MATH5102M01

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Examination for the Module MATH5102M
(May–June 2010)

ADVANCED LOGIC

Time allowed: **3 hours**

ANSWERS .

All questions carry equal marks.

1. (a) We suppose that t_1, t_2 and t_3 are any terms of $\mathcal{L}_{\mathcal{PA}}$. Then (i) is a theorem of $\mathcal{PA}$ since

1) $\vdash_{\mathcal{PA}}\ x_1 = x_2 \to (x_1 = x_3 \to x_2 = x_3)$ (PA1)

2) $\vdash_{\mathcal{PA}}\ \forall x_1(x_1 = x_2 \to (x_1 = x_3 \to x_2 = x_3))$ (Gen,1)

3) $\vdash_{\mathcal{PA}}\ \forall x_1(x_1 = x_2 \to (x_1 = x_3 \to x_2 = x_3)) \to (t_1 = x_2 \to (t_1 = x_3 \to x_2 = x_3))$ (PA5)

4) $t_1 = x_2 \to (t_1 = x_3 \to x_2 = x_3)$ (MP,2,3)

Now repeat the above steps with the wf of 4. and t_2/x_2, and then repeat this process once again with the resulting wf and t_3/x_3 to obtain:

10) $\vdash_{\mathcal{PA}}\ t_1 = t_2 \to (t_1 = t_3 \to t_2 = t_3)$ $(\!(\mathbf{2\ marks})\!)$

Likewise, (ii) is a theorem of $\mathcal{PA}$ since:

1) $\vdash_{\mathcal{PA}}\ \overline{0} \neq x_1'$ (PA3)

2) $\vdash_{\mathcal{PA}}\ \forall x_1(\overline{0} \neq x_1')$ (Gen,1)

3) $\vdash_{\mathcal{PA}}\ \forall x_1(\overline{0} \neq x_1') \to \overline{0} \neq t_1'$ (PC5)

4) $\vdash_{\mathcal{PA}}\ \overline{0} \neq t_1'$ (MP,2,3).

We conclude that both (i) and (ii) are theorems of $\mathcal{PA}$. $(\!(\mathbf{2\ marks})\!)$

(b) $x_1 = x_1$ is a theorem of $\mathcal{PA}$ because

1) $\vdash_{\mathcal{PA}}\ x_1 + \overline{0} = x_1$ (PA5)

2) $\vdash_{\mathcal{PA}}\ x_1 + \overline{0} = x_1 \to (x_1 + \overline{0} = x_1 \to x_1 = x_1)$ (instance of (a) (i) with $t_1 = x_1 + \overline{0}, t_2 = t_3 = x_1$).

3) $\vdash_{\mathcal{PA}}\ x_1 + \overline{0} = x_1 \to x_1 = x_1$ (MP,1,2)

4) $\vdash_{\mathcal{PA}}\ x_1 = x_1$ (MP,1,3). $(\!(\mathbf{4\ marks})\!)$

(c) Suppose that $\mathfrak{N} \models \mathcal{PA}$ and that for each $m \in \mathbb{N}$, $\vdash_{\mathcal{PA}} \varphi(\overline{m})$.

Now, since $\mathfrak{N} \models \mathcal{PA}$, $\mathfrak{N} \models \varphi(\overline{m})$ for all $m \in \omega$. Thus it cannot be the case that $\mathfrak{N} \models \exists x_1 \neg\varphi(x_1)$ (since otherwise for some m, both $\mathfrak{N} \models \varphi(\overline{m})$ and $\mathfrak{N} \models \neg\varphi(\overline{m})$, a contradiction). Hence $\mathfrak{N} \models \neg\exists x_1 \neg\varphi(x_1)$ (as either $\mathfrak{N} \models \chi$ or $\mathfrak{N} \models \neg\chi$ for every sentence χ). Therefore it is not the case that $\vdash_{\mathcal{PA}} \exists x_1 \neg\varphi(x_1)$, since otherwise, as $\mathfrak{N} \models \mathcal{PA}$, we would have $\mathfrak{N} \models \exists x_1 \neg\varphi(x_1)$. Again a contradiction. Thus, if

$$\vdash_{\mathcal{PA}} \exists x_i \neg\varphi(x_i) \quad \underline{\text{then}} \quad \text{“not”} \vdash_{\mathcal{PA}} \varphi(\overline{m}), \text{ for some } m. (\!(\mathbf{4\ marks})\!)$$

(d) Suppose that $\mathcal{T}$ is not consistent and that ψ is a wf of $\mathcal{L}_{\mathcal{T}}$. Hence there exists a wf φ such that

1) $\vdash_{\mathcal{T}} \varphi \wedge \neg\varphi$.

