SOLUTIONS (5163, 2008)

+ MARKING SCHEME

Q1) (a) $x \times 0 = 0$

(2) $x \times (y+1) = x \times y + x$

So only need to show + is prim:

$$x + 0 = x$$

$$x + (y+1) = (x+y)'.$$

Also $sg(0) = 0$

(1) $sg(x+1) = 1.$

Then $rm(x,0) = 0.$

$$rm(x, y+1) = rm(x,y)' \times sg(|x - rm(x,y)'|). \quad (3)$$

problems sheet

(b) A suitable machine would have program:

new

$q_0 1 0 q_1$ ← delete leftmost 1

$q_1 0 R q_2$ ← move right in search of further 1's

$q_2 0 0 q_3$ ← if not found, print output 0 and halt

$q_2 1 R q_4$ ← if found, prepare to start subroutine

(4)

$q_4 1 0 q_5$ } ← subroutine consists of deleting
$q_5 0 R q_4$ } all but one remaining 1's and halting

(3)

Given TM computes the function $2(n+1)$

Similar problem sheet

similar problem sheet

Let T compute $\varphi_x(x)$ — so T halts on input $x \Leftrightarrow \varphi_x(x)\downarrow \Leftrightarrow x \in K$. So the halting problem for T is unsolvable.

Now modify T by:

1. Replacing every occurrence of S_k as the 3rd entry of a quadruple by $S_{k'}$, say, not occurring in P.
2. For each $q_i S_k A q_j$ in the modified P, add a quadruple $q_i S_{k'} A q_j$ (so the new program does not distinguish between S_k and $S_{k'}$, but only prints $S_{k'}$).
3. Adding a quadruple $q_i S_m S_k q_j$ to the resulting program for every q_i, S_m occurring in it for which there is not yet a quadruple $q_i S_m A q_{j'}$, where q_j does not occur as the start of a quadruple.

Then the new machine T^*, say, prints S_k on input $x \Leftrightarrow T$ halts on input x. So the printing problem for T^* is unsolvable.

7

Q2 (a) A is c.e. iff there is a computable list of all its members: $a_0, a_1, \ldots$ (or $A = \emptyset$). 1

Assume A, B c.e., with effective enumerations $a_0, a_1, \ldots$, and $b_0, b_1, \ldots$, respectively. Then 2

1. $a_0, b_0, a_1, b_1, \ldots$ is an effective enumeration of $A \cup B$.
2. Enumerate $A \cap B$ by enumerating a_s or b_s into $A \cap B$ whenever we find (respy.) $a_s = b_i$ or $b_s = a_j$, some $i, j \leq s$. 2

(if A or $B = \emptyset$, have $A \cup B = A$ or B, $A \cap B = \emptyset$)

as homework

1. Say A, and hence $\overline{A}$, computable. Then $A, \overline{A}$ are c.e. 1
2. Say $A, \overline{A}$ c.e. with effective enumerations $a_0, a_1, \ldots$, $a_0', a_1', \ldots$, respectively. Effectively decide whether $x \in A$ or not (any $x \in \mathbb{N}$) by enumerating A and $\overline{A}$ until we find an $i(x)$ such that $x = a_{i(x)}$ or $a'_{i(x)}$ ($i(x)$ exists since $\mathbb{N} = A \cup \overline{A}$). Then $x \in A \Leftrightarrow x = a_{i(x)}$. 2

as homework

(b) Write $y \in \mathrm{Rng}\, \varphi_e \Leftrightarrow \exists \langle s, x \rangle\ \varphi_{e,s}(x) = y$ 1

$\langle x, y \rangle \in \mathrm{Graph}\, \varphi_e \Leftrightarrow (\exists s)\ \varphi_{e,s}(x) = y$, 1

and then argue that each set is c.e. because it can be written as a Σ^0_1 set.

