MATH-516301

This question paper consists of 4 printed pages, each of which is identified by the reference MATH 516301

No calculators allowed

Examination for the Module MATH 5163

(January 2008)

ADVANCED COMPUTABILITY AND UNSOLVABILITY

Time allowed : 3 hours

Do not answer more than *FOUR* questions.

All questions carry equal marks.

1. (**a**) Show that $f(x,y) = x \times y$ and

$$\mathrm{sg}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0 \end{cases}$$

are primitive recursive functions.

Deduce that if the function

$$|x - y| = \begin{cases} x - y & \text{if } x \geq y, \\ y - x & \text{if } x < y \end{cases}$$

is known to be primitive recursive, then so is the **remainder function** $\mathrm{rm}(x,y)$ defined by:

$$\mathrm{rm}(x,y) = \begin{cases} \text{the remainder upon division of } y \text{ by } x, & \text{if } x \neq 0, \\ y & \text{otherwise.} \end{cases}$$

(**b**) Write a Turing program for the function $\mathrm{sg}(n)$ defined in part (a), and briefly explain why your program works.

Say which function $f(n)$ is computed by the Turing machine below, which uses extra tape symbols ε, η as counters:

$q_0\,1\,\varepsilon\,q_1$	$q_2\,\eta\,1\,q_3$	$q_6\,1\,L\,q_6$
$q_1\,\varepsilon\,R\,q_1$	$q_3\,1\,R\,q_4$	$q_6\,0\,L\,q_7$
$q_1\,1\,R\,q_1$	$q_4\,0\,1\,q_5$	$q_7\,1\,L\,q_7$
$q_1\,0\,R\,q_2$	$q_5\,1\,R\,q_5$	$q_7\,\varepsilon\,0\,q_7$
$q_2\,0\,\eta\,q_2$	$q_5\,0\,\eta\,q_5$	$q_7\,0\,R\,q_0$
$q_2\,1\,R\,q_2$	$q_5\,\eta\,L\,q_6$	

(**c**) The **Printing Problem** for a Turing machine T and a symbol S_k is the problem of determining, for any given input x, whether T ever prints the symbol S_k.

Find a Turing machine T for which the printing problem is unsolvable.

[You may assume that the universal Turing machine U has unsolvable halting problem.]

2. (**a**) Define: A is **computably enumerable**.

Show that if $A, B \subseteq \mathbb{N}$ are computably enumerable, then $A \cup B$ and $A \cap B$ are also computably enumerable.

Prove the **Complementation Theorem**: A is computable if and only if both A and $\overline{A}$ (the complement of A) are computably enumerable.

(**b**) Show that the following sets are computably enumerable:

(i) Range $\varphi_e = \{y \mid \exists x \varphi_e(x) = y\}$,

(ii) Graph $\varphi_e = \{\langle x, y\rangle \mid \varphi_e(x) = y\}$.

where, for each $x \in \mathbb{N}$, φ_x is the x^{th} partial computable function in some standard listing.

Deduce that a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is partial computable if and only if Graph(f) is computably enumerable.

(**c**) For each $i \geq 0$, let $W_{i,s}$ be the standard computable approximation to the i^{th} computably enumerable set W_i.

Let $X = W_e$, $Y = W_f$ be given computably enumerable sets, where we write $X^s = W_{e,s}$, $Y^s = W_{f,s}$.

Let $X \searrow Y = \{z \mid \exists s(z \in X^s - Y^s)\}$ and $X \searrow\!\!\searrow Y = (X \searrow Y) \cap Y$. Show that:

(i) Both $X \searrow Y$ and $X \searrow\!\!\searrow Y$ are computably enumerable sets.

(ii) $X \searrow Y = (X - Y) \cup (X \searrow\!\!\searrow Y)$.

(iii) If $X \searrow\!\!\searrow Y$ is finite then $X - Y$ is computably enumerable.

(iv) Assuming that numbers are enumerated into *at most one* of $X^s = W_{e,s}$, $Y^s = W_{f,s}$ at any given stage s, and taking $\hat{X} = X \searrow Y$, $\hat{Y} = Y \searrow X$, prove the **Reduction Principle** for X, Y:

Given computably enumerable sets X and Y, there exist computably enumerable sets $\hat{X} \subseteq X$ and $\hat{Y} \subseteq Y$ such that $\hat{X} \cap \hat{Y} = \emptyset$ and $\hat{X} \cup \hat{Y} = X \cup Y$.

3. (**a**) Define: $A \subseteq \mathbb{N}$ is *simple*.

(i) Show that no simple set is computable.

(ii) Show that if A is a simple set and W is an infinite c.e. set then $A \cap W$ is an infinite c.e. set.

