3163/5164: Computability and Unsolvability

SOLUTIONS TO PROBLEMS 6

1) ① $\le_T$ is reflexive (notes), so so is $\equiv_T$.

② $\le_T$ is transitive. PROOF: Say $A \le_T B$, $B \le_T C$. Then, $\forall x \in \mathbb{N}$, $A(x)$ can be computed using a finite number of values of (the characteristic fn. of) B — say $B(y_1), \dots, B(y_k)$. But, since $B \le_T C$, each such $B(y_i)$ can, in turn, be computed using a finite number of values $C(z_1^i), \dots, C(z_{l_i}^i)$. But this means that $A(x)$ is computable using the finite no. of values $C(z_{j'}^j)$, $1 \le j \le k$, $1 \le j' \le l_j$. So $A \le_T C$. It follows that $\equiv_T$ is also transitive.

③ Symmetry of $\equiv_T$ follows straight from defn.

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2) ① Reflexivity of $\le$ follows directly from that of $\le_T$.

② Transitivity: $deg(A) \le deg(B)$ & $deg(B) \le deg(C)$
$\Rightarrow A \le_T B$ & $B \le_T C \Rightarrow A \le_T C$ (by Q1)
$\Rightarrow deg(A) \le deg(C)$.

③ Anti symmetry: $deg(A) \le deg(B)$ & $deg(B) \le deg(A)$
$\Rightarrow A \le_T B$ & $B \le_T A \Rightarrow A \equiv_T B \Rightarrow deg(A) = deg(B)$.

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3) (a) i) $A \le_m A \oplus B$ & $B \le_m A \oplus B$ (Problems 5 no 6), so $A \le_T A \oplus B$ & $B \le_T A \oplus B$.

ii) Say $A = \Phi_i^C$ & $B = \Phi_j^C$, some p.c. Φ_i, Φ_j.

Then $(A \oplus B)(x) = \begin{cases} \Phi_i^C(\frac{x}{2}) & \text{if } x \text{ even} \\ \Phi_j^C(\frac{x-1}{2}) & \text{if } x \text{ odd}. \end{cases}$

So $A \oplus B \leq_T C$.

(b) By (a) i), $\underset{\sim}{a} \leq \underset{\sim}{a} \cup \underset{\sim}{b}$, $\underset{\sim}{b} \leq \underset{\sim}{a} \cup \underset{\sim}{b}$, so $\mathrm{lub}\{\underset{\sim}{a}, \underset{\sim}{b}\} \leq \underset{\sim}{a} \cup \underset{\sim}{b}$.

By (a) ii) – If $\underset{\sim}{a} \leq \underset{\sim}{c}$ & $\underset{\sim}{b} \leq \underset{\sim}{c}$, then $\underset{\sim}{a} \cup \underset{\sim}{b} \leq \underset{\sim}{c}$.

So $\underset{\sim}{a} \cup \underset{\sim}{b} \leq \mathrm{lub}\{\underset{\sim}{a}, \underset{\sim}{b}\}$ – hence get $\underset{\sim}{a} \cup \underset{\sim}{b} = \mathrm{lub}\{\underset{\sim}{a}, \underset{\sim}{b}\}$.

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4) Similar to proof of corollary 7.8 of notes:

Let $\underset{\sim}{0} = \{A \mid A \text{ computable}\}$. Then ① $A, B \in \underset{\sim}{0} \Rightarrow$ $A \leq_T B$ and $B \leq_T A$ $\Rightarrow$ $\underset{\sim}{0} \subseteq$ some Turing degree.

② If $A \equiv_T B \in \underset{\sim}{0}$, then $A \leq_T B$ computable, so A computable, and hence $\in \underset{\sim}{0}$. So, from ①, $\underset{\sim}{0}$ is a Turing degree.

③ Given any $\deg(A)$, have $B \leq_T A$, each $B \in \underset{\sim}{0}$, giving $\underset{\sim}{0} \leq \deg(A)$.