similar homework

By (ii), if φ_e is p.c. then Graph φ_e is c.e. new
Conversely, say f has c.e. graph. Then to compute $f(x)$, enumerate Graph(f), and if some $\langle x,y\rangle$ is enumerated, define $f(x)=y$. 2

c) (i) $z \in X \setminus Y \Leftrightarrow \exists s\,(z \in X_s - Y_s) \in \Sigma^0_1$, so c.e.
$X \searrow Y = (X \setminus Y) \cap Y$ = intersection of two c.e. sets, so c.e. 2

(ii) $x \in X \setminus Y \Leftrightarrow \exists s\,(x \in X_s - Y)$ or $x \in (X\setminus Y)\cap Y$

$\Leftrightarrow x \in X - Y$ or $x \in X \searrow Y$

$\Leftrightarrow x \in (X-Y) \cup (X \searrow Y)$ 2

(iii) Say $X \searrow Y$ finite. Then $(X \searrow Y) - (X - Y)$ is a finite set S, say. Then can enumerate $X - Y$ by enumerating all members of $X \setminus Y$ 2 into $X - Y$, except for those appearing in S.

(iv) $(X\setminus Y) \cap (Y \setminus X) = \phi$ since if $x \in X \setminus Y$ and $x \in Y \setminus X$, $x \in X \cap Y$ and x is in X before Y and x is in Y before X (impossible by choice of X_s, Y_s).
Also $(X\setminus Y) \cup (Y \setminus X) = X \cup Y$ since $\subseteq$ immediate, and $(X\setminus Y) \cup (Y\setminus X) \supseteq X \cup Y$ since if $x \in X \cup Y$, $x \in X \setminus Y$ or $x \in Y \setminus X$. 2

similar homework

3. (a) A is *simple* iff c.e., $\overline{A}$ inf., & 1

$\forall W_i$ inf., $W_i \nsubseteq \overline{A}$.

bookwork

(i) If S simple, computable, then $\overline{S}$ computable, so c.e. 2

$= W_i$ say. Since S simple, W_i inf., $\subseteq \overline{S}$, contra. $\boxtimes$

(ii) Say A simple, W infinite c.e. Then $A \cap W$ is c.e. Say $A \cap W$ finite. Then $X = W - (A \cap W)$ is also infinite and c.e., and $X \subseteq \overline{A}$, contradicting A being simple. 2

homework

(iii) Say A, B simple (and so c.e.)

Then (1) $A \cap B$ is c.e., and (2) If W infinite and c.e., then $A \cap W$ is inf. and c.e. (by (ii))

Then $B \cap (A \cap W)$ is infinite c.e. (by (ii) again).

So $(A \cap B) \cap W \neq \phi$ — and (3) $\overline{A \cap B}$ is infinite, since $\overline{A \cap B} \supseteq \overline{A}$ infinite. 3

(b) If
$$\varphi_{\varphi_x(x,\vec{y})} = \begin{cases} \varphi_{\varphi_x(x,\vec{y})} & \text{if } \varphi_x(x,\vec{y})\downarrow \\ \text{totally undefined} & \text{o.w.} \end{cases}$$

can find a computable d s.t. $\varphi_{d(x,\vec{y})} = \varphi_{\varphi_x(x,\vec{y})}$

Then $f(d(x,\vec{y}),\vec{y})$ is computable, $= \varphi_e(x,\vec{y})$, some e.

in problem sheet

So $\varphi_{f(d(x,\vec{y}),\vec{y})} = \varphi_{\varphi_e(x,\vec{y})}$.

Taking $x = e$, get

$$\varphi_{f(d(e,\vec{y}),\vec{y})} = \varphi_{\varphi_e(e,\vec{y})} = \varphi_{d(e,\vec{y})}.$$

So have $\varphi_{k(\vec{y})} = \varphi_{f(k(\vec{y}),\vec{y})}$ with $k(\vec{y}) = d(e,\vec{y})$.

Hence $W_{k(\vec{y})} = \mathrm{dom}\, \varphi_{k(\vec{y})} = \mathrm{dom}\, \varphi_{f(k(\vec{y}),\vec{y})}$

$$= W_{f(k(\vec{y}),\vec{y})}.$$

6

in problem sheet

(c) (i) To see that $g(x,y)$ is computable, just need to notice one obtains $W_{g(x,y)}$ uniformly by (1) enumerating nothing into $W_{g(x,y)}$ until y enters A, in which case just enumerate $f(x)$ into $W_{g(x,y)}$.

