MATH-316301

This question paper consists of 4 printed pages, each of which is identified by the reference MATH 316301

No calculators allowed

Examination for the Module MATH 3163
(January 2008)

COMPUTABILITY AND UNSOLVABILITY

Time allowed : 3 hours

Do not answer more than *FOUR* questions.
All questions carry equal marks.

1. (**a**) Show that $f(x, y) = x \times y$ and

$$\mathrm{sg}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0 \end{cases}$$

are primitive recursive functions.

Deduce that if the function

$$|x - y| = \begin{cases} x - y & \text{if } x \geq y, \\ y - x & \text{if } x < y \end{cases}$$

is known to be primitive recursive, then so is the **remainder function** $\mathrm{rm}(x, y)$ defined by:

$$\mathrm{rm}(x, y) = \begin{cases} \text{the remainder upon division of } y \text{ by } x, & \text{if } x \neq 0, \\ y & \text{otherwise.} \end{cases}$$

(**b**) Write a Turing program for the function $\mathrm{sg}(n)$ defined in part (a), and briefly explain why your program works.

Say which function $f(n)$ is computed by the Turing machine below, which uses extra tape symbols ε, η as counters:

$q_0\,1\,\varepsilon\,q_1$
$q_1\,\varepsilon\,R\,q_1$
$q_1\,1\,R\,q_1$
$q_1\,0\,R\,q_2$
$q_2\,0\,\eta\,q_2$
$q_2\,1\,R\,q_2$
$q_2\,\eta\,1\,q_3$
$q_3\,1\,R\,q_4$
$q_4\,0\,1\,q_5$
$q_5\,1\,R\,q_5$
$q_5\,0\,\eta\,q_5$
$q_5\,\eta\,L\,q_6$
$q_6\,1\,L\,q_6$
$q_6\,0\,L\,q_7$
$q_7\,1\,L\,q_7$
$q_7\,\varepsilon\,0\,q_7$
$q_7\,0\,R\,q_0$

Question 1 continues ...

(**c**) Let U be the **universal Turing machine**.

A function $\psi(x, y)$ is defined by

$$\psi(x,y) = \begin{cases} 0 & \text{if } y = 1\text{, or if } y = 0 \text{ and } U \text{ halts on input } x, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Using Church's thesis, or otherwise, show that ψ is partial recursive.

A **weak μ-operator** μ^* is defined by:

$$\mu^* y[\psi(x,y) = 0] =_{\text{defn.}} \begin{cases} \text{the least } y \text{ such that } \psi(x,y) = 0 & \text{if such a } y \text{ exists,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Show that the partial recursive functions are not closed under all applications of the weak μ-operator μ^*.

[You may assume that U has unsolvable halting problem.]

2. (**a**) Define: A is **computably enumerable**.

Show that if $A, B \subseteq \mathbb{N}$ are computably enumerable, then $A \cup B$ and $A \cap B$ are also computably enumerable.

Prove the **Complementation Theorem**: A is computable if and only if both A and $\overline{A}$ (the complement of A) are computably enumerable.

(**b**) Show that the following sets are computably enumerable:

(i) Range $\varphi_e = \{y \mid \exists x \varphi_e(x) = y\}$,

(ii) Graph $\varphi_e = \{\langle x, y\rangle \mid \varphi_e(x) = y\}$.

where, for each $x \in \mathbb{N}$, φ_x is the x^{th} partial computable function in some standard listing.

Deduce that a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is partial computable if and only if Graph(f) is computably enumerable.

(**c**) For each $i \geq 0$, let $W_{i,s}$ be the standard computable approximation to the i^{th} computably enumerable set W_i.

Let $X = W_e$, $Y = W_f$ be given computably enumerable sets, where we write $X^s = W_{e,s}$, $Y^s = W_{f,s}$.

Let $X \searrow Y = \{z \mid \exists s(z \in X^s - Y^s)\}$ and $X \searrow\!\!\!\searrow Y = (X \searrow Y) \cap Y$. Show that:

(i) Both $X \searrow Y$ and $X \searrow\!\!\!\searrow Y$ are computably enumerable sets.