However, $\varphi \wedge \neg\varphi \to \psi$ is a tautology and so

2) $\vdash_{\mathcal{T}} \varphi \wedge \neg\varphi \to \psi$.

Hence by Modus Ponens applied to 1 and 2, $\vdash_{\mathcal{T}} \psi$. $(\!(\mathbf{2\ marks})\!)$

Now let $\varphi(x_i)$ be a wf of $\mathcal{L}_{\mathcal{PA}}$. Then as $\mathcal{T}$ is ω-consistent either

$$\text{“not”} \vdash_{\mathcal{T}} \varphi(\overline{m}) \quad \text{for some } m \in \mathbb{N},$$

or

$$\text{“not”} \vdash_{\mathcal{T}} \exists x_i \neg\varphi(x_i).$$

Hence $\mathcal{T}$ is consistent since not every wf is provable in $\mathcal{T}$. $(\!(\mathbf{2\ marks})\!)$

(e) For a contradiction suppose that $\mathcal{T}$ is as stated in the question. Then

1) $\vdash_{\mathcal{T}} \neg\big((\overline{0}+\overline{1})' = \overline{0}+\overline{1}'\big)$

since $\vdash_{\mathcal{PA}} \big((\overline{0}+\overline{1})' = \overline{0}+\overline{1}'\big)$.

However, by logical validity,

2) $\vdash_{\mathcal{T}} \neg\big((\overline{0}+\overline{1})' = \overline{0}+\overline{1}'\big) \;\to\; \exists x_1 \neg\big((\overline{0}+\overline{1})' = \overline{0}+x_1'\big)$.

Hence

3) $\vdash_{\mathcal{T}} \exists x_1 \neg\big((\overline{0}+\overline{1})' = \overline{0}+x_1'\big)$ (call this χ)

by Modus Ponens applied to 1 and 2.

However, χ is also a theorem of $\mathcal{PA}$ since $\vdash_{\mathcal{PA}} \neg\big((\overline{0}+\overline{1})' = \overline{0}+\overline{0}'\big)$. Moreover, χ is not logically valid since if $\mathfrak{M} = \langle \{0\}, +, \times, ', \overline{0}, = \rangle$ is a singleton model in which $'$ is interpreted as the identity function, then it is not the case that $\mathfrak{M} \models \chi$. This contradicts the definition of $\mathcal{T}$. $(\!(\mathbf{4\ marks})\!)$

2. (a) (i) Let $R(\overrightarrow{m})$ be a $k+1$-place relation. We say that R is *representable* in $\mathcal{PA}$ iff there is a wf φ of $\mathcal{L}_{\mathcal{PA}}$ for which

$$\begin{aligned} R(\overrightarrow{m}) \quad &\Rightarrow \quad \vdash_{\mathcal{PA}} \varphi(\overrightarrow{\overline{m}}) \\ \neg R(\overrightarrow{m}) \quad &\Rightarrow \quad \vdash_{\mathcal{PA}} \neg\varphi(\overrightarrow{\overline{m}}) \end{aligned}$$

Question 2 continued...

⦅**1 mark**⦆

(ii) We say that $f : \mathbb{N} \to \mathbb{N}$ is *representable* in $\mathcal{PA}$ if the graph of f is representable in $\mathcal{PA}$. ⦅**1 mark**⦆

(iii) We say that $S \subseteq \mathbb{N}$ is *representable* in $\mathcal{PA}$ if the relation "$m \in S$" is representable in $\mathcal{PA}$. ⦅**1 mark**⦆

(iv) $S \subseteq \mathbb{N}$ is *semi-representable* in $\mathcal{PA}$ iff there exists a wf $\varphi(x_i)$ of $\mathcal{L}_{\mathcal{PA}}$ such that

$$m \in S \qquad \Leftrightarrow \qquad \vdash_{\mathcal{PA}} \varphi(\overline{m})\,.$$

⦅**1 mark**⦆

(b) Let $\varphi(x_0, x_1)$ be the formula $x_1 = \overline{1}$. Consider any $m_0, m_1 \in \mathbb{N}$.

