THEOREM (Friedberg-Muchnik): There exist c.e. degrees \(a\) and \(b\) such that \(a \mid b\).

PROOF:

• We construct c.e. sets \(A\) and \(B\) such that \(A \not\leq_T B\) and \(B \not\leq_T A\).

• We break these conditions up into an infinite list of requirements:

\[
\begin{align*}
\mathcal{R}_{2i} &: A \neq \Phi_i^B \\
\mathcal{R}_{2i+1} &: B \neq \Phi_i^A.
\end{align*}
\]

• We get \(A\) and \(B\) c.e. via c.e. approximating sequences \(\{A^s\}_{s \geq 0}, \{B^s\}_{s \geq 0}\) to \(A\) and \(B\).

• The construction will take place at stages 0, 1, ..., \(s+1\), ...

• At stage \(s+1\) we will computably construct \(A^{s+1}, B^{s+1} \supseteq A^s, B^s\), so as to help satisfy just one requirement.

The strategy for satisfying \(\mathcal{R}_{2i}\)

At a general stage \(s+1\) we focus, in turn, on just one phase of the following:

1. Choose a potential witness \(x\) to \(A \neq \Phi_i^B\), where \(x\) is not yet in \(A\) — we aim to make \(A(x) \neq \Phi_i^B(x)\).

2. Do nothing more — unless we get a stage \(s+1\) at which \(A^s(x) = 0 = \Phi_i^B(x)[s]\) ...

3. In which case enumerate \(x\) into \(A^{s+1}\). And if \(\Phi_i^B(x)[s]\) has use \(z\), preserve \(B^s \uparrow z = B \uparrow z\) for evermore. We call this \(z\) a \(B\)-restraint.

The analysis of outcomes for the strategy

The only outcomes are:

\(u\): The strategy waits forever at (2) for \(A^s(x) = 0 = \Phi_i^B(x)[s]\) — in which case either \(\Phi_i^B(x) \uparrow\) or \(\Phi_i^B(x) = 1 \neq A(x)\).

• So \(\mathcal{R}_{2i}\) is satisfied.

\(s\): The strategy halts at (3) with

\[A(x) = A^{s+1}(x) = 1 \neq 0 = \Phi_i^B(x)[s].\]

• Since we preserve \(B^s \uparrow z = B \uparrow z\), we have \(\Phi_i^B(x) \neq A(x)\), so again \(\mathcal{R}_{2i}\) is satisfied.
The strategy for $R_{2i+1}$ below that for $R_{2i}$

(1) Choose a witness $y$ to $B \neq \Phi^A_i$, where $y$ has not yet appeared in the strategy — either as a witness or below any restraint previously set up.

We say $y$ is fresh.

(2) At each later stage $s + 1$ at which $R_{2i}$ is halted at (1) or (2) of its strategy, first check if

(a) $y < a$ restraint set up by $R_{2i}$ — in which case we throw $y$ away and go back to (1) to get a fresh witness.

We say $R_{2i+1}$ has been injured.

Otherwise, ask:

(b) Is $B^s(y) = 0 = \Phi^A_i(y)[s]$?

If “no” go back to 2(a) at the next stage, if “yes” go straight to (3).

(3) Enumerate $y$ into $B^{s+1}$. And set up an $A$ restraint $w = \varphi^A_i(y)[s]$.

If $R_{2i}$ injures $R_{2i+1}$ at a later stage — that is, enumerates $x$ into $A$ with $x < w$ — throw $y$ and $w$ away, and return to (1) to start all over again.

The analysis of outcomes for the injurable strategy (for $R_{2i+1}$)

- For $R_{2i}$, the outcomes are exactly as before.
- For $R_{2i+1}$ they are very similar:

  $[w]$ : The strategy waits forever at (2) for $B^s(y) = 0 = \Phi^A_i(y)[s]$, and $R_{2i+1}$ is satisfied.

  $[s]$ : The strategy halts at (3) with

  $$ B(x) = B^{s+1}(y) = 1 \neq 0 = \Phi^A_i(y)[s]. $$

We set up the $A$ restraint $w$, and $R_{2i+1}$ is satisfied.

- But what about the injuries — surely they introduce another outcome in which $R_{2i}$ keeps on injuring $R_{2i+1}$?

Not at all.

- $R_{2i}$, of course, is never injured. It only has outcomes $[w]$ and $[s]$, each of which mean $R_{2i}$ never again chooses a fresh witness or sets up a new restraint.

- So there is a stage after which $R_{2i+1}$ too is never injured. This means that for $R_{2i+1}$ also there are only two outcomes $[w]$ and $[s]$.

- The strategy is finite injury, and the use of priority has enabled us to satisfy both requirements.

The strategy for $R_{2i+1}$ with all the other requirements

- It is easy to see now that all the requirements can successfully pursue their own copies of the strategy.

- $R_0$ is never injured.

- So after some stage $s_0$ $R_1$ is never injured, and so never injures after some stage $s_1 \geq s_0$.

- Inductively we get $R_{2i+1}$ is not injured after some stage, and so gets satisfied via $[w]$ or $[s]$. 