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5) (a) Adapt soln. to Problems 3, no. 1.

(b) Adapt soln. to Problems 3, no. 4.

(c) ① X is A-c.e. $\Rightarrow X = W_e^A$, some e (defn. 8.3)

$\Rightarrow X = \{x \mid x \in W_e^A\} \Rightarrow X = \{x \mid \exists s\, [\underbrace{x \in W_{e,s}^A}_{A\text{-computable}}]\}$

$\Rightarrow X \in \Sigma_1^A$.

② Conversely, $X \in \Sigma_1^A \Rightarrow X = \{x \mid \exists s\, R^A(x,s)\}$, some A-computable R^A.

Define $\psi^A(x) = \begin{cases} 0 & \text{if } \exists s\, R^A(x,s) \\ \text{undefined} & \text{o.w.} \end{cases}$

Then ψ is a partial function computable from A.

So $\psi^A = \Phi_e^A$, some e, giving $X = \mathrm{dom}\, \psi^A$ $= \mathrm{dom}\, \Phi_e^A = W_e^A$ – so X is A-c.e.

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6) (a) X is A-c.e. $\Leftrightarrow X = W_e^A$, some e

$$\Leftrightarrow X = \{x \mid x \in W_e^A\} = \{x \mid \langle x, e\rangle \in A'\}$$

$$\Rightarrow X \leq_m A' \text{ via } f(x) = \langle x, e\rangle.$$

Conversely, $X \leq_m A'$ via f, say, $\Rightarrow X = \{x \mid f(x) \in A'\}$

$$= \{x \mid \langle y_0(x), y_1(x)\rangle \in A'\}, \text{ where } f(x) = \langle y_0(x), y_1(x)\rangle$$

$$= \{x \mid y_0(x) \in W^A_{y_1(x)}\} = \{x \mid \exists s\, \underbrace{[y_0(x) \in W^A_{y_1(x),s}]}_{A\text{-computable}}\}$$

$$\Rightarrow X \in \Sigma_1^A \Rightarrow X \text{ } A\text{-c.e. by Q 5) (c)}.$$

(b) $x \in K^A \Leftrightarrow \exists s\, [x \in W^A_{x,s}] \in \Sigma_1^A$ — so K^A is A-c.e.
And if K^A is A-computable, $\overline{K}^A$ is A-computable, so is A-c.e.
So $\overline{K}^A = W_i^A$, some i. But then $i \in W_i^A \Leftrightarrow i \in \overline{K}^A \Leftrightarrow i \notin W_i^A$ $\boxtimes$

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7) We construct $A_0, A_1, \ldots$ with $A_i \not\leq_T A_j$, each $i \neq j$.

Requirements:

$$R_{\langle i,j,k\rangle}: \quad A_i \neq \Phi_k^{A_j}, \quad \text{each } i, j, k \geq 0, \ (i \neq j).$$

Build sequences
$$\left.\begin{array}{l} \sigma_0 \subset \sigma_1 \subset \cdots \subset A_i \\ \tau_0 \subset \cdots\cdots \subset A_j \end{array}\right\} \text{each } i, j$$

The construction Let $s = \langle i, j, k\rangle$, $i \neq j$.

Stage 0 Define $\sigma_0 = \tau_0 = \phi$

Stage $s+1$ (assuming σ_s, τ_s already defined)

Let $x = |\sigma_s|$. Look for $\tau \supset \tau_s$ s.t. $\Phi_k^\tau(x)\downarrow$.

CASE I. τ exists, with $\Phi_k^\tau(x)\downarrow = y$, say.

Define $\sigma_{s+1} = \sigma_s{}^\frown(1-y)$, $\tau_{s+1} = \tau \supset \tau_s$

CASE II. No such τ exists.

Define $\sigma_{s+1} = \sigma_s{}^\frown 0$, $\tau_{s+1} = \tau_s{}^\frown 0$.