1) Then $\mathbf{1}(m_0) = m_1$ implies that $m_1 = 1$. So, using (I) we have that $\vdash_{\mathcal{PA}} \overline{m_1} = \overline{1}$, i.e. that $\vdash_{\mathcal{PA}} \varphi(m_0, m_1)$.

2) On the other hand, $\mathbf{1}(m_0) \neq m_1$ implies that $m_1 \neq 1$. So by (II) $\vdash_{\mathcal{PA}} \neg(\overline{m_1} = \overline{1}$, i.e. that $\vdash_{\mathcal{PA}} \neg\varphi(m_0, m_1)$.

Thus $\varphi(x_0, x_1)$ represents the constant function $\mathbf{1} : m \mapsto 1$. ⦅**4 marks**⦆

(c) (i) We have that $max(m_1, m_2) = m_1 \times sg(m_1 \dot{-} m_2) + m_2 \times \overline{sg}(m_1 \dot{-} m_2)$.

Hence, $max(m_1, m_2)$ is primitive recursive as it is the composition of primitive recursive functions (the substitution rule). ⦅**2 marks**⦆

(ii) We use induction to show that $max\{m_1, \ldots, m_n\}$ is primitive recursive for $n \geq 2$.

Base Case $n = 2$: $max\{m_1, m_2\} = max(m_1, m_2)$.

Case $n > 2$: $max\{m_1, \ldots, m_n\} = max(max\{m_1, \ldots, m_{n-1}\}, m_n)$.

By the induction hypothesis $max\{m_1, \ldots, m_{n-1}\}$ is primitive recursive. Thus $max\{m_1, \ldots, m_n\} = max(max\{m_1, \ldots, m_{n-1}\}, m_n)$ is primitive recursive by the substitution rule. ⦅**2 marks**⦆

(d) Choose $m \in \mathbb{N}$ and suppose that $Th_{\mathcal{PA}}(m)$. Thus $\exists R(p, m)$. Hence $R(n, m)$ holds for some $n \in \mathbb{N}$. Therefore

(i) $\vdash_{\mathcal{PA}} \psi(\overline{n}, \overline{m})$ since ψ represents R.

However $\psi(\overline{n}, \overline{m}) \to \exists x_0 \psi(x_0, \overline{m})$ is logically valid. Thus

(ii) $\vdash_{\mathcal{PA}} \psi(\overline{n}, \overline{m}) \to \exists x_0 \psi(x_0, \overline{m})$

and it follows, by applying Modus Ponens to (i) and (ii) that

$\vdash_{\mathcal{PA}} \exists x_0 \psi(x_0, \overline{m})$, i.e. that $\vdash_{\mathcal{PA}} \varphi(\overline{m})$. ⦅**4 marks**⦆

(e) Suppose that $\vdash_{\mathcal{PA}} \varphi(\overline{m})$. In other words $\vdash_{\mathcal{PA}} \exists x_0 \psi(x_0, \overline{m})$.

However $\exists x_0 \psi(x_0, \overline{m}) \to \exists x_0 \neg\big(\neg\psi(x_0, \overline{m})\big)$ is logically valid and so provable in $\mathcal{PA}$. Therefore using Modus Ponens we know that,

$\vdash_{\mathcal{PA}} \exists x_0 \neg\big(\neg\psi(x_0, \overline{m})\big)$.

Thus, by ω-consistency of $\mathcal{PA}$, "not" $\vdash_{\mathcal{PA}} \neg\psi(\overline{m}, \overline{n})$ for some $n \in \mathbb{N}$. So $R(m, n)$ holds, since ψ represents R. This means that $\exists p R(p, m)$, i.e. $Th_{\mathcal{PA}}(m)$.

By (d) and (e) we know that, for all $m \in \mathbb{N}$,

$$\vdash_{\mathcal{PA}} \varphi(\overline{m}) \Leftrightarrow Th_{\mathcal{PA}}(m).$$

Hence $Th_{\mathcal{PA}}(m)$ is semi-representable. $\langle\!\langle$**4 marks**$\rangle\!\rangle$

3. (a) (i) We have that, for all $m \in \mathbb{N}$,

$$\begin{aligned} m \in \mathcal{K} &\Leftrightarrow m \in W_m \\ &\Leftrightarrow \exists p T_1(m, p, m). \end{aligned}$$

So $\mathcal{K}$ is Σ_1^0 since $T_1(m, p, m)$ is computable in m, p. Thus $\mathcal{K}$ is c.e. by the first part of Question 4(e). $\langle\!\langle$**2 marks**$\rangle\!\rangle$

