

THEOREM (Friedberg-Muchnik): There exist c.e. degrees \mathbf{a} and \mathbf{b} such that $\mathbf{a} \mid \mathbf{b}$.

PROOF:

- We construct c.e. sets A and B such that $A \not\leq_T B$ and $B \not\leq_T A$.
- We break these conditions up into an infinite list of requirements:

$$\begin{aligned} \mathcal{R}_{2i} &: A \neq \Phi_i^B \\ \mathcal{R}_{2i+1} &: B \neq \Phi_i^A. \end{aligned}$$

- We get A and B c.e. via c.e. approximating sequences $\{A^s\}_{s \geq 0}$, $\{B^s\}_{s \geq 0}$ to A and B .
- The construction will take place at stages $0, 1, \dots, s+1, \dots$.
- At stage $s+1$ we will computably construct $A^{s+1}, B^{s+1} \supseteq A^s, B^s$, so as to help satisfy just one requirement.

The strategy for satisfying \mathcal{R}_{2i}

At a general stage $s+1$ we focus, in turn, on just one phase of the following:

- (1) Choose a potential *witness* x to $A \neq \Phi_i^B$, where x is not yet in A — we aim to make $A(x) \neq \Phi_i^B(x)$.
- (2) Do nothing more — unless we get a stage $s+1$ at which $A^s(x) = 0 = \Phi_i^B(x)[s] \dots$
- (3) In which case enumerate x into A^{s+1} . And if $\Phi_i^B(x)[s]$ has use z , preserve $B^s \upharpoonright z = B \upharpoonright z$ for evermore. We call this z a *B-restraint*.

The analysis of outcomes for the strategy

The only outcomes are:

\boxed{w} : The strategy waits forever at (2) for $A^s(x) = 0 = \Phi_i^B(x)[s]$ — in which case either $\Phi_i^B(x) \uparrow$ or $\Phi_i^B(x) = 1 \neq A(x)$.

- So \mathcal{R}_{2i} is satisfied.

\boxed{s} : The strategy halts at (3) with

$$A(x) = A^{s+1}(x) = 1 \neq 0 = \Phi_i^B(x)[s].$$

- Since we preserve $B^s \upharpoonright z = B \upharpoonright z$, we have $\Phi_i^B(x) \neq A(x)$, so again \mathcal{R}_{2i} is satisfied.

The strategy for \mathcal{R}_{2i+1} below that for \mathcal{R}_{2i}

(1) Choose a witness y to $B \neq \Phi_i^A$, where y has not yet appeared in the strategy — either as a witness or below any restraint previously set up.

We say y is *fresh*.

(2) At each later stage $s + 1$ at which \mathcal{R}_{2i} is halted at (1) or (2) of its strategy, first check if
 (a) $y < a$ B restraint set up by \mathcal{R}_{2i} — in which case we throw y away and go back to (1) to get a fresh witness.

We say \mathcal{R}_{2i+1} has been *injured*.

Otherwise, ask:

(b) Is $B^s(y) = 0 = \Phi_i^A(y)[s]$?

If “no” go back to 2(a) at the next stage, if “yes” go straight to (3).

(3) Enumerate y into B^{s+1} . And set up an A restraint $w = \varphi_i^A(y)[s]$.

If \mathcal{R}_{2i} *injures* \mathcal{R}_{2i+1} at a later stage — that is, enumerates x into A with $x < w$ — throw y and w away, and return to (1) to start all over again.

The analysis of outcomes for the injurable strategy (for \mathcal{R}_{2i+1})

- For \mathcal{R}_{2i} the outcomes are exactly as before.
- For \mathcal{R}_{2i+1} they are very similar:

\boxed{w} : The strategy waits forever at (2) for $B^s(y) = 0 = \Phi_i^A(y)[s]$, and \mathcal{R}_{2i+1} is satisfied.

\boxed{s} : The strategy halts at (3) with

$$B(x) = B^{s+1}(y) = 1 \neq 0 = \Phi_i^A(y)[s].$$

We set up the A restraint w , and \mathcal{R}_{2i+1} is satisfied.

- But what about the injuries — surely they introduce another outcome in which \mathcal{R}_{2i} keeps on injuring \mathcal{R}_{2i+1} ?

Not at all.

- \mathcal{R}_{2i} , of course, is *never* injured. It only has outcomes \boxed{w} and \boxed{s} , each of which mean \mathcal{R}_{2i} *never again* chooses a fresh witness or sets up a new restraint.
- So there is a stage after which \mathcal{R}_{2i+1} too is *never* injured. This means that for \mathcal{R}_{2i+1} also there are only two outcomes \boxed{w} and \boxed{s} .
- The strategy is *finite injury*, and the use of priority has enabled us to satisfy both requirements.

The strategy for \mathcal{R}_{2i+1} with all the other requirements

- It is easy to see now that all the requirements can successfully pursue their own copies of the strategy.
- \mathcal{R}_0 is never injured.
- So after some stage s_0 \mathcal{R}_1 is never injured, and so never injures after some stage $s_1 \geq s_0$.
- Inductively we get \mathcal{R}_{2i+1} is not injured after some stage, and so gets satisfied via \boxed{w} or \boxed{s} .