MATH310101

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH3101**

No calculators allowed

Examination for the ModuleMATH3101
(May-June 1997)

MATHEMATICAL LOGIC II

Time allowed: **2 hours**

Do not answer more than *FOUR* questions.

All questions carry equal marks.

Throughout this paper it can be assumed that the first order theory $\mathcal{N}$ for arithmetic has the special axioms:
(N1) $(x_1 = x_2 \rightarrow (x_1 = x_3 \rightarrow x_2 = x_3))$
(N2) $(x_1 = x_2 \rightarrow (x_1' = x_2'))$
(N3) $(\overline{0} \neq x_1')$
(N4) $(x_1' = x_2' \rightarrow x_1 = x_2)$
(N5) $(x_1 + \overline{0} = x_1)$
(N6) $(x_1 + x_2' = (x_1 + x_2)')$
(N7) $(x_1 \times \overline{0} = \overline{0})$
(N8) $(x_1 \times x_2' = x_1 \times x_2 + x_1)$
and the axiom scheme
(N9) If $\phi(x_1)$ is a wf of $\mathcal{L}_{\mathcal{N}}$, then

$$(\phi(\overline{0}) \rightarrow ((\forall x_i)(\phi(x_i) \rightarrow \phi(x_i')) \rightarrow (\forall x_i)\phi(x_i)))$$

is an axiom of $\mathcal{N}$.

1. (a) Show that if t_1, t_2 and t_3 are any terms of $\mathcal{L}_{\mathcal{N}}$, then

$$(t_1 = t_2 \rightarrow (t_1 = t_3 \rightarrow t_2 = t_3))$$

and

$$(t_1 + \overline{0} = t_1)$$

are theorems of $\mathcal{N}$.

(b) Show that if m and n are natural numbers and $\overline{m}$ and $\overline{n}$ are the corresponding numerals, then

$$\vdash_{\mathcal{N}} \ (\overline{m} = \overline{m})$$

and

$$\vdash_{\mathcal{N}} \ (\overline{m} = \overline{n} \rightarrow \overline{n} = \overline{m}).$$

(c) If $\mathbf{M} = \langle M, 0, ', +, \times =\rangle$ is a model of $\mathcal{N}$, and $c \in M$, we say that c is a *non-standard element* of $\mathbf{M}$ if and only if c is not the interpretation of any numeral $\overline{m}$.

Show that if $\mathbf{M}$ is a model of $\mathcal{N}$, and c is a non-standard element of $\mathbf{M}$, then c' (the successor of c in $\mathbf{M}$) is also a non-standard element of $\mathbf{M}$.

Deduce that any non-standard model of $\mathcal{N}$ has infinitely many non-standard elements.
[You may assume that $(x_1 \neq x_1')$ is a theorem of $\mathcal{N}$.]

2. (a) Say what is meant by the terms *representable function*, *representable set*.

Show that if the sets S and T are representable in $\mathcal{N}$, then so is $S \cup T$.

(b) Assuming that $+$ (addition of pairs of natural numbers) is a primitive recursive function, show that $\times$ (multiplication of pairs of natural numbers) is also primitive recursive.

Show that if $f(\overrightarrow{n}, m)$ is primitive recursive, then so is the *bounded product* $h(\overrightarrow{n}, p)$ defined by

$$h(\overrightarrow{n}, p) = \prod_{m \leq p} f(\overrightarrow{n}, m).$$

(c) Let gn be a standard Gödel numbering for $\mathcal{N}$. Assume the following definitions of number theoretic relations and sets:

Form$(m) \Leftrightarrow_{\text{defn}} \ gn^{-1}(m)$ is a wf of $\mathcal{L}_{\mathcal{N}}$,

MP$(m, n, p) \Leftrightarrow_{\text{defn}} \ gn^{-1}(p)$ is derived from $gn^{-1}(m)$ and $gn^{-1}(n)$ via Modus Ponens,

Proof$_{\mathcal{N}}(m) \Leftrightarrow_{\text{defn}} \ gn^{-1}(m)$ is a proof of $\mathcal{N}$, and

$T_{\mathcal{N}} = \{m \mid \vdash_{\mathcal{N}} \ gn^{-1}(m)\}$.

Show that: (i) If Form is a recursive relation, then so is MP, and

(ii) If Proof$_{\mathcal{N}}$ is a recursive relation, then $T_{\mathcal{N}}$ is recursively enumerable.
[You may assume that if S is any set for which $m \in S$ is a Σ_1^0-relation, then S is recursively enumerable.]

3. Write an essay on *Gödel's Incompleteness Theorem*, covering not more than about two sides.
[In it, you should mention any points of particular interest or difficulty in the proof of the theorem, and any mathematical, or more general, consequences of the theorem you can think of.]

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4. (a) Define: A is *many-one reducible* to B $(A \leq_m B)$.

Show that if $S \leq_m S'$ with S' recursively enumerable, then S is recursively enumerable.
[You may assume that a set S is recursively enumerable if and only if $m \in S$ is a Σ^0_1-relation]

(b) Show that if S is semi-representable in a recursively axiomatisable theory $\mathcal{T}$ then $S \leq_m T_{\mathcal{T}}$ (the set of Gödel numbers of theorems of $\mathcal{T}$), and hence S is recursively enumerable.
[You may assume that for any recursively axiomatisable $\mathcal{T}$ we have $T_{\mathcal{T}}$ recursively enumerable.]

(c) Define: $\mathcal{T}$ is ***ω**-consistent*.

Show that if $\mathcal{T}$ is an ω-consistent theory in the language of $\mathcal{N}$ in which every recursive relation is representable, then every recursively enumerable set is semi-representable in $\mathcal{T}$.

Deduce that for any recursively axiomatisable theory $\mathcal{T}$ we have $T_{\mathcal{T}} \leq_m T_{\mathcal{N}}$.
[You may assume $\mathcal{N}$ to be an ω-consistent theory in which every recursive relation is representable.]

5. (a) Let $\mathcal{T}'$ be a finite extension of a first order theory $\mathcal{T}$.

Show that $T_{\mathcal{T}'} \leq_m T_{\mathcal{T}}$.

(b) We say a first order theory $\mathcal{T}$ is *strongly undecidable* if and only if $\mathcal{T}$ is finitely axiomatisable and every theory $\mathcal{T}'$ in the language of $\mathcal{T}$ that is consistent with $\mathcal{T}$ (that is, such tht $\mathcal{T} \cup \mathcal{T}'$ is consistent) is undecidable.

Show that if RR is a finitely axiomatisable theory in the language of $\mathcal{N}$ in which every recursive relation is representable, then RR is strongly undecidable.

[You may assume *Rosser's Theorem* in the form: If $\mathcal{T}'$ is a consistent axiomatisable first order theory in which every recursive function is representable, then $\mathcal{T}'$ is incomplete and undecidable.]

Deduce that there is no algorithm for deciding of any given wf φ in the language of $\mathcal{N}$ whether or not it is logically valid or not.

END