(ii) Now, for a contradiction suppose that $\mathcal{K}$ is computable. Then $\overline{\mathcal{K}}$ is c.e. by Basic Fact 2. So, for some i, $\overline{\mathcal{K}} = W_i$. Thus, for all $m \in \mathbb{N}$, $m \in \overline{\mathcal{K}}$ iff $m \in W_i$. Putting $m = i$, we have that,

$$\begin{aligned} i \in \overline{\mathcal{K}} &\Leftrightarrow i \in W_i && \text{since } W_i = \overline{\mathcal{K}} \\ &\Leftrightarrow i \in \mathcal{K} && \text{by definition of } \mathcal{K}. \end{aligned}$$

A contradiction. So $\mathcal{K}$ is not computable. $\langle\!\langle$**2 marks**$\rangle\!\rangle$

(b) Suppose that X is a c.e. set. Then there exists j such that $X = W_j$. Define the function $f : \mathbb{N} \to \mathbb{N}$ by $f(m) = \langle m, j\rangle$ for all $m \in \mathbb{N}$. Then f is computable since the pairing function is computable and j is fixed (formally in terms of recursive function this is substitution of the constant function **j** : $m \mapsto j$ in the recursive function $\langle\, , \rangle$). Moreover, by definition, for all $m \in \mathbb{N}$, $m \in X$ iff $\langle m, j\rangle \in \mathcal{K}^*$. Thus

$$m \in X \quad \Leftrightarrow \quad f(m) \in \mathcal{K}^*$$

for all $m \in \mathbb{N}$. $\langle\!\langle$**2 marks**$\rangle\!\rangle$

We know that $\leq_{\mathrm{m}}$ is transitive (Question 4(a)). Also by the above $X \leq_{\mathrm{m}} \mathcal{K}^*$ and we are give that $\mathcal{K}^* \leq_{\mathrm{m}} \mathcal{K}$. Thus $X \leq_{\mathrm{m}} \mathcal{K}$. $\langle\!\langle$**1 mark**$\rangle\!\rangle$

Now suppose that $\overline{X}$ is also c.e. Then, by Basic Fact 2, X is computable. Suppose also that $\mathcal{K} \leq_{\mathrm{m}} X$. Then there exists a computable function g such that, for all $m \in \mathbb{N}$, $n \in \mathcal{K}$ iff $g(m) \in X$. Let C_X be the characteristic function of X. Then the characteristic function of $\mathcal{K}$ is the function $C_X(g(n))$ which is computable, being the composition of two computable functions. A contradiction. So it is not the case that $\mathcal{K} \leq_{\mathrm{m}} X$ in this case. $\langle\!\langle$**1 mark**$\rangle\!\rangle$

(c) Suppose that S is representable in $\mathcal{PA}$ via the wf $\varphi(x_0)$. Then, by definition,

$$\begin{aligned} m \in S &\Rightarrow \vdash_{\mathcal{PA}} \varphi(\overline{m}) && (1) \\ m \notin S &\Rightarrow \vdash_{\mathcal{PA}} \neg\varphi(\overline{m}) && (2) \end{aligned}$$

for all $m \in \mathbb{N}$.

Now, it is not the case that both $\vdash_{\mathcal{PA}} \varphi(\overline{m})$ and $m \notin S$ for some $m \in \mathbb{N}$ since by (2) this would imply that both $\vdash_{\mathcal{PA}} \varphi(\overline{m})$ and $\vdash_{\mathcal{PA}} \neg\varphi(\overline{m})$ contradicting consistency of $\mathcal{PA}$. Hence, for all $m \in \mathbb{N}$, $m \in S$ iff $\vdash_{\mathcal{PA}} \varphi(\overline{m})$.

Likewise it is not the case that both $\vdash_{\mathcal{PA}} \neg\varphi(\overline{m})$ and $m \in S$ for some $m \in \mathbb{N}$ otherwise by (1) both $\vdash_{\mathcal{PA}} \neg\varphi(\overline{m})$ and $\vdash_{\mathcal{PA}} \varphi(\overline{m})$ again contradicting consistency of $\mathcal{PA}$. Hence, for all $m \in \mathbb{N}$, $m \in \overline{S}$ iff $\vdash_{\mathcal{PA}} \neg\varphi(\overline{m})$.

