

## 7 Arithmetising Arithmetic: Gödel Numbers.

**Motivation:** If we code up the wfs, proofs etc. of  $\mathcal{PA}$  with numbers (called Gödel numbers), we can talk about  $\mathcal{PA}$  in the language  $\mathcal{L}_{\mathcal{PA}}$ . And, in particular we can prove facts about  $\mathcal{PA}$  in  $\mathcal{PA}$  using the representability of the recursive functions.

### Gödel numbers of the symbols of $\mathcal{PA}$

We define  $gn : \mathcal{L}_{\mathcal{PA}} \rightarrow \mathbb{N}$  with  $gn$  1-1.

$gn( )$	=	$p_0$	=	2	$gn(\bar{0})$	=	$p_9$	=	29
$gn( ( )$	=	$p_1$	=	3	$gn(')$	=	$p_{10}$	=	31
$gn(\neg)$	=	$p_2$	=	5	$gn(+)$	=	$p_{11}$	=	37
$gn(\wedge)$	=	$p_3$	=	7	$gn(\times)$	=	$p_{12}$	=	41
$gn(\vee)$	=	$p_4$	=	11	$gn(=)$	=	$p_{13}$	=	43
$gn(\rightarrow)$	=	$p_5$	=	13	$gn(x_i)$	=	$p_{14+i}$		$(i = 0, 1,$
$gn(\leftrightarrow)$	=	$p_6$	=	17					
$gn(\forall)$	=	$p_7$	=	19					
$gn(\exists)$	=	$p_8$	=	23					

*Note.* This Gödel numbering can be modified to cover larger languages than  $\mathcal{L}_{\mathcal{PA}}$  (e.g. to get a Gödel numbering for the predicate calculus  $\mathcal{K}$ ). For example, suppose that we have infinitely many function letters  $f_i$  in  $\mathcal{L}$ . Then we can define

$$\begin{aligned} gn(x_i) &= p_{14+2i} \\ gn(f_i) &= p_{15+2i} \end{aligned}$$

### Gödel numbers for wfs and proofs of $\mathcal{PA}$ .

1) If  $\varphi = s_1 s_2 \dots s_n$  is a list of symbols of  $\mathcal{L}_{\mathcal{PA}}$  (so in particular any wf) then define

$$gn(\varphi) = p_0^{gn(s_0)} \times p_1^{gn(s_1)} \times \dots \times p_n^{gn(s_n)}.$$

2) If  $\vec{\varphi} = \varphi_0 \varphi_1 \dots \varphi_n$  is a list of formulas (so in particular a sequence of wfs forming a proof in  $\mathcal{PA}$ ), then we define

$$gn(\vec{\varphi}) = p_0^{gn(\varphi_0)} \times p_1^{gn(\varphi_1)} \times \dots \times p_n^{gn(\varphi_n)}.$$

*Example 7.1.* Find  $gn(\varphi)$  where  $\varphi$  is the wf  $(\forall x_1 (x_3 = x_1))$ .

*Solution.*

$$\begin{aligned}
 gn(\varphi) &= 2^{gn(()} 3^{gn(\forall)} 5^{gn(x_1)} 7^{gn(()} 11^{gn(x_3)} 13^{gn(=)} 17^{gn(x_1)} 19^{gn(()} 23^{gn())} \\
 &= 2^{p_1} 3^{p_7} 5^{p_{15}} 7^{p_1} 11^{p_{17}} 13^{p_{13}} 17^{p_{15}} 19^{p_0} 23^{p_0} \\
 &= 2^3 3^{19} 5^3 7^3 11^{61} 13^{43} 17^{53} 19^2 23^2 \quad \square
 \end{aligned}$$

The inverse  $gn^{-1}$  of the Gödel numbering.

*Note 7.2.* Since  $gn$  is 1-1 (by the unique factorisation theorem for natural numbers), we can define an inverse function  $gn^{-1} : \mathbb{N} \rightarrow \mathcal{L}_{\mathcal{PA}} \cup \{\perp\}$ .

**Definition 7.3.**

$$gn^{-1}(m) = \begin{cases} s & \text{if } s \text{ is a symbol of } \mathcal{L}_{\mathcal{PA}} \text{ and } gn(s) = m, \\ s_0s_1 \dots s_n & \text{if } gn(s_0s_1 \dots s_n) = m \text{ (with } s_0, s_1, \dots, s_n \text{ being symbols)}, \\ \varphi_0 \dots \varphi & \text{if } gn(\vec{\varphi}) = m \text{ (with } \vec{\varphi} \text{ a sequence of wfs)}, \\ \perp & \text{otherwise.} \end{cases}$$

*Note 7.4.*  $gn^{-1}(m)$  is computable.

*Example 7.5.* Find  $gn^{-1}(m)$  for  $m = p_0^{15} p_1^{13} p_2^{29} p_3^{31} p_4^{31}$ .

