THEOREM (Friedberg-Muchnik): There exist c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \mid \mathbf{b}$.

PROOF:

• We start off as before, needing to construct sets A and B such that $A \not\leq_T B$ and $B \not\leq_T A$.

• Again we break these conditions up into an infinite list of requirements:

$$\mathcal{R}_{2i}: \quad A \neq \Phi_i^B$$
$$\mathcal{R}_{2i+1}: \quad B \neq \Phi_i^A.$$

• The way to get A and B c.e. though is to actually make the construction produce c.e. approximating sequences $\{A^s\}_{s\geq 0}$, $\{B^s\}_{s\geq 0}$ to A and B — and this is where we have to discard any oracles that might spoil the computability of these enumerations.

NOTE: We call a computable sequence
$A^0 \subset A^1 \subset \ldots \subset A^s \subset \ldots$ of finite sets a **c.e. approximating sequence** to $A = \cup_{s\geq 0} A^s$.

- The construction will take place at stages $0, 1, \ldots, s+1, \ldots$ as before.
- At stage $s+1$ we will computably construct $A^{s+1}, B^{s+1} \supseteq A^s, B^s$, so as to help satisfy just one requirement, although this time we will not be able to decide ahead of time exactly *which* requirement.
- Without oracles we will blunder around making mistakes which later have to be put right.

NOTES: (1) We write $A \upharpoonright z$ for the string $A(0)A(1)\ldots A(z)$.

(2) If $\Phi_i^A(x)\downarrow$, then the **use** $z = \varphi_i^A(x)$ of this computation is

$$z = \mu z[\Phi_i^{A \upharpoonright z}(x)\downarrow].$$

The strategy for satisfying $\mathcal{R}_{2i}$

In isolation, $\mathcal{R}_{2i}$ is easy to satisfy, even without oracles. At a general stage $s+1$ we focus, in turn, on just one phase of the following:

(1) Choose a potential *witness* x to $A \neq \Phi_i^B$, where x is not yet in A — we aim to make $A(x) \neq \Phi_i^B(x)$.

(2) Do nothing more — unless we get a stage $s+1$ at which $A^s(x) = 0 = \Phi_i^B(x)[s] \ldots$

(3) In which case enumerate x into A^{s+1}. And if $\Phi_i^B(x)[s]$ has use z, preserve $B^s \upharpoonright z = B \upharpoonright z$ for evermore. We call this z a *B-restraint*.

The analysis of outcomes for the strategy

The only outcomes are:

$\boxed{w}$: The strategy waits forever at (2) for $A^s(x) = 0 = \Phi_i^B(x)[s]$ — in which case either $\Phi_i^B(x)\uparrow$ or $\Phi_i^B(x) = 1 \neq A(x)$.

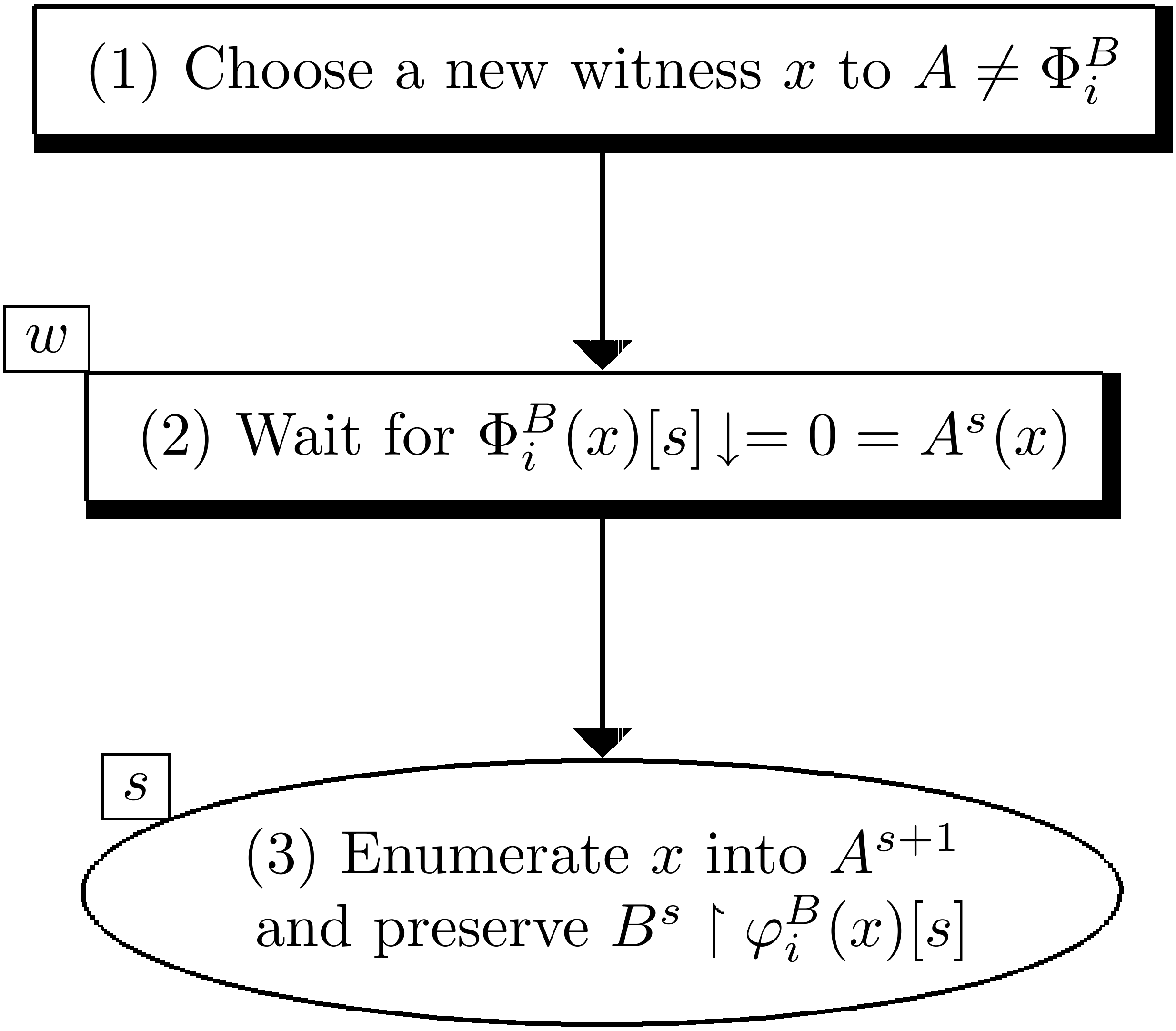
- So $\mathcal{R}_{2i}$ is satisfied.