We can conclude therefore that both S and $\overline{S}$ are semi-representable and so c.e. Hence by Basic Fact 2 we deduce that S is computable. (**(4 marks)**)

(d) We are given that, for all $m \in \mathbb{N}$,

$$m \in \mathcal{K} \qquad \text{iff} \qquad \vdash_{\mathcal{PA}} \varphi(\overline{m}).$$

By part (a) we know that $\mathcal{K}$ is c.e. but not computable. Thus, by part (c) $\mathcal{K}$ is not representable in $\mathcal{PA}$. Hence there exists $m \in \mathbb{N}$ such that

$$m \in \overline{\mathcal{K}} \qquad \text{but it is not the case that} \qquad \vdash_{\mathcal{PA}} \neg\varphi(\overline{m})$$

since otherwise $\varphi(x_0)$ would represent $\mathcal{K}$ in $\mathcal{PA}$. However for such $m \in \overline{\mathcal{K}}$ we also know that it is not the case that $\vdash_{\mathcal{PA}} \varphi(\overline{m})$ by (4). Thus neither $\varphi(\overline{m})$ nor $\neg\varphi(\overline{m})$ is provable in $\mathcal{PA}$. (**(3 marks)**)

We conclude that $\mathcal{PA}$ is incomplete. (**(1 mark)**)

(e) Let $\mathcal{T}$ and $\mathcal{T}'$ be as in the Question. Then for all $m \in \mathbb{N}$

$$\begin{array}{llll} m \in T_{\mathcal{T}'} & \Leftrightarrow & \vdash_{\mathcal{T}'} gn^{-1}(m) & \text{by definition,} \\ & \Leftrightarrow & \varphi_1, \ldots, \varphi_n \vdash_{\mathcal{T}} gn^{-1}(m) & \text{since } \mathcal{T}' = \mathcal{T} \cup \Sigma, \\ & \Leftrightarrow & \varphi_1 \wedge \cdots \wedge \varphi_n \vdash_{\mathcal{T}} gn^{-1}(m) & \end{array}$$

since $\varphi_1, \ldots \varphi_n \vdash_{\mathcal{T}} \varphi_1 \wedge \cdots \wedge \varphi_n$, whereas for all $1 \leq i \leq n$ it is the case that $\varphi_1 \wedge \cdots \wedge \varphi_n \vdash_{\mathcal{T}} \varphi_i$,

$$\Leftrightarrow \qquad \vdash_{\mathcal{T}} \varphi_1 \wedge \cdots \wedge \varphi_n \to gn^{-1}(m)$$

by the Deduction Theorem,

$$\Leftrightarrow \qquad \underbrace{gn\Big(\varphi_1 \wedge \cdots \wedge \varphi_n \to gn^{-1}(m)\Big)}_{\text{computable function of } m} \in T_{\mathcal{T}}\,.$$

Thus $T_{\mathcal{T}'} \leq_{\mathrm{m}} T_{\mathcal{T}}$ as required. (**(4 marks)**)

4. (a) Let $S, S', S'' \subseteq \mathbb{N}$ be any sets. Then,

(i) $S \leq_{\mathrm{m}} S$ via the identity function, so $\equiv_{\mathrm{m}}$ is reflexive. (**(1 mark)**)

(ii) $S \equiv_{\mathrm{m}} S' \quad \Rightarrow \quad S' \equiv_{\mathrm{m}} S$ by the definition of $\equiv_{\mathrm{m}}$ and so $\equiv_{\mathrm{m}}$ is symmetric. (**(1 mark)**)

(iii) Suppose that $S \leq_{\mathrm{m}} S'$ via f and $S' \leq_{\mathrm{m}} S''$ via g. Then $m \in S$ iff $f(m) \in S'$ iff $g(f(m)) \in S''$.

Thus $S \leq_{\mathrm{m}} S''$ via $g \circ f$. Therefore

$$\begin{aligned} S \equiv_{\mathrm{m}} S' \ \&\ S' \equiv_{\mathrm{m}} S'' \quad & \Rightarrow \quad S \leq_{\mathrm{m}} S'' \ \&\ S'' \leq_{\mathrm{m}} S \\ & \Rightarrow \quad S \equiv_{\mathrm{m}} S''. \end{aligned}$$

So $\equiv_{\mathrm{m}}$ is transitive.

