Gödel’s Incompleteness Theorem - background
David Hilbert at 1900
ICM in Paris

SUR LES
PROBLÈMES FUTURS DES MATHEMATIQUES,
PAR M. DAVID HILBERT (Göttingen),
TRADUIRE PAR M. L. LAUGEL (1).

Qui ne souleverait volontiers le voile qui nous cache l'avenir afin
de jeter un coup d'oeil sur les progrès de notre Science et les secrets
de son développement ultérieur durant les siècles futurs? Dans ce
champ si fécond et si vaste de la Science mathématique, quels seront
les buts particuliers que tenteront d'atteindre les guides de la

Hilbert's lecture from the Paris Proceedings
Emerging Themes

(a) The scope of algorithms in Mathematics: - Are there algorithms for solving certain general classes of problems in Mathematics? - Example: Hilbert's 10th Problem

Example: Is there an algorithm for deciding for a given sentence of predicate calculus whether it is logically valid or not?

More generally, do there exist unsolvable problems in Mathematics?
Emerging Themes

- The role of formalism in Mathematics: Developing consistent axiomatic theories for important areas to:

  - Put Mathematics on a sure footing, and

  - Provide a route to systematically finding solutions to important open problems.

Example: Aim to solve Cantor’s Continuum Problem of deciding how many real numbers there are
AIM: Capture Mathematics in COMPLETE, CONSISTENT theories

Bertrand Russell’s Paradox (1901): Define $S = \{ x : x \notin x \}$ ... then: Is $S \in S$?

... an allowable definition in the set theory of Georg Cantor and Gottlob Frege

Principia Mathematica (1901 -1910, with A.N. Whitehead)
Hilbert’s Programme
(1904 -1928)

“For the mathematician there is no Ignorabimus, and, in my opinion, not at all for natural science either. ... The true reason why [no one] has succeeded in finding an unsolvable problem is, in my opinion, that there is no unsolvable problem. In contrast to the foolish Ignorabimus, our credo avers:

We must know,
We shall know. “

- David Hilbert’s opening address to the Society of German Scientists and Physicians,
Königsberg, September 1930
1931- Gödel’s Incompleteness Thm

Any ‘sensible’ theory containing enough basic facts about arithmetic must be **incomplete** (i.e. incapable of proving all true statements about arithmetic).

And even be **incapable** of proving its own consistency!

- Many consequences ...
Various Formalisations of ‘Computable’

Recursive functions (Gödel, Kleene)

\( \lambda \)-computable functions (Church, Kleene)

Turing Computable functions (1936)
Gödel Numbering, Programs As Data

- Coding and listing the Computable functions
- The computing revolution - from calculating machine to stored program computer
Discovery of Unsolvable Problems

- But - techniques for presenting machines give the Universal Turing machine - and emergence of incomputable objects

- Turing: Unsolvability of the Halting Problem for Turing machines

- Church/Turing: Undecidability of logical validity, and PA

- An example ...
Now we witnessed ...
a certain extraordinarily complicated looking set, namely the Mandelbrot set.
Although the rules which provide its definition are surprisingly simple, the set itself exhibits an endless variety of highly elaborate structures.

Roger Penrose
in “The Emperor’s New mind”, Oxford Univ. Press, 1994

Mandelbrot Set
Computable?