MATH5102M01

This question paper consists of 5 printed pages, each of which is identified by the reference **MATH5102M**

No calculators allowed

Examination for the Module MATH5102M

(May–June 2010)

ADVANCED LOGIC

Time allowed: **3 hours**

Do not answer more than *FOUR* questions.

All questions carry equal marks.

Preliminaries.

(I) Throughout this paper it can be assumed that the first order theory $\mathcal{PA}$ for arithmetic has the special axioms:

(PA1) $(x_1 = x_2 \to (x_1 = x_3 \to x_2 = x_3))$
(PA2) $(x_1 = x_2 \to (x_1' = x_2'))$
(PA3) $\neg(\overline{0} = x_1')$
(PA4) $(x_1' = x_2' \to x_1 = x_2)$
(PA5) $(x_1 + \overline{0} = x_1)$
(PA6) $(x_1 + x_2' = (x_1 + x_2)')$
(PA7) $(x_1 \times \overline{0} = \overline{0})$
(PA8) $(x_1 \times x_2' = x_1 \times x_2 + x_1)$

and the axiom scheme

(PA9) If $\varphi(x_1)$ is a wf of $\mathcal{L}_{\mathcal{PA}}$, then

$$(\varphi(\overline{0}) \to ((\forall x_i)(\varphi(x_i) \to \varphi(x_i')) \to (\forall x_i)\varphi(x_i)))$$

is an axiom of $\mathcal{PA}$.

You are also reminded of the following axiom of Predicate Calculus:

(PC5) $(\forall x_i)\mathscr{A}(x_i) \to \mathscr{A}(t)$ if t is free for x_i in $\mathscr{A}(x_i)$

(II) We also assume that gn is the standard Gödel numbering of $\mathcal{L}_{\mathcal{PA}}$ (such that both gn and gn^{-1} are computable).

(III) For any function $f : \mathbb{N} \to \mathbb{N}$, $Ran(f)$ denotes the range of f. We say that a set X is computably enumerable (c.e.) if $X = \emptyset$ or if there is a computable function f such that $Ran(f) = X$, and is Σ_1^0 if there is a computable relation R such that, for all $m \in \mathbb{N}$, $m \in$

$X \Leftrightarrow \exists p R(p,m)$. You are reminded that a set X is computable iff both X and $\overline{X}$ are c.e. (Basic Fact 2).

(IV) We assume that $\langle n,m\rangle$ is a standard computable one-one, onto pairing function over the integers (so that $\langle\ ,\ \rangle : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$, $Ran(\langle\ ,\ \rangle) = \mathbb{N}$ and $\langle m,n\rangle \neq \langle p,q\rangle$ if either $m\neq p$ or $n\neq q$) with computable inverse functions $(\ )_0$ and $(\ )_1$ such that $(\langle m,n\rangle)_0 = m$ and $(\langle m,n\rangle)_1 = n$.

(V) For any first order theory $\mathcal{T}$ in $\mathcal{L}_{\mathcal{PA}}$ you are reminded of the following number theoretic relations relative to $\mathcal{L}_{\mathcal{PA}}/\mathcal{T}$.

$Form(m)$	$\Leftrightarrow_{\text{defn}}$	$gn^{-1}(m)$ is a wf of $\mathcal{L}_{\mathcal{PA}}$
$Ax_{\mathcal{T}}(m)$	$\Leftrightarrow_{\text{defn}}$	$gn^{-1}(m)$ is an axiom of $\mathcal{T}$
$MP(m,n,p)$	$\Leftrightarrow_{\text{defn}}$	$Form(m), Form(n), Form(p)$ and $gn^{-1}(n) = gn^{-1}(m) \to gn^{-1}(p)$
$Gen(m,n)$	$\Leftrightarrow_{\text{defn}}$	$Form(m), Form(n)$ and, for some i, $gn^{-1}(n) = (\forall x_i) gn^{-1}(m)$
$Proof_{\mathcal{T}}(m)$	$\Leftrightarrow_{\text{defn}}$	$gn^{-1}(m)$ is a proof of $\mathcal{T}$
$Th_{\mathcal{T}}(m)$	$\Leftrightarrow_{\text{defn}}$	$\vdash_{\mathcal{T}} gn^{-1}(m)$.

Note also that $T_{\mathcal{T}}$ is used to denote the set of (codes of) theorems of $\mathcal{T}$. In other words,

$$T_{\mathcal{T}} \ =_{\text{def}} \ \{ m \mid Th_{\mathcal{T}}(m) \}.$$

(VI) Finally remember that by Church's Thesis a function is recursive iff it is computable. Accordingly in this paper the terms "recursive" and "computable" are considered to be interchangeable.

1. (a) Show that if t_1, t_2 and t_3 are any terms of $\mathcal{L}_{\mathcal{PA}}$ then (i) and (ii) below are theorems of $\mathcal{PA}$.