*Solution.*

$$\begin{aligned} gn^{-1}(m) &= gn^{-1}(15)gn^{-1}(13)gn^{-1}(29)gn^{-1}(31)gn^{-1}(31) \\ &= \text{“ } x_1 = \bar{0} \text{”} \\ &= \text{“ } x_1 = \bar{2} \text{”} \end{aligned}$$

□

*Example 7.6.* Find  $gn^{-1}(p_0^{26} p_1^{13} p_2^{43})$ .

*Solution.*  $p_0^{26} p_1^{13} p_2^{43} = p_0^{2 \times 13} p_1^{13} p_2^{43}$ , so it is not a Gödel number. Thus  $gn^{-1}(p_0^{26} p_1^{13} p_2^{43}) = \perp$ . □

## Important number theoretic relations related to $\mathcal{PA}$ .

- (1)  $Sym(m) \Leftrightarrow_{\text{defn}} gn^{-1}(m)$  is a symbol of  $\mathcal{L}_{\mathcal{PA}}$   
(i.e.  $m$  is the Gödel number of a symbol),
- (2)  $Term(m) \Leftrightarrow_{\text{defn}} gn^{-1}(m)$  is a term of  $\mathcal{L}_{\mathcal{PA}}$ ,
- (3)  $Form(m) \Leftrightarrow_{\text{defn}} gn^{-1}(m)$  is a wf of  $\mathcal{L}_{\mathcal{PA}}$ ,
- (4)  $Sent(m) \Leftrightarrow_{\text{defn}} gn^{-1}(m)$  is a sentence of  $\mathcal{L}_{\mathcal{PA}}$ ,
- (5)  $Ax_{\mathcal{PA}}(m) \Leftrightarrow_{\text{defn}} gn^{-1}(m)$  is an axiom of  $\mathcal{PA}$ ,
- (6)  $Ax_{\mathcal{PA}}^1(m) \Leftrightarrow_{\text{defn}} gn^{-1}(m)$  is a logical axiom of  $\mathcal{PA}$ ,  
 $Ax_{\mathcal{PA}}^2(m) \Leftrightarrow_{\text{defn}} gn^{-1}(m)$  is a special axiom of  $\mathcal{PA}$ ,
- (7)  $MP(m, n, p) \Leftrightarrow_{\text{defn}} Form(m), Form(n), Form(p)$  and  
 $gn^{-1}(p)$  is derived from  $gn^{-1}(m)$  and  $gn^{-1}(n)$  via MP,
- (8)  $Gen(m, n) \Leftrightarrow_{\text{defn}} Form(m), Form(n)$  and  $gn^{-1}(n)$  is  
derived from  $gn^{-1}(m)$  via Gen,
- (9)  $Proof_{\mathcal{PA}}(m) \Leftrightarrow_{\text{defn}} gn^{-1}(m)$  is a proof of  $\mathcal{PA}$ ,
- (10)  $Th_{\mathcal{PA}}(m) \Leftrightarrow_{\text{defn}} \vdash_{\mathcal{PA}} gn^{-1}(m)$ .

**Theorem 7.7.** *All of the relations (1) - (9) are recursive, and so computable.*

*Proof.* We just prove (9).

*Notation.* We write:

- (1)  $lh(m) =$  largest  $i$  such that  $p_i | m$  (e.g.  $lh(6) = lh(2 \times 3) = 1$ ).
- (2)  $(m)_i =$  the exponent of  $p_i$  in the factorisation of  $m$  (e.g.  $(12)_0 = (p_0^2 p_1^1)_0 = 2$ ).

Then

$$\begin{aligned} Proof_{\mathcal{PA}}(m) &\Leftrightarrow gn^{-1}(m) \text{ is a proof of } \mathcal{PA}, \\ &\Leftrightarrow gn^{-1}((m)_0) gn^{-1}((m)_1) \dots gn^{-1}((m)_{lh(m)}) \text{ is a proof of } \mathcal{PA}. \end{aligned}$$

Then, to check if  $Proof(m)$  holds:

- (i) Check that  $Form((m)_0), Form((m)_1), \dots, Form((m)_{lh(m)})$  (if not then  $Proof(m)$  is false),

(ii) For each  $(m)_i$ , check that either  $Ax_{\mathcal{P}\mathcal{A}}((m)_i)$  or for some  $p, q < i$ , one of  $MP((m)_p, (m)_q, (m)_i)$  or  $Gen((m)_p, (m)_i)$ .

Note that this algorithm is effective—since  $Form$ ,  $Ax_{\mathcal{P}\mathcal{A}}$ ,  $MP$  and  $Gen$  are already proved to be recursive.  $\square$

*Question 7.8.* What about  $Th_{\mathcal{P}\mathcal{A}}$ ?