To prove – $\boxed{R_{\langle i,j,k\rangle} \text{ satisfied for all } i \neq j, k}$

PROOF: Assume $A_i = \Phi_k^{A_j}$, some $i \neq j$, some k.
Look at stage $\langle i,j,k\rangle + 1 = s+1$ of the construction, where $x = |\sigma_s|$. Since $A_i(x) = \Phi_k^{A_i}(x)$, $\exists\, \tau \supset \tau_s$ s.t. $\tau \subset A_j$ and $\Phi_k^{\tau}(x)\downarrow$. So by Case I of stage $s+1$, we have $\tau_{s+1} = \tau$, $\Phi_k^{\tau}(x)\downarrow = y$, say, and $\Phi_k^{A_j}(x)\downarrow = \Phi_k^{\tau_{s+1}}(x) = y$.
But $A_i(x) = \sigma_{s+1}(x) = 1 - y \neq \Phi_k^{A_j}(x)$. So $A_i \neq \Phi_k^{A_j}$. ⊠

Hence $A_i \not\leq_T A_j$ each $i \neq j$, so $\deg(A_i) \not\leq \deg(A_j)$, all i, j, $i \neq j$.
So $\{\deg(A_0), \deg(A_1), \dots\}$ is the required set of mutually incomparable Turing degrees.

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8) ① Write $X_i = \{\bar{\sigma} \mid \sigma \in \hat{W}_i\}$. Then $X_i \leq_m \hat{W}_i$ via $f(\sigma) = \bar{\sigma}$, so (by Thm. 7.3) X_i is c.e.

② Assume $A \subseteq \mathbb{N}$ is 1-generic, and let $\hat{W}_i$ be any c.e. set of strings. Then (choosing X_i as above) $A \Vdash X_i$ – ie, $\exists\, \sigma \subset A$ s.t. either $\sigma \in X_i$
or $\forall \tau \supset \sigma\, [\tau \notin X_i]$.

That is, either $\bar{\sigma} \in \hat{W}_i$ or $\forall \bar{\tau} \supset \bar{\sigma}\, [\bar{\tau} \notin \hat{W}_i]$, where $\bar{\sigma} \subset \bar{A}$. But this means $\bar{A} \Vdash \hat{W}_i$ —

so $\overline{A}$ is 1-generic. (Similarly, if $\overline{A}$ is 1-generic, then so is A.)

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9) ① $Y_i = \{\sigma \mid \exists x\, [\sigma(x)=0\ \&\ x \in W_i]\}$
$= \{\sigma \mid \exists x,s\, [\underbrace{\sigma(x)=0\ \&\ x \in W_{i,s}}_{\text{computable reln. of } \sigma, x, s}]\}$

So $Y_i \in \Sigma^0_1$, so Y_i c.e.

② So, if A is 1-generic, $A \Vdash Y_i$.

That is, $\exists \sigma \subset A$ s.t. <u>either</u> $\sigma \in Y_i$
<u>or</u> $\forall \tau \supset \sigma\, [\tau \notin Y_i]$

But if W_i is infinite, $\forall \sigma \exists \tau \supset \sigma\, [\tau \in Y_i]$ (just choose $x \in W_i$ with $x > |\sigma|$, and then choose $\tau \supset \sigma$ s.t. $\tau(x) = 0$).

Hence $\exists \sigma \subset A$ with $\sigma \in Y_i$, where $\exists x\, [\sigma(x) = 0\ \&\ x \in W_i]$.

So $\exists x \in \overline{A} \cap W_i$, ie., $W_i \nsubseteq A$.

Follows that A is either finite or immune.

③ But by Q 7) above, A 1-generic $\Leftrightarrow$ $\overline{A}$ 1-generic.

So if A is finite, must have
$\overline{A} = \mathbb{N} - A$ = an infinite c.e. set $\subseteq \overline{A}$,
contradicting $\overline{A}$ finite or immune.

So deduce that every 1-generic is immune (and hence not c.e.).