MATH316301

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH3163**

No calculators allowed

Examination for the Module MATH3163
(January 2009)

COMPUTABILITY AND UNSOLVABILITY

Time allowed: **3 hours**

Do not answer more than *FOUR* questions.

All questions carry equal marks.

1. **(a)** Show that $f(x,y) = x + y$ and $g(x,y) = x \times y$ are primitive recursive functions.

If $g_1, g_2, \ldots, g_n \in \mathrm{PRIM}$, and if $R_1, R_2, \ldots, R_n$ are primitive recursive relations such that for each $x \in \mathbb{N}$ exactly one of $R_i(x)$ holds, $1 \leq i \leq n$, show that f is primitive recursive where

$$f(x) = \begin{cases} g_1(x) & \text{if } R_1(x) \text{ holds}\,, \\ g_2(x) & \text{if } R_2(x) \text{ holds}\,, \\ \vdots & \\ g_n(x) & \text{if } R_n(x) \text{ holds}\,. \end{cases}$$

(b) Write a Turing program for

$$h(x,y) = x \dot{-} y\,,$$

and briefly describe why your program works.

(c) Let $\{\varphi_x\}_{x\in\mathbb{N}}$ be a standard list of all the partial computable functions.

If the total function f is defined by

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_x \text{ is total}, \\ 0 & \text{otherwise}, \end{cases}$$

explain why f cannot be computable.

Deduce that if

$$\mathrm{Tot} = \{x \in \mathbb{N} \mid \varphi_x \text{ is total}\},$$

then Tot is not computable.

2. **(a)** For each $e \in \mathbb{N}$, we define computable approximations for φ_e (the e^{th} partial computable function in some standard listing) by:

$$\varphi_{e,s}(x) = y \iff_{\text{defn.}} \begin{array}{l} x, y, e < s, \text{ and } y \text{ is the output of } \varphi_e(x) \\ \text{in} < s \text{ steps of the Turing program } P_e. \end{array}$$

Show that $\varphi_{e,s}(x)\downarrow$ ("$\varphi_{e,s}(x)$ is defined") is a computable relation over $e, s, x \in \mathbb{N}$,

(b) Define what is meant by saying that a set A is *computably enumerable*.

Prove the *Normal Form Theorem* giving the equivalence of the following three statements:

i) A is computably enumerable,

ii) A is a Σ^0_1 set,

iii) A is the domain of some p.c. function φ_e.

(c) Show that:

(i) $\varphi_x(x)\downarrow$ is a Σ^0_1 relation, but

(ii) $\varphi_x(x)\downarrow$ is not a computable relation.

Deduce that there exists a c.e. set which is not computable.

[You may assume in part (c) that if a set A is computable, then it is also computably enumerable.]

3. **(a)** We say $A \subseteq \mathbb{N}$ is *creative* if and only if

1) A is c.e., and

2) There is a computable function f such that for each e

$$W_e \subseteq \overline{A} \Rightarrow f(e) \in \overline{A} - W_e\,,$$

where $\{W_e\}_{e\in\mathbb{N}}$ is a standard list of all c.e. sets.

(i) Show the existence of a creative set.

(ii) Prove that if A is a computably enumerable set and ψ is a partial computable function, then $\psi^{-1}(A)$ is also c.e.

(iii) Show that if C is creative, and A is a c.e. set such that $C \leq_m A$ (C is *many-one reducible to* A), then A is also creative.

. (b) Define: $A \subseteq \mathbb{N}$ is *simple*.

(i) Show that no simple set is computable.

(ii) Show that if A is a simple set and W is an infinite c.e. set then $A \cap W$ is an infinite c.e. set.

(iii) Deduce that if A and B are simple sets then $A \cap B$ is also simple.

[You may assume that the intersection $X \cap Y$ of two computably enumerable sets X and Y is also computably enumerable.]

4. Let $A, B \subseteq \mathbb{N}$ be sets other than $\mathbb{N}$ or $\emptyset$.

. (a) Define : $A \leq_m B$ (A is *many-one reducible* to B).

Show that $\leq_m$ is a reflexive and transitive relation over the sets of numbers.

Deduce that the ordering $\leq$ on the many-one degrees defined by

$$\text{deg}_m(A) \leq \text{deg}_m(B) \Leftrightarrow A \leq_m B$$

is a partial ordering (that is, is reflexive, transitive and anti-symmetric).

. (b) Let $\mathbf{a}$ be a Turing degree.

Show that $\{X \mid X \in \mathbf{a}\}$ is countably infinite, and that $\{\mathbf{b} \mid \mathbf{b} \leq \mathbf{a}\}$ is countable.

Show, however, that $\boldsymbol{\mathcal{D}}$ (= the set of all Turing degrees) is uncountable.

5. Show that there exists a pair of incomparable Turing degrees below $\mathbf{0}'$.

6. Write an essay, covering **not more than three pages**, describing the background to, and consequences of, Alan Turing's discovery of the existence of a Universal Turing Machine.

Your answer should contain enough mathematical content to show a good grasp of the notions and results involved, and enough discussion of these to show an understanding of the broader context.

END