1. (a) Show that \( f(x, y) = x + y \) and \( g(x, y) = x \times y \) are primitive recursive functions.

If \( g_1, g_2, \ldots, g_n \in \text{PRIM} \), and if \( R_1, R_2, \ldots, R_n \) are primitive recursive relations such that for each \( x \in \mathbb{N} \) exactly one of \( R_i(x) \) holds, \( 1 \leq i \leq n \), show that \( f \) is primitive recursive where

\[
\begin{cases} 
  g_1(x) & \text{if } R_1(x) \text{ holds,} \\
  g_2(x) & \text{if } R_2(x) \text{ holds,} \\
  \vdots \\
  g_n(x) & \text{if } R_n(x) \text{ holds.}
\end{cases}
\]

(b) Write a Turing program for

\[ h(x, y) = x \div y, \]

and briefly describe why your program works.

(c) Let \( \{\varphi_x\}_{x \in \mathbb{N}} \) be a standard list of all the partial computable functions.

If the total function \( f \) is defined by

\[
\begin{cases} 
  \varphi_x(x) + 1 & \text{if } \varphi_x \text{ is total,} \\
  0 & \text{otherwise,}
\end{cases}
\]

explain why \( f \) cannot be computable.

Deduce that if

\[ \text{Tot} = \{ x \in \mathbb{N} \mid \varphi_x \text{ is total} \}, \]

then \( \text{Tot} \) is not computable.
2. (a) For each $e \in \mathbb{N}$, we define computable approximations for $\varphi_e$ (the $e^{th}$ partial computable function in some standard listing) by:

$$
\varphi_{e,s}(x) = y \iff \text{defn. } x, y, e < s, \text{ and } y \text{ is the output of } \varphi_e(x) \text{ in } < s \text{ steps of the Turing program } P_e.
$$

Show that $\varphi_{e,s}(x) \downarrow$ ("$\varphi_{e,s}(x)$ is defined") is a computable relation over $e, s, x \in \mathbb{N}$.

(b) Define what is meant by saying that a set $A$ is computably enumerable.

Prove the Normal Form Theorem giving the equivalence of the following three statements:

i) $A$ is computably enumerable,

ii) $A$ is a $\Sigma^0_1$ set,

iii) $A$ is the domain of some p.c. function $\varphi_e$.

(c) Show that:

i) $\varphi_e(x) \downarrow$ is a $\Sigma^0_1$ relation, but

ii) $\varphi_e(x) \downarrow$ is not a computable relation.

Deduce that there exists a c.e. set which is not computable.

[You may assume in part (c) that if a set $A$ is computable, then it is also computably enumerable.]

3. (a) We say $A \subseteq \mathbb{N}$ is creative if and only if

1) $A$ is c.e., and

2) There is a computable function $f$ such that for each $e$

$$
W_e \subseteq \overline{A} \Rightarrow f(e) \in \overline{A} - W_e,
$$

where $\{W_e\}_{e \in \mathbb{N}}$ is a standard list of all c.e. sets.

(i) Show the existence of a creative set.

(ii) Prove that if $A$ is a computably enumerable set and $\psi$ is a partial computable function, then $\psi^{-1}(A)$ is also c.e.

(iii) Show that if $C$ is creative, and $A$ is a c.e. set such that $C \leq_m A$ ($C$ is many-one reducible to $A$), then $A$ is also creative.

(b) Define: $A \subseteq \mathbb{N}$ is simple.

(i) Show that no simple set is computable.

(ii) Show that if $A$ is a simple set and $W$ is an infinite c.e. set then $A \cap W$ is an infinite c.e. set.

(iii) Deduce that if $A$ and $B$ are simple sets then $A \cap B$ is also simple.

[You may assume that the intersection $X \cap Y$ of two computably enumerable sets $X$ and $Y$ is also computably enumerable.]
4. Let $A, B \subseteq \mathbb{N}$ be sets other than $\mathbb{N}$ or $\emptyset$.

(a) Define : $A \leq_m B$ ($A$ is *many-one reducible* to $B$).
Show that $\leq_m$ is a reflexive and transitive relation over the sets of numbers.
Deduce that the ordering $\leq$ on the many-one degrees defined by

$$\deg_m(A) \leq \deg_m(B) \iff A \leq_m B$$

is a partial ordering (that is, is reflexive, transitive and anti-symmetric).

(b) Let $a$ be a Turing degree.
Show that $\{X \mid X \in a\}$ is countably infinite, and that $\{b \mid b \leq a\}$ is countable.
Show, however, that $\mathcal{D}$ (= the set of all Turing degrees) is uncountable.

5. Show that there exists a pair of incomparable Turing degrees below $0'$.

6. Write an essay, covering **not more than three pages**, describing the background to, and consequences of, Alan Turing’s discovery of the existence of a Universal Turing Machine.

Your answer should contain enough mathematical content to show a good grasp of the notions and results involved, and enough discussion of these to show an understanding of the broader context.

END