(iii) Deduce that if A and B are simple sets then $A \cap B$ is also simple.

[You may assume that the intersection $X \cap Y$ of two computably enumerable sets X and Y is also computably enumerable.]

(**b**) Prove the **Fixed Point Theorem with Parameters**:

If $f(x, \overrightarrow{y})$ is a computable function, then there exists a computable $k(\overrightarrow{y})$ such that

$$\varphi_{f(k(\overrightarrow{y}),\overrightarrow{y})} = \varphi_{k(\overrightarrow{y})}.$$

Given $f(x, \overrightarrow{y})$ computable, deduce that there exists a computable $k(\overrightarrow{y})$ such that

$$W_{f(k(\overrightarrow{y}),\overrightarrow{y})} = W_{k(\overrightarrow{y})}.$$

(**c**) We say $C \subseteq \mathbb{N}$ is **creative** if and only if

1) C is c.e., and

2) There is a computable function f, called a **creative function** for C, such that for each e

$$W_e \subseteq \bar{C} \Rightarrow f(e) \in \bar{C} - W_e\,,$$

where $\{W_e\}_{e \in \mathbb{N}}$ is a standard list of all c.e. sets.

Let C be a creative set with creative function f.

(i) Show that if A is any c.e. set then there is a computable g for which

$$W_{g(x,y)} = \begin{cases} \{f(x)\} & \text{if } y \in A, \\ \emptyset & \text{if } y \notin A, \end{cases}$$

and a computable k such that $W_{k(y)} = W_{g(k(y),y)}$.

(ii) Show that $y \in A \Rightarrow fk(y) \in C$, and $y \notin A \Rightarrow fk(y) \in \overline{C}$, and hence that $A \leq_m C$.

Deduce that every creative set C is m-complete.

4. (**a**) Let $A, B \subseteq \mathbb{N}$ be sets other than $\mathbb{N}$ or $\emptyset$.

Define : $A \leq_m B$ (A is **many-one reducible** to B).

(i) Let $A \oplus B = \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\}$.

Show that for all $A, B \subseteq \mathbb{N}$:

1. $A \leq_m A \oplus B$ and $B \leq_m A \oplus B$, and
2. If $A \leq_m C$ and $B \leq_m C$ then $A \oplus B \leq_m C$.

Question 4 continues . . .

(ii) If we define the **join** $\mathbf{a}_m \cup \mathbf{b}_m$ of two m-degrees $\mathbf{a}_m = \deg_m(A)$, $\mathbf{b}_m = \deg_m(B)$ by $\mathbf{a}_m \cup \mathbf{b}_m = \deg_m(A \oplus B)$, deduce that $\mathbf{a}_m \cup \mathbf{b}_m = \text{lub}\{\mathbf{a}_m, \mathbf{b}_m\}$ (the least upper bound of $\mathbf{a}_m$ and $\mathbf{b}_m$ in the ordering of the many-one degrees).

(**b**) Define the notions "$A \leq_T B$" (A is **Turing reducible to** B), and "$A \equiv_T B$" (A is **Turing equivalent to** B), where $A, B \subseteq \mathbb{N}$.

Show that:

(i) If $X \subseteq \mathbb{N}$ is A-computable then X is A-c.e.

(ii) $X \subseteq \mathbb{N}$ is A-computable if, and only if, X and $\overline{X}$ are A-c.e.

(iii) $X \subseteq \mathbb{N}$ is A-c.e. if, and only if, $X \in \Sigma_1^A$.

5. (**a**) Let $\mathbf{a}$ be a Turing degree.

Show that $\{X \mid X \in \mathbf{a}\}$ is countably infinite.

(**b**) Define: A is *1-generic*.

The *join* $\mathbf{a} \cup \mathbf{b}$ of Turing degrees $\mathbf{a} = \deg(A)$, $\mathbf{b} = \deg(B)$ is defined by $\mathbf{a} \cup \mathbf{b} = \deg(A \oplus B)$, in which case one has that $\mathbf{a} \cup \mathbf{b} = \text{lub}\{\mathbf{a}, \mathbf{b}\}$.

Show that if $\mathbf{a} \in \boldsymbol{\mathcal{D}}$ is 1-generic, then $\mathbf{a}$ is the join $\mathbf{a}_0 \cup \mathbf{a}_1$ of two incomparable Turing degrees $\mathbf{a}_0, \mathbf{a}_1$.

6. "There is nothing *useful* one can say about incomputability" — discuss.

Write an essay answering the above question, covering *not more than three pages*.

Your answer should contain enough mathematical content to show a good grasp of the notions and results involved in analysing incomputable sets, relations and functions, and enough discussion of these to show an understanding of the broader context.

END