We conclude that $\equiv_{\mathrm{m}}$ is an equivalence relation. $\langle\!\langle$**2 marks**$\rangle\!\rangle$

(b) Let $S, S' \notin \{\emptyset, \mathbb{N}\}$ be computable sets. Choose $p \in S'$ and $\overline{p} \in \overline{S'}$. Define

$$f(m) \quad = \quad \begin{cases} p & \text{if } m \in S \\ \overline{p} & \text{if } m \notin S. \end{cases}$$

Then f is computable since S is and $S' \leq_{\mathrm{m}} S$ via f. Likewise $S \leq_{\mathrm{m}} S'$. Thus $S \equiv_{\mathrm{m}} S'$. So the set of computable sets is subsumed by the same many one degree $\mathbf{0}_{\mathrm{m}}$. Choose any set $X \in \mathbf{0}_{\mathrm{m}}$. Then $X \leq_{\mathrm{m}} S$ by definition. Suppose that f is the computable function that witnesses this and that C_S is the characteristic function of S. Then $C_S \circ f$ is the characteristic function of X. However $C_S \circ f$ is the composition of computable functions and so computable. Thus X is computable. We conclude that $\mathbf{0}_{\mathrm{m}}$ is exactly the set of all computable sets. $\langle\!\langle$**4 marks**$\rangle\!\rangle$

(c) (i) Firstly we know that the absolute value function is primitive recursive since $|m - n| = (m \dot{-} n) + (n \dot{-} m)$, i.e. it is the composition of two primitive recursive functions and so is itself primitive recursive. Secondly supposing that C_S, C_R are the characteristic functions of primitive recursive sets $S, R \subseteq \mathbb{N}$ we can see that the characteristic function $C_{R \cup S}$ of $R \cup S$ is defined by $C_{R \cup S} = sg(C_R + C_S)$, again a composition of primitive recursive functions and so primitive recursive. Now every singleton set $\{m\}$ has primitive recursive characteristic function $C_{\{m\}}(n) = \overline{sg}(|m - n|)$. Thus by induction over the cardinality and the fact that the union of two primitive recursive sets is primitive recursive we deduce that all finite sets are primitive recursive. (If D is a finite set of cardinality $n \geq 2$ then $D = D^* \cup \{p\}$ with the cardinality of D^* being $n - 1$. So by induction C_{D^*} is primitive recursive and we know that $C_{\{p\}}$ is primitive recursive. Thus, as union also preserves primitive recursiveness, C_D is primitive recursive (and so D is).

From the above we know that every nonempty finite set is primitive recursive and hence recursive, i.e. computable. $\langle\!\langle$**2 marks**$\rangle\!\rangle$

(ii) By definition, if for any sets X, Y, $X \leq_1 Y$ then $X \leq_{\mathrm{m}} Y$. Hence also if $X \equiv_1 Y$ then $X \equiv_{\mathrm{m}} Y$. It follows that $\mathbf{0}_{\mathrm{m}}$ contains the one-degrees of the computable sets not in $\{\emptyset, \mathbb{N}\}$. Consider any non empty finite sets D, D^* and suppose that $D \equiv_1 D^*$. Then $D \leq_1 D^*$ and $D^* \leq_1 D$ via one-one computable functions f and g respectively. But the cardinality of the range of f (g) is the same as the cardinality of the domain of f (g), since these functions are one-one. Hence we know, respectively that $|D| \leq |D^*|$ and $|D^*| \leq |D|$, i.e. that $|D^*| = |D|$. We conclude that the class $\mathcal{E}_{n+1}$ of all finite sets of cardinality $n + 1$ subsumes a one-one degree (and it is easy to see that this is precisely a single one-one degree). Hence by the first part of this question we know that $\mathbf{0}_{\mathrm{m}}$ contains at least a countably infinite number of one-one degrees. $\langle\!\langle$**2 marks**$\rangle\!\rangle$

(d) We know that, for any $n \in \mathbb{N}$,

$$\begin{aligned} Proof_{\mathcal{T}} &\Leftrightarrow gn^{-1}(n) \text{ is a proof in } \mathcal{T}, \\ &\Leftrightarrow gn^{-1}(n) \text{ is a sequence of wfs} \\ & \varphi_1, \ldots, \varphi_k \text{ for some } k \geq 1, \end{aligned}$$

such that, for each $1 \leq i \leq k$, $Form(gn(\varphi_i)$ holds and either $Ax_{\mathcal{T}}(gn(\varphi_i))$ or $Gen(gn(\varphi_j), gn(\varphi_i))$ or $MP(gn(\varphi_l), gn(\varphi_p), gn(\varphi_i))$ for some $1 \leq j, p, l < i$. However each of $Form$, $Ax_{\mathcal{T}}$, MP and Gen is computable. Therefore we can define an algorithm that, on input n,

(i) Decodes n, and tests whether n codes a sequence of strings of symbols, and using $Form$ whether these strings are wfs of $\mathcal{L}_{\mathcal{PA}}$.