$\boxed{s}$: The strategy halts at (3) with

$$A(x) = A^{s+1}(x) = 1 \neq 0 = \Phi_i^B(x)[s].$$

- Since we preserve $B^s \upharpoonright z = B \upharpoonright z$, we have $\Phi_i^B(x) \neq A(x)$, so again $\mathcal{R}_{2i}$ is satisfied.

We can set this strategy out on a flow diagram:



The strategy for $\mathcal{R}_{2i+1}$ below that for $\mathcal{R}_{2i}$

(1) Choose a witness y to $B \neq \Phi_i^A$, where y has not yet appeared in the strategy — either as a witness or below any restraint previously set up.

We say y is *fresh*.

(2) At each later stage $s+1$ at which $\mathcal{R}_{2i}$ is halted at (1) or (2) of its strategy, first check if

(a) $y <$ a B restraint set up by $\mathcal{R}_{2i}$ — in which case we throw y away and go back to (1) to get a fresh witness.

We say $\mathcal{R}_{2i+1}$ has been *injured*.

Otherwise, ask:

(b) Is $B^s(y) = 0 = \Phi_i^A(y)[s]$?

If "no" go back to 2(a) at the next stage, if "yes" go straight to (3).

(3) Enumerate y into B^{s+1}. And set up an A restraint $w = \varphi_i^A(y)[s]$.

If $\mathcal{R}_{2i}$ *injures* $\mathcal{R}_{2i+1}$ at a later stage — that is, enumerates x into A with $x < w$ — throw y and w away, and return to (1) to start all over again.

The analysis of outcomes for the injurable strategy (for $\mathcal{R}_{2i+1}$)

- For $\mathcal{R}_{2i}$ the outcomes are exactly as before.
- For $\mathcal{R}_{2i+1}$ they are very similar:

$\boxed{w}$: The strategy waits forever at (2) for $B^s(y) = 0 = \Phi_i^A(y)[s]$, and $\mathcal{R}_{2i+1}$ is satisfied.

$\boxed{s}$: The strategy halts at (3) with

$$B(x) = B^{s+1}(y) = 1 \neq 0 = \Phi_i^A(y)[s].$$

We set up the A restraint w, and $\mathcal{R}_{2i+1}$ is satisfied.

- But what about the injuries — surely they introduce another outcome in which $\mathcal{R}_{2i}$ keeps on injuring $\mathcal{R}_{2i+1}$?

Not at all!

• $\mathcal{R}_{2i}$, of course, is *never* injured. It only has outcomes $\boxed{w}$ and $\boxed{s}$, each of which mean $\mathcal{R}_{2i}$ *never again* chooses a fresh witness or sets up a new restraint.

• So there is a stage after which $\mathcal{R}_{2i+1}$ too is *never* injured. This means that for $\mathcal{R}_{2i+1}$ also there are only two outcomes $\boxed{w}$ and $\boxed{s}$.

• The strategy is *finite injury*, and the use of priority has enabled us to satisfy both requirements.

The strategy for $\mathcal{R}_{2i+1}$ with all the other requirements

• It is easy to see now that all the requirements can successfully pursue their own copies of the strategy.

• $\mathcal{R}_0$ is never injured.

• So after some stage s_0 $\mathcal{R}_1$ is never injured, and so never injures after some stage $s_1 \geq s_0$.

• Inductively we get $\mathcal{R}_{2i+1}$ is not injured after some stage, and so gets satisfied via $\boxed{w}$ or $\boxed{s}$.