MATH-316301

This question paper consists of 3 printed pages, each of which is identified by the reference MATH 316301

No calculators allowed

Examination for the Module MATH 3163

(January 2006)

COMPUTABILITY AND UNSOLVABILITY

Time allowed : 3 hours

Do not answer more than *FOUR* questions.

All questions carry equal marks.

1. (**a**) Show that $f(x,y) = x + y$ and $g(x,y) = x \,.\, y$ are primitive recursive functions.

If $g_1, g_2, \ldots, g_n \in \text{PRIM}$, and if $R_1, R_2, \ldots, R_n$ are primitive recursive relations such that for each $x \in \mathbb{N}$ exactly one of $R_i(x)$ holds, $1 \leq i \leq n$, show that f is primitive recursive where

$$f(x) = \begin{cases} g_1(x) & \text{if } R_1(x) \text{ holds}\,, \\ g_2(x) & \text{if } R_2(x) \text{ holds}\,, \\ \vdots & \\ g_n(x) & \text{if } R_n(x) \text{ holds}\,. \end{cases}$$

(**b**) Write a Turing program for

$$h(x,y) = x \dot{-} y\,,$$

and briefly describe why your program works.

(**c**) Let $\{\varphi_x\}_{x\in\mathbb{N}}$ be a standard list of all the partial computable functions.

If the total function f is defined by

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_x \text{ is total,} \\ 0 & \text{otherwise,} \end{cases}$$

explain why f cannot be computable.

Deduce that if

$$\text{Tot} = \{x \in \mathbb{N} \mid \varphi_x \text{ is total}\},$$

then Tot is not computable.

2. (**a**) We define: A is *computably enumerable* (*c.e.*) if and only if $A = \emptyset$, or A is the range of some computable function.

Show that: (i) If $A \subseteq \mathbb{N}$ is computable, then A is computably enumerable, and (ii) Every c.e. set is the halting set W_i of some Turing machine.

[You may assume that every partial computable function is Turing computable.]

(**b**) Show that the following sets are c.e:

(i) $K_1 = \{x \mid W_x \neq \emptyset\}$, where you can assume that, for each $x \in \mathbb{N}$, W_x is the x^{th} computably enumerable set in some standard listing, and

(ii) $K = \{x \mid x \in W_x\}$.

Show that there exists a computably enumerable set which is not computable.

[You should carefully state any basic results of computability theory which you use.]

(**c**) Show that there exists a Turing machine T with an unsolvable halting problem.

Deduce that the halting problem for the *Universal Turing Machine* is unsolvable.

3. (**a**) We say $A \subseteq \mathbb{N}$ is *creative* if and only if

1) A is c.e., and

2) There is a computable function f such that for each e

$$W_e \subseteq \bar{A} \Rightarrow f(e) \in \bar{A} - W_e\,,$$

where $\{W_e\}_{e\in\mathbb{N}}$ is a standard list of all c.e. sets.

Show that if C is a creative set then

(i) $\overline{C} \neq \emptyset$,

(ii) For each $n \in \mathbb{N}$, if there exist n members of $\overline{C}$ then there exist $n+1$ such members,

(iii) $\overline{C}$ contains an infinite c.e. subset.

[You may assume that for any finite set X we can effectively find an i such that $X = W_i$.]

(**b**) (i) Prove the *Fixed Point Theorem* for a computable function f.

(ii) Explain why there is a computable function f such that $W_{f(n)} = \{n\}$ for every $n \in \mathbb{N}$.

Say also why it is that every c.e. set A has infinitely many distinct indices e with $A = W_e$.

(iii) We say that a set A is an *index set* if for all $x, y \in \mathbb{N}$ we have

$$[x \in A \;\&\; W_x = W_y] \Longrightarrow y \in A.$$

Let $K = \{x \mid x \in W_x\}$. By using the fixed point theorem, or otherwise, show that K is not an index set.

4. (**a**) Define the notions $A \leq_T B$ (that is, A is *Turing reducible to* B), and $A \equiv_T B$ (that is, A is *Turing equivalent to* B), where $A, B \subseteq \mathbb{N}$.

(i) Show that $\equiv_T$ is an equivalence relation over the sets of natural numbers.

(ii) Show that $\leq$ (the ordering induced by $\leq_T$ on the equivalence classes under $\equiv_T$) is a partial ordering on $\boldsymbol{\mathcal{D}}$ (the set of all Turing degrees).

(**b**) The *Turing jump* B' of $B \subseteq \mathbb{N}$ is defined to be

$$B' = \{\langle m, n\rangle \mid m \in W_n^B\},$$

where $\{W_n^B\}_{n\in\mathbb{N}}$ is a standard list of all B-c.e. sets.

(i) Show that B' is c.e. in B, and that if X is c.e. in B then $X \leq_m B'$.

(ii) Show that if $B \leq_T A$ and X is computably enumerable in B, then X is computably enumerable in A.

Deduce that if $A \equiv_T B$ then $A' \equiv_T B'$.

5. **Either:**

(**a**) Show that there exists a pair of incomparable Turing degrees below $\mathbf{0}'$.

Or:

(**b**) Outline briefly a proof of the following extension of the *Friedberg-Muchnik Theorem:*

There exists an infinite sequence $\{\mathbf{a}_i \mid i \geq 0\}$ *of computably enumerable Turing degrees, such that for each* $i \neq j$ *we have* $\mathbf{a}_i \not\leq \mathbf{a}_j$.

6. Write an essay, covering **not more than three pages**, describing the background to, and consequences of, Alan Turing's discovery of the existence of a Universal Turing Machine.

Your answer should contain enough mathematical content to show a good grasp of the notions and results involved, and enough discussion of these to show an understanding of the broader context.

END