

MATH516301

This question paper consists of 3 printed pages, each of which is identified by the reference MATH5163

No calculators allowed

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Examination for the Module MATH5163

(January 2009)

ADVANCED COMPUTABILITY AND UNSOLVABILITY

Time allowed: **3 hours**

Do not answer more than *FOUR* questions.

All questions carry equal marks.

1. (a) Show that $f(x, y) = x + y$ and $g(x, y) = x \times y$ are primitive recursive functions.

If $g_1, g_2, \dots, g_n \in \text{PRIM}$, and if R_1, R_2, \dots, R_n are primitive recursive relations such that for each $x \in \mathbb{N}$ exactly one of $R_i(x)$ holds, $1 \leq i \leq n$, show that f is primitive recursive where

$$f(x) = \begin{cases} g_1(x) & \text{if } R_1(x) \text{ holds,} \\ g_2(x) & \text{if } R_2(x) \text{ holds,} \\ \vdots & \\ g_n(x) & \text{if } R_n(x) \text{ holds.} \end{cases}$$

- (b) Write a Turing program for

$$h(x, y) = x \dot{-} y,$$

and briefly describe why your program works.

- (c) Let $\{\varphi_x\}_{x \in \mathbb{N}}$ be a standard list of all the partial computable functions.

If the total function f is defined by

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_x \text{ is total,} \\ 0 & \text{otherwise,} \end{cases}$$

explain why f cannot be computable.

Deduce that if

$$\text{Tot} = \{x \in \mathbb{N} \mid \varphi_x \text{ is total}\},$$

then Tot is not computable.

2. (a) We define: A is *computably enumerable (c.e.)* if and only if $A = \emptyset$, or A is the range of some computable function.

Show that: (i) If $A \subseteq \mathbb{N}$ is computable, then A is computably enumerable, and
(ii) Every c.e. set is the halting set W_i of some Turing machine.

[You may assume that every partial computable function is Turing computable.]

(b) Show that the following sets are c.e.:

(i) $K_1 = \{x \mid W_x \neq \emptyset\}$, where you can assume that, for each $x \in \mathbb{N}$, W_x is the x^{th} computably enumerable set in some standard listing, and

(ii) $K = \{x \mid x \in W_x\}$.

Show that there exists a computably enumerable set which is not computable.

[You should carefully state any basic results of computability theory which you use.]

(c) Show that there exists a Turing machine T with an unsolvable halting problem.

Deduce that the halting problem for the *Universal Turing Machine* is unsolvable.

3. (a) We say $A \subseteq \mathbb{N}$ is *creative* if and only if

1) A is c.e., and

2) There is a computable function f such that for each e

$$W_e \subseteq \bar{A} \Rightarrow f(e) \in \bar{A} - W_e,$$

where $\{W_e\}_{e \in \mathbb{N}}$ is a standard list of all c.e. sets.

Show that if C is a creative set then

(i) $\bar{C} \neq \emptyset$,

(ii) For each $n \in \mathbb{N}$, if there exist n members of \bar{C} then there exist $n + 1$ such members,

(iii) \bar{C} contains an infinite c.e. subset.

[You may assume that for any finite set X we can computably find an i such that $X = W_i$.]

- (b) (i) Prove the *Fixed Point Theorem* for a computable function f .

(ii) Explain why there is a computable function f such that $W_{f(n)} = \{n\}$ for every $n \in \mathbb{N}$.

Say also why it is that every c.e. set A has infinitely many distinct indices e with $A = W_e$.

(iii) We say that a set A is an *index set* if for all $x, y \in \mathbb{N}$ we have

$$[x \in A \ \& \ W_x = W_y] \implies y \in A.$$

Let $K = \{x \mid x \in W_x\}$. By using the fixed point theorem, or otherwise, show that K is not an index set.

4. (a) Define the notions $A \leq_T B$ (that is, A is *Turing reducible to* B), and $A \equiv_T B$ (that is, A is *Turing equivalent to* B), where $A, B \subseteq \mathbb{N}$.

(i) Show that \equiv_T is an equivalence relation over the sets of natural numbers.

(ii) Show that \leq (the ordering induced by \leq_T on the equivalence classes under \equiv_T) is a partial ordering on \mathcal{D} (the set of all Turing degrees).

(b) The *Turing jump* B' of $B \subseteq \mathbb{N}$ is defined to be

$$B' = \{\langle m, n \rangle \mid m \in W_n^B\},$$

where $\{W_n^B\}_{n \in \mathbb{N}}$ is a standard list of all B -c.e. sets.

(i) Show that B' is c.e. in B , and that if X is c.e. in B then $X \leq_m B'$.

(ii) Show that if $B \leq_T A$ and X is computably enumerable in B , then X is computably enumerable in A .

Deduce that if $A \equiv_T B$ then $A' \equiv_T B'$.

5. (a) Define: A is *1-generic*.

Prove that there exists a 1-generic set A , with $\deg(A) \leq \mathbf{0}'$.

(b) We say that $A \subseteq \mathbb{N}$ is *immune* if and only if A is infinite and contains no infinite computably enumerable subsets.

Let $S =$ the set of all finite 0 – 1 valued strings.

Show that, for each $i \geq 0$, $Y_i = \{\sigma \in S \mid \exists x[\sigma(x) = 0 \ \& \ x \in W_i]\}$ is a computably enumerable set of strings.

Show that if A is 1-generic then A forces each such Y_i (that is, $A \Vdash Y_i$), and hence that A is either finite or immune.

Deduce that every 1-generic set is immune, and hence not computably enumerable.

6. “**There is nothing useful one can say about incomputability**” — discuss.

Write an essay answering the above question, covering *not more than three pages*.

Your answer should contain enough mathematical content to show a good grasp of the notions and results involved in analysing incomputable sets, relations and functions, and enough discussion of these to show an understanding of the broader context.

END