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MATH 3163/5164M COMPUTABILITY AND UNSOLVABILITY
Problems 6

Read **CT**, pages 139–154, 161–166, and 287–296

1) Show that $\equiv_T$ is an equivalence relation.

2) Show that $\leq$ is a partial ordering on $\boldsymbol{\mathcal{D}}$.

3) (a) Show that for all $A, B \subseteq \mathbb{N}$
i) $A \leq_T A \oplus B$ and $B \leq_T A \oplus B$, and
ii) If $A \leq_T C$ and $B \leq_T C$ then $A \oplus B \leq_T C$.
(b) If we define the *join* $\mathbf{a} \cup \mathbf{b}$ of Turing degrees $\mathbf{a} = \deg(A)$, $\mathbf{b} = \deg(B)$ by $\mathbf{a} \cup \mathbf{b} = \deg(A \oplus B)$, deduce that $\mathbf{a} \cup \mathbf{b} = \text{lub}\{\mathbf{a}, \mathbf{b}\}$.

4) Show that there is a *least* Turing degree $\mathbf{0}$ = the set of all computable sets.

5) Show that:
(a) If $X \subseteq \mathbb{N}$ is A-computable then X is A-c.e.
(b) X is A-computable if and only if X and $\overline{X}$ are A-c.e. (if and only if $X \in \Delta_1^A$, where we write $X \in \Delta_1^A$ for $X \in \Sigma_1^A$ *and* $X \in \Pi_1^A$).
(c) X is A-c.e. if and only if $X \in \Sigma_1^A$.

6) Show that:
(a) $X \leq_m A'$ if and only if X is A-c.e.
(b) If $K^A = \{x \mid x \in W_x^A\}$, then K^A is A-c.e. but not A-computable.

7) Show that there exists an infinite sequence $\mathbf{a}_0, \mathbf{a}_1, \ldots$ of degrees $\leq \mathbf{0}'$ such that for each $i \neq j$ we have $\mathbf{a}_i \mid \mathbf{a}_j$.

For MATH5164M only:

8)$^{\mathbf{D}}$ For any string σ let $\bar{\sigma}$ be given by

$$\bar{\sigma} = \begin{cases} 0 & \text{if } \sigma(x) = 1, \\ 1 & \text{if } \sigma(x) = 0. \end{cases}$$

Show that if $\widehat{W}_i$ is a c.e. set of strings then so is $\{\bar{\sigma} \mid \sigma \in \widehat{W}_i\}$.
Hence, or otherwise, show that $A \subseteq \mathbb{N}$ is 1-generic if and only if $\overline{A}$ is 1-generic.

9)$^{\mathbf{D}}$ We say that $A \subseteq \mathbb{N}$ is *immune* if and only if A is infinite and contains no infinite c.e. subsets.
Show that for each i $Y_i = \{\sigma \mid \exists x[\sigma(x) = 0 \,\&\, x \in W_i]\}$ is a c.e. set of strings.
Show that if A is 1-generic then A forces each such Y_i, and hence that A is either finite or immune.
Deduce (using the result of question 8 above) that every 1-generic set is immune, and hence not c.e.

HAND IN SOLUTIONS TO **THREE** QUESTIONS