(i) $t_1 = t_2 \to (t_1 = t_3 \to t_2 = t_3)$,

(ii) $\overline{0} \neq t_1'$.

(b) Use (PA5) and part (a)(i) (with $t_1 = x_1 + \overline{0}$, $t_2 = x_1$ and $t_3 = x_1$) to show that $x_1 = x_1$ is a theorem of $\mathcal{PA}$. Deduce that, for any term t of $\mathcal{L}_{\mathcal{PA}}$, $t = t$ is a theorem of $\mathcal{PA}$.

(c) Given that the standard structure $\mathfrak{N} = \langle \mathbb{N}, 0, ', +, \times, = \rangle$ is a model of $\mathcal{PA}$, show that $\mathcal{PA}$ is ω-consistent. That is, for each wf $\varphi(x_1)$ of $\mathcal{L}_{\mathcal{PA}}$ if

$$\vdash_{\mathcal{PA}} \exists x_i \neg\varphi(x_i) \quad \underline{\text{then}} \quad \text{“not”} \vdash_{\mathcal{PA}} \varphi(\overline{m}), \text{ for some } m.$$

(d) Show that if a first order theory $\mathcal{T}$ is not consistent, then every wf ψ of $\mathcal{L}_{\mathcal{T}}$ is provable in $\mathcal{T}$. Hence deduce that that if a theory $\mathcal{T}$, with $\mathcal{L}_{\mathcal{T}} = \mathcal{L}_{\mathcal{PA}}$ is ω-consistent, then $\mathcal{T}$ is consistent.

(e) Show that there is no axiomatisable first order theory $\mathcal{T}$ whose theorems are the wfs of $\mathcal{PA}$ which are *either* logically valid *or* whose negations are provable in $\mathcal{PA}$. *(You may assume that every logically valid wf φ of $\mathcal{L}_{\mathcal{PA}}$ is provable in $\mathcal{PA}$.)*

2. (a) In the context of $\mathcal{PA}$ define (i)-(iv) below.

(i) Representable $k+1$-place relation,

(ii) Representable function $f : \mathbb{N} \to \mathbb{N}$,

(iii) Representable set $S \subseteq \mathbb{N}$,

(iv) Semi-representable set $S \subseteq \mathbb{N}$.

(b) Show that the constant function $\mathbf{1} : m \mapsto 1$ (for all $m \in \mathbb{N}$) is representable in $\mathcal{PA}$ via the wf $\varphi(x_0, x_1) \ =_{\text{def}} \ x_1 = \overline{1}$. $\big($Use (I) Question 1.(b) and (II) the fact that, for any $m, n \in \mathbb{N}$, if $m \neq n$, then $\vdash_{\mathcal{PA}} \neg(\overline{m} = \overline{n})$.$\big)$

(c) Given that the recursive difference

$$n \dot{-} m \quad = \quad \begin{cases} n - m & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases}$$

and

$$sg(m) \quad = \quad \begin{cases} 0 & \text{if } m = 0 \\ 1 & \text{if } m \neq 0\,. \end{cases}$$

are primitive recursive, and hence that also $\overline{sg}(m) = 1 \dot{-} sg(m)$ is primitive recursive, show that the function

$$max(m, n) \quad = \quad \begin{cases} m & \text{if } m \geq n \\ n & \text{otherwise} \end{cases}$$

is primitive recursive. Hence show by induction on $n \geq 2$ that the function:

$$max\{m_1, \ldots, m_n\} \quad = \quad \text{largest of the numbers } m_1, \ldots, m_n$$

is primitive recursive.

(d) You are given that the relation $Th_{\mathcal{PA}}(m)$ is Σ^0_1. In other words there is a computable relation $R(p, m)$ such that

$$Th_{\mathcal{PA}}(m) \quad \Leftrightarrow \quad \exists p R(p, m)\,.$$

Let $\psi(x_0, x_1)$ represent $R(p, m)$ in $\mathcal{PA}$ and let $\varphi(x_1) =_{\text{def}} \exists x_0 \psi(x_0, x_1)$. Show that, for any $m \in \mathbb{N}$,

$$Th_{\mathcal{PA}}(m) \quad \Rightarrow \quad \vdash_{\mathcal{PA}} \varphi(\overline{m})\,.$$

(e) Show that, under the assumption that $\mathcal{PA}$ is ω-consistent (see Question 1.(c)) then, for all $m \in \mathbb{N}$,

$$\vdash_{\mathcal{PA}} \varphi(\overline{m}) \quad \Rightarrow \quad Th_{\mathcal{PA}}(m)\,.$$

Using this result and part (d) explain which of the definitions of part (a) applies to $Th_{\mathcal{PA}}(m)$.

 Question 3 continued...

