 8. ■

We have proved the theorem.

Remark. The construction can be easily modified to construct A_0 and A_1 such that both $A_0 \oplus L$ and $A_1 \oplus L$ are low sets.

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