2

Then get k from the Fixed Pt. Thm. with Parameters.

(ii) $y \in A \Rightarrow fk(y) \in W_{k(y)} \Rightarrow W_{k(y)} \nsubseteq \overline{C} \Rightarrow fk(y) \in C$

and $y \notin A \Rightarrow W_{k(y)} = \phi \Rightarrow fk(y) \in \overline{C}$ (since $\phi \subseteq \overline{C}$).

So (i) and (ii) give $A \leq_m C$ via fk.

4

It follows every creative set C is m-complete, since if C is creative, every c.e. $A \leq_m C$.

24) (a) $A \le_m B$ iff $f(A) \subseteq B$, $f(\overline{A}) \subseteq \overline{B}$, some computable f. 1

(i) 1 $A \le_m A \oplus B$ via $f(x) = 2x$, and
$B \le_m A \oplus B$ via $g(x) = 2x+1$. 2

2 Say $A \le_m C$ via f computable,
$B \le_m C$ via g "

Then $A \oplus B \le_m C$ via h where

$$h(x) = \begin{cases} f\left(\frac{x}{2}\right) & \text{if } x \text{ even}, \\ g\left(\frac{x-1}{2}\right) & \text{if } x \text{ odd}. \end{cases}$$ 3

(ii) By (i) 1, $\underset{\sim}{a}_m \le \underset{\sim}{a}_m \cup \underset{\sim}{b}_m$, $\underset{\sim}{b}_m \le \underset{\sim}{a}_m \cup \underset{\sim}{b}_m$,
so $\mathrm{lub}\{\underset{\sim}{a}_m, \underset{\sim}{b}_m\} \le \underset{\sim}{a}_m \cup \underset{\sim}{b}_m$.

By (i) 2 - If $\underset{\sim}{a}_m \le \underset{\sim}{c}_m$, $\underset{\sim}{b}_m \le \underset{\sim}{c}_m$, then $\underset{\sim}{a}_m \cup \underset{\sim}{b}_m \le \underset{\sim}{c}_m$.
So $\underset{\sim}{a}_m \cup \underset{\sim}{b}_m \le \mathrm{lub}\{\underset{\sim}{a}_m, \underset{\sim}{b}_m\}$ — so it follows that $\underset{\sim}{a}_m \cup \underset{\sim}{b}_m = \mathrm{lub}\{\underset{\sim}{a}_m, \underset{\sim}{b}_m\}$. 3

bookwork + similar problem sheet

(b) $A \le_T B$ iff there is an algorithm, which given $x \in \mathbb{N}$, computes whether or not $x \in A$ via a computation using at most finitely many answers to questions of the form "is $y \in B$?"

1 $A \equiv_T B \iff A \le_T B$ & $B \le_T A$.

new

(i) If X is A-computable, either $X = \phi$ (so A-c.e.)

or can define an A-computable f by

$$f(0) = \text{least } x \in X$$

3 $$f(n+1) = \text{least } x \in X \text{ with } x > f(n),$$

s.t. $X = \text{range } f$ (so A-c.e.)

(ii) ($\Rightarrow$) X A-computable $\Rightarrow$ $\overline{X}$ A-computable.

So $X, \overline{X}$ A-c.e. (by (i))

($\Leftarrow$) If X or $\overline{X} = \phi$, result follows.

o.w. let $X = \text{range } f$, f A-computable

$\overline{X} = \text{range } g$, g A-computable

3 Then $x \in X \Leftrightarrow f(\mu y[f(y) = x \vee g(y) = x]) = x$.

(iii) 1 X is A-c.e. $\Rightarrow X = W_e^A$, some e

$\Rightarrow X = \{x \mid x \in W_e^A\} \Rightarrow X = \{x \mid \exists s\, [\underbrace{x \in W_{e,s}^A}_{A\text{-computable}}]\}$

$\Rightarrow X \in \Sigma_1^A$.