(ii) If n does indeed encode a sequence of wfs $\varphi_1, \ldots, \varphi_k$ say, then the algorithm tests whether $Ax_{\mathcal{T}}(\varphi_1)$, and then subsequently for each of φ_i with $1 < i < k$, makes the appropriate tests as mentioned above using $Ax_{\mathcal{T}}$, MP and Gen.

(iii) If at any stage in this process one of the checks fails the algorithm halts and rejects (output 0). If however all the checks are positive, the algorithm accepts on reaching φ_k (output 1).

We conclude that $Proof_{\mathcal{T}}$ is computable. $(\!(\mathbf{4\ marks})\!)$

(e) (i) Let X be a non empty Σ^0_1 set. Then there exists a binary computable relation R such that for all $m \in \mathbb{N}$,

$$m \in X \qquad \Leftrightarrow \qquad \exists p R(p, m) \,.$$

Choose $m_0 \in X$ and define $f : \mathbb{N} \rightarrow \mathbb{N}$ so that it effects a downward search as follows. $f(0) = m_0$, $f(n+1) =$ the least $m \leq n$ such that $m \notin \{f(0), \ldots, f(n)\}$ and there exists $p \leq n$ so that $R(p, m)$. If there is no such p set $f(n+1) = m_0$. Then it is clear that f is computable. Moreover the range of f lies inside X since if f only maps n to a number m say if $R(p, m)$ holds for some p. Moreover f is clearly also onto X, (If not, then there will be a least $n \in X$ such that n is not in the range of f. An easy argument shows that f will eventually output n. A contradiction.)

(Another easy way of doing this is using the standard pairing function $\langle \, , \, \rangle$ and defining f such that $f(\langle p, m\rangle) = m$ if $R(p, m)$ holds and $f(\langle p, m\rangle) = m_0$ otherwise.) $(\!(\mathbf{1.5\ marks})\!)$

(ii) For any $m \in \mathbb{N}$,

$$\begin{aligned} m \in T_{\mathcal{T}} &\Leftrightarrow Th_{\mathcal{T}}(m) \text{ holds (by definition)}, \\ &\Leftrightarrow gn^{-1}(m) \text{ is a theorem of } \mathcal{T}, \\ &\Leftrightarrow \text{there is a proof of } gn^{-1}(m) \text{ in } \mathcal{T}, \\ &\Leftrightarrow \text{there exists } p \text{ such that } Proof_{\mathcal{T}}(p) \text{ and } m = l(p), \end{aligned}$$

in other words,

$$\Leftrightarrow \exists p \, [\, m = l(p) \;\; \& \;\; Proof_{\mathcal{T}}(p)\,].$$

Thus we can define $R(p, m) =$ "$m = l(p) \quad \& \quad Proof_{\mathcal{T}}(p)$" and we note that R is computable by part (c). Hence, $T_{\mathcal{T}}$ is Σ^0_1 and thus c.e. by the first part of this question. $\left(\!\left(\mathbf{2.5\ marks}\right)\!\right)$

5. Some of the things that might be included in an answer to the question on Gödel's Theorem are as follows.

(i) A statement of the theorem.
(ii) Generalisations (e.g. adding more axioms, dropping the requirement of ω-consitency).
(iii) What is representability and what does it do?
(iv) What are Gödel numbers and why are they useful (mention of self-reference, Russell's Paradox)
(v) Representability of $Proof_{\mathcal{PA}}(m)$ in $\mathcal{PA}$.
(vi) The final proof of the theorem (including the role of c.e. sets).
(vii) The undecidability of $\mathcal{PA}$ and related theories.
(viii) Other applications, e.g. the undecidability of predicate calculus.
(ix) Informal consequences of the theorem (e.g. computers cannot do anything).
(x) Examples of undecidability in Mathematics.
(xi) Anything else you think interesting or relevant. $(\!(\mathbf{20\ marks})\!)$

Note that marks are out of 80.

End