(ii) $X \searrow Y = (X - Y) \cup (X \searrow\!\!\!\searrow Y)$.

(iii) If $X \searrow\!\!\!\searrow Y$ is finite then $X - Y$ is computably enumerable.

(iv) Assuming that numbers are enumerated into *at most one* of $X^s = W_{e,s}$, $Y^s = W_{f,s}$ at any given stage s, and taking $\hat{X} = X \setminus Y$, $\hat{Y} = Y \setminus X$, prove the **Reduction Principle** for X, Y:

Given computably enumerable sets X and Y, there exist computably enumerable sets $\hat{X} \subseteq X$ and $\hat{Y} \subseteq Y$ such that $\hat{X} \cap \hat{Y} = \emptyset$ and $\hat{X} \cup \hat{Y} = X \cup Y$.

3. (**a**) A computably enumerable set S is said to be **simple** if and only if

1. $\bar{S}$ is infinite, and
2. $(\forall i)\,[W_i \text{ infinite} \Rightarrow W_i \cap S \neq \emptyset]$.

Show that no simple set can be computable, and prove the existence of a simple set.

(**b**) Prove the **Fixed Point Theorem with Parameters**:

If $f(x, \overrightarrow{y})$ is a computable function, then there exists a computable $k(\overrightarrow{y})$ such that

$$\varphi_{f(k(\overrightarrow{y}), \overrightarrow{y})} = \varphi_{k(\overrightarrow{y})}.$$

Given $f(x, \overrightarrow{y})$ computable, deduce that there exists a computable $k(\overrightarrow{y})$ such that

$$W_{f(k(\overrightarrow{y}), \overrightarrow{y})} = W_{k(\overrightarrow{y})}.$$

4. (**a**) Let $A, B \subseteq \mathbb{N}$ be sets other than $\mathbb{N}$ or $\emptyset$.

Define : $A \leq_m B$ (A is **many-one reducible** to B).

(i) Let $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

Show that for all $A, B \subseteq \mathbb{N}$:

1. $A \leq_m A \oplus B$ and $B \leq_m A \oplus B$, and
2. If $A \leq_m C$ and $B \leq_m C$ then $A \oplus B \leq_m C$.

(ii) If we define the **join** $\mathbf{a}_m \cup \mathbf{b}_m$ of two m-degrees $\mathbf{a}_m = \deg_m(A)$, $\mathbf{b}_m = \deg_m(B)$ by $\mathbf{a}_m \cup \mathbf{b}_m = \deg_m(A \oplus B)$, deduce that $\mathbf{a}_m \cup \mathbf{b}_m = \text{lub}\{\mathbf{a}_m, \mathbf{b}_m\}$ (the least upper bound of $\mathbf{a}_m$ and $\mathbf{b}_m$ in the ordering of the many-one degrees).

(**b**) Define the notions "$A \leq_T B$" (A is **Turing reducible to** B), and "$A \equiv_T B$" (A is **Turing equivalent to** B), where $A, B \subseteq \mathbb{N}$.

Show that:

(i) If $X \subseteq \mathbb{N}$ is A-computable then X is A-c.e.

(ii) $X \subseteq \mathbb{N}$ is A-computable if, and only if, X and $\overline{X}$ are A-c.e.

Question 4 continues . . .

(iii) $X \subseteq \mathbb{N}$ is A-c.e. if, and only if, $X \in \Sigma_1^A$.

5. Show that there exists an infinite sequence $\mathbf{a}_0, \mathbf{a}_1, \ldots$ of degrees $\leq \mathbf{0}'$ such that for each $i \neq j$ we have $\mathbf{a}_i \mid \mathbf{a}_j$ (that is, $\mathbf{a}_i$ is incomparable with $\mathbf{a}_j$).

6. Write an essay, covering **not more than three pages**, describing the background to, and consequences of, Alan Turing's discovery of the existence of a Universal Turing Machine.

Your answer should contain enough mathematical content to show a good grasp of the notions and results involved, and enough discussion of these to show an understanding of the broader context.

END