3. (a) Let $W_0, W_1, \ldots$ be a standard listing of all c.e. sets and let $T_1(i, p, m)$ be the computable relation such that, for all $m \in \mathbb{N}$,

$$m \in W_i \quad \Leftrightarrow \quad \exists p \, T_1(i, p, m) .$$

Define $\mathcal{K} = \{ n \mid n \in W_n \}$.

Show that $\mathcal{K}$ is c.e., but that $\mathcal{K}$ is not computable. (*You may use the result of the first part of Question 4(e).*)

(b) Define $\mathcal{K}^* = \{ \langle m, i \rangle \mid m \in W_i \}$. We know that $\mathcal{K}^* \leq_{\mathrm{m}} \mathcal{K}$.

Show that, if X is a c.e. set, there is a computable function f such that

$$m \in X \quad \Leftrightarrow \quad f(m) \in \mathcal{K}^* .$$

Using the relevant property of $\leq_{\mathrm{m}}$ subsumed by the result of Question 4(a), deduce that $X \leq_{\mathrm{m}} \mathcal{K}$. Show that, if $\overline{X}$ is also c.e. then $\mathcal{K} \not\leq_{\mathrm{m}} X$.

(c) You are given that every set semi-representable in $\mathcal{PA}$ is c.e. Using this fact, show that, if the set S is representable in $\mathcal{PA}$, then both S and $\overline{S}$ are c.e. and hence that S is computable.

(d) Suppose that $\mathcal{K}$ is semi-represented by $\varphi(x_0)$ in $\mathcal{PA}$. Deduce, using parts (a) and (c) that there is a number m such that neither $\varphi(\overline{m})$ nor $\neg\varphi(\overline{m})$ is provable in $\mathcal{PA}$. What can we therefore say about $\mathcal{PA}$?

(e) Let T and $T' = T \cup \Sigma$ be theories in the $\mathcal{L}_{\mathcal{PA}}$ such that $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$ is a finite set of wfs. Prove that

$$T_{\mathcal{T}'} \leq_{\mathrm{m}} T_{\mathcal{T}} .$$

4. (a) Define $S \equiv_{\mathrm{m}} S' \quad \Leftrightarrow_{\mathrm{defn}} \quad S \leq_{\mathrm{m}} S'$ and $S' \leq_{\mathrm{m}} S$. Show that $\equiv_{\mathrm{m}}$ is an equivalence relation (where the equivalence classes are called *many one degrees*).

(b) Show that the class of computable (i.e. recursive) sets—other than $\emptyset$ and $\mathbb{N}$—forms a many one degree (which we call $\mathbf{0}_{\mathrm{m}}$).

(c) Prove that, for any nonempty $X \subseteq \mathbb{N}$, if X is finite then X is primitive recursive. (*Make use of the functions given in Question 2(c). You might start by showing that the absolute value function $|m-n|$ is primitive recursive, using this for the case of singleton sets, and showing that the union of two primitive recursive sets is primitive recursive.*) Deduce that every nonempty finite set $X \subseteq \mathbb{N}$ is computable.

Define $S \leq_1 S'$ if $S \leq_{\mathrm{m}} S'$ via a one-one computable function f (i.e. for all $m, n, p \in \mathbb{N}$, $m \in S$ iff $f(m) \in S'$ and $f(n) \neq f(p)$ if $n \neq p$). Define $S \equiv_1 S'$ iff $S \leq_1 S'$ and $S' \leq_1 S$, and note that this is also an equivalence relation (i.e. you do not have to prove this). Call the resulting equivalence classes *one-one degrees*. Prove that there are infinitely many one-one degrees inside $\mathbf{0}_{\mathrm{m}}$.

In parts (d) and (e) below, you are given that $\mathcal{T}$ is a computably axiomatisable first order theory in $\mathcal{L}_{\mathcal{PA}}$.

(d) Prove that the relation $Proof_{\mathcal{T}}(n)$ is computable.

Question 4 continued...

(e) Prove that, for any $X \subseteq \mathbb{N}$, if X is Σ^0_1 then X is c.e. (you may ignore the trivial case of the c.e./Σ^0_1 set $\emptyset$).

You are given that the function

$$l(p) \quad = \quad \begin{cases} m & \text{if } gn^{-1}(p) \text{ is a sequence of wfs of } \mathcal{L}_{\mathcal{PA}} \\ & \text{and } gn^{-1}(m) \text{ is the last wf of this sequence,} \\ 0 & (\neq \text{the Gödel number of any wf) otherwise.} \end{cases}$$

is computable. Prove that there is a computable relation $R(p, m)$ such that $m \in T_{\mathcal{T}} \Leftrightarrow \exists p\, R(p, m)$. Deduce that $T_{\mathcal{T}}$ is c.e.

5. Write an essay on *Gödel's Incompleteness Theorem*, covering not more than about three sides.

In it you should mention any points of particular interest or difficulty in the proof of the theorem, and any consequences you can think of for mathematics, and our understanding of the world in general.

End