MA TH5163

This question paper consists of 3 printed pages, each of which is identified by the reference MA TH5163

No calculators allowed

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Examination for the Module MA TH5163

(January 2009)

ADVANCED COMPUTABILITY AND UNSOLVABILITY

Time allowed: 3 hours

Do not answer more than FOUR questions.

All questions carry equal marks.

1. (a) Show that \( f(x, y) = x + y \) and \( g(x, y) = x \times y \) are primitive recursive functions.

If \( g_1, g_2, \ldots, g_n \in \text{PRIM} \), and if \( R_1, R_2, \ldots, R_n \) are primitive recursive relations such that for each \( x \in \mathbb{N} \) exactly one of \( R_i(x) \) holds, \( 1 \leq i \leq n \), show that \( f \) is primitive recursive where

\[
    f(x) = \begin{cases} 
    g_1(x) & \text{if } R_1(x) \text{ holds,} \\
    g_2(x) & \text{if } R_2(x) \text{ holds,} \\
    \vdots & \\
    g_n(x) & \text{if } R_n(x) \text{ holds.}
    \end{cases}
\]

(b) Write a Turing program for

\[
    h(x, y) = x \div y ,
\]

and briefly describe why your program works.

(c) Let \( \{ \varphi_x \}_{x \in \mathbb{N}} \) be a standard list of all the partial computable functions.

If the total function \( f \) is defined by

\[
    f(x) = \begin{cases} 
    \varphi_x(x) + 1 & \text{if } \varphi_x \text{ is total,} \\
    0 & \text{otherwise,}
    \end{cases}
\]

explain why \( f \) cannot be computable.

Deduce that if

\[
    \text{Tot} = \{ x \in \mathbb{N} \mid \varphi_x \text{ is total} \},
\]

then Tot is not computable.

CONTINUED...
2. (a) We define: \( A \) is \textit{computably enumerable (c.e.)} if and only if \( A = \emptyset \), or \( A \) is the range of some computable function.

Show that: (i) If \( A \subseteq \mathbb{N} \) is computable, then \( A \) is computably enumerable, and (ii) Every c.e. set is the halting set \( W_i \) of some Turing machine.

[You may assume that every partial computable function is Turing computable.]

(b) Show that the following sets are c.e.:

\( K_1 = \{ x \mid W_x \neq \emptyset \} \), where you can assume that, for each \( x \in \mathbb{N} \), \( W_x \) is the \( x \)th computably enumerable set in some standard listing, and

\( K = \{ x \mid x \in W_x \} \).

Show that there exists a computably enumerable set which is not computable.

[You should carefully state any basic results of computability theory which you use.]

(c) Show that there exists a Turing machine \( T \) with an unsolvable halting problem. Deduce that the halting problem for the \textit{Universal Turing Machine} is unsolvable.

3. (a) We say \( A \subseteq \mathbb{N} \) is \textit{creative} if and only if

1) \( A \) is c.e., and

2) There is a computable function \( f \) such that for each \( e \)

\[ W_e \subseteq \overline{A} \implies f(e) \in \overline{A} - W_e, \]

where \( \{W_e\}_{e \in \mathbb{N}} \) is a standard list of all c.e. sets.

Show that if \( C \) is a creative set then

(i) \( \overline{C} \neq \emptyset \),

(ii) For each \( n \in \mathbb{N} \), if there exist \( n \) members of \( \overline{C} \) then there exist \( n + 1 \) such members,

(iii) \( \overline{C} \) contains an infinite c.e. subset.

[You may assume that for any finite set \( X \) we can computably find an \( i \) such that \( X = W_i \).]

(b) (i) Prove the \textit{Fixed Point Theorem} for a computable function \( f \).

(ii) Explain why there is a computable function \( f \) such that \( W_{f(n)} = \{n\} \) for every \( n \in \mathbb{N} \).

Say also why it is that every c.e. set \( A \) has infinitely many distinct indices \( e \) with \( A = W_e \).

(iii) We say that a set \( A \) is an \textit{index set} if for all \( x, y \in \mathbb{N} \) we have

\[ [x \in A \& W_x = W_y] \implies y \in A. \]

Let \( K = \{ x \mid x \in W_x \} \). By using the fixed point theorem, or otherwise, show that \( K \) is not an index set.
4. (a) Define the notions $A \leq_T B$ (that is, $A$ is Turing reducible to $B$), and $A \equiv_T B$ (that is, $A$ is Turing equivalent to $B$), where $A, B \subseteq \mathbb{N}$.

(i) Show that $\equiv_T$ is an equivalence relation over the sets of natural numbers.

(ii) Show that $\leq$ (the ordering induced by $\leq_T$ on the equivalence classes under $\equiv_T$) is a partial ordering on $\mathcal{D}$ (the set of all Turing degrees).

(b) The Turing jump $B'$ of $B \subseteq \mathbb{N}$ is defined to be

$$B' = \{ \langle m, n \rangle \mid m \in W^B_n \},$$

where $\{W^B_n\}_{n \in \mathbb{N}}$ is a standard list of all $B$-c.e. sets.

(i) Show that $B'$ is c.e. in $B$, and that if $X$ is c.e. in $B$ then $X \leq_m B'$.

(ii) Show that if $B \leq_T A$ and $X$ is computably enumerable in $B$, then $X$ is computably enumerable in $A$.

Deduce that if $A \equiv_T B$ then $A' \equiv_T B'$.

5. (a) Define: $A$ is 1-generic.

Prove that there exists a 1-generic set $A$, with $\text{deg}(A) \leq 0'$.

(b) We say that $A \subseteq \mathbb{N}$ is immune if and only if $A$ is infinite and contains no infinite computably enumerable subsets.

Let $S = \text{the set of all finite 0–1 valued strings}.$

Show that, for each $i \geq 0$, $Y_i = \{ \sigma \in S \mid \exists x[\sigma(x) = 0 \& x \in W_i]\}$ is a computably enumerable set of strings.

Show that if $A$ is 1-generic then $A$ forces each such $Y_i$ (that is, $A \vDash Y_i$), and hence that $A$ is either finite or immune.

Deduce that every 1-generic set is immune, and hence not computably enumerable.

6. “There is nothing useful one can say about incomputability” — discuss.

Write an essay answering the above question, covering not more than three pages.

Your answer should contain enough mathematical content to show a good grasp of the notions and results involved in analysing incomputable sets, relations and functions, and enough discussion of these to show an understanding of the broader context.

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