2 Conversely, $X \in \Sigma_1^A \Rightarrow X = \{x \mid \exists s\, R^A(x,s)\}$

some A-computable R^A,

Define $\psi^A(x) = \begin{cases} 0 & \text{if } \exists s\, R^A(x,s) \\ \text{undefined} & \text{o.w.} \end{cases}$

Then ψ is a partial function computable from A.

4 So $\psi^A = \Phi_e^A$, some e, giving $X = \text{dom } \psi^A$

$= \text{dom } \Phi_e^A = W_e^A$ — so X is A-c.e.

all on problem sheets

5) (a) $\underset{\sim}{a} = \{X \subseteq \mathbb{N} \mid X \equiv_T A\}$

$\subseteq \{X \mid X \leq_T A$

$= \{\Phi_i^A \mid \Phi_i^A \text{ total}\} \subseteq \{\Phi_i^A \mid i \geq 0\}$

3 Since $\underset{\sim}{a}$ a subset of a countable set, $\underset{\sim}{a}$ countable.

$\underset{\sim}{a}$ is also infinite:

Write $A_i = \begin{cases} A \cup \{i\} & \text{if } i \notin A \\ A - \{i\} & \text{if } i \in A \end{cases}$

So for each $i \neq i$, $A_i(i) \neq A_j(i)$, so for each $i \neq j$, $A_j \neq A_i$.

But $A_i \equiv_T A_j \equiv_T A$ each i, j.

3 So $\underset{\sim}{a}$ infinite.

2 (b) A 1-generic $\Leftrightarrow$ for every c.e. set X of strings, either (a) $\exists \tau \subseteq A[\tau \in X]$ or (b) $(\exists \tau \subseteq A)(\forall \sigma \supseteq \tau)[\sigma \notin X]$.

Let A be 1-generic, $A = A_0 \oplus A_1$ (so $A_0, A_1 \leq_T A$).

We need to show $A_0 \not\leq_T A_1$ & $A_1 \not\leq_T A_0$.

Assume, WLOG, that $A_0 \leq_T A_1$, where $A_0 = \Phi_i^{A_1}$.

Define

$$X = \{\sigma \mid \exists x < |\sigma_0| [\Phi_i^{\sigma_1}(x) \downarrow \neq \sigma_0(x)]$$

(where $\sigma = \sigma_0 \oplus \sigma_1$).

Then $\boxed{X \text{ is c.e.}}$, since

$$\sigma \in X \iff \exists s\, \exists x < |\sigma_0| \left[\Phi_{i,s}^{\sigma_1}(x)\downarrow \neq \sigma_0(x) \right],$$

so $X \in \Sigma_1^0$.

So $A \Vdash X$ — so $\exists \sigma \subset A$ s.t. $\sigma \Vdash X$.

$\boxed{\text{We show that } \sigma \in X}$:

Assume o.w. — so $\forall \tau \supset \sigma\, [\tau \notin X]$.

Let $x \geq |\sigma_0|$, and let $\Phi_i^{A_1}(x) = \delta$, say.

So $\Phi_i^{\tau_1}(x) = \delta$, some $\tau_1 \supset \sigma_1$.

But since $x \geq |\sigma_0|$, can choose a $\tau_0 \supset \sigma_0$ with $\tau_0(x) \neq \delta$ — so $\tau \supset \sigma$ and $\tau \in X$, a contradiction. $\boxtimes$

<u>BUT</u> — if $\sigma \in X$, get

$$\exists x < |\sigma_0| \left[\Phi_i^{\sigma_1}(x)\downarrow \neq \sigma_0(x) \right]$$

— from which get $\Phi_i^{A_1}(x)\downarrow \neq A_0(x)$, contradicting $\Phi_i^{A_1} = A_0$. $\boxtimes$ (10)

From choice of A_0, A_1 have

$\underset{\sim}{a} = \deg(A_0) \cup \deg(A_1) = \underset{\sim}{a}_0 \cup \underset{\sim}{a}_1$, say. (2)