**Definition 7.9.** (1)  $R(\vec{x})$  is a  $\Sigma_1$ -relation iff it can be expressed in the form

$$R(\vec{x}) = \exists y P(\vec{x}, y)$$

with  $P$  computable.

(2)  $S \subseteq \mathbb{N}$  is  $\Sigma_1$  iff the relation  $x \in S$  is  $\Sigma_1$ .

**Theorem 7.10.**  $Th_{\mathcal{P}\mathcal{A}}(m)$  is  $\Sigma_1$ .

*Proof.*

$$Th_{\mathcal{P}\mathcal{A}} \Leftrightarrow \exists p [Proof_{\mathcal{P}\mathcal{A}}(p) \ \& \ (p)_{lh(p)} = m]$$

and so it suffices to notice that “ $Proof_{\mathcal{P}\mathcal{A}}(p) \ \& \ (p)_{lh(p)} = m$ ” is computable by Theorem 7.7 etc.  $\square$

**Definition 7.11.** Theory  $\mathcal{T}$  is computably axiomatisable iff  $Ax_{\mathcal{T}}$  is computable.

**Corollary 7.12.** *If  $\mathcal{T}$  is computably axiomatisable, then  $Th_{\mathcal{T}}$  is  $\Sigma_1^0$ .*

*Proof.* Show that  $Proof_{\mathcal{T}}$  is computable just as in the case for  $Proof_{\mathcal{P}\mathcal{A}}(m)$  etc. (see Theorem 7.7).  $\square$

## 8 Computationally Enumerable Sets and Many One Computability.

**Definition 8.1.** Let  $S \subseteq \mathbb{N}$ . We say that  $S$  is computationally enumerable (c.e.) iff  $S = \text{Range}(f)$  for some computable function  $f$ , or  $S = \emptyset$ . Equivalently iff we can computably list  $\{a_0, a_1, \dots\}$  the members of  $S$ .

Intuition:  $S$  is c.e. iff we can computably list the members of  $S$ .

**Definition 8.2.** If  $A, B \subseteq \mathbb{N}$ ,  $A$  is many one reducible to  $B$  (written  $A \leq_m B$ ) iff there exists a computable function  $f$  such that, for all  $m \in \mathbb{N}$ ,

$$m \in A \quad \Leftrightarrow \quad f(m) \in B. \quad (8.1)$$

Intuition:  $\leq_m$  gives us a way of comparing the computability of different sets. Note also that  $A \leq_m B$  iff  $\overline{A} \leq_m \overline{B}$ .

*Question 8.3.* What is the relation between “recursive/computable” and “c.e.”.

**Basic Fact 1.**  $S$  computable  $\Rightarrow S$  c.e.

*Proof.* Suppose that  $S$  is computable, so we can computably decide if  $n \in S$  for all  $n \in \mathbb{N}$ . Then  $S$  can be computably listed as follows.

- Ask for each  $n$  if  $n \in S$  (in order).
- Each time that the answer to this question is “yes,  $n \in S$ ”, add  $n = a_i$  (say) to the list  $a_0, a_1, \dots$  □

**Basic Fact 2.**  $S$  computable  $\Leftrightarrow S$  and  $\overline{S}$  c.e.

*Proof.* ( $\Rightarrow$ ) By Basic Fact 2  $S$  computable  $\Rightarrow S$  c.e. whereas  $S$  computable  $\Rightarrow \overline{S}$  computable  $\Rightarrow \overline{S}$  c.e.

( $\Leftarrow$ ) Suppose that  $a_0, a_1, \dots$  is a computable list of  $S$  and  $b_0, b_1, \dots$  is a computable list of  $\overline{S}$ . Then to computably decide whether  $n \in S$  or  $n \notin S$  search for  $n$  in these two lists (and  $n$  must appear in one of them. Now, if  $n = a_i$  for some  $i$  then  $n \in S$  whereas if  $n = b_j$  for some  $j$  then  $n \notin S$ . □

**Basic Fact 3.**  $S$  c.e.  $\Leftrightarrow S$  is  $\Sigma_1^0$ .

*Proof.* ( $\Rightarrow$ ) Suppose that  $a_0, a_1, \dots$  is a computable list of  $S$ . Then,

$$n \in S \quad \Leftrightarrow \quad \exists i \left( \underbrace{a_i = n}_{\text{computable relation}} \right).$$

( $\Leftarrow$ ) Suppose that  $n \in S \Leftrightarrow \exists m (P(n, m))$  with  $P$  a computable relation. Then computably list all  $2^n 3^m$  such that  $P(n, m)$  holds, say

$$c_0, c_1, c_2, \dots$$

Then define  $a_i = (c_i)_0$ , giving a computable list

$$a_0, a_1, a_2, \dots$$

of members of  $S$ . So  $S$  is c.e. □

**Basic Fact 4.**  $A \leq_m B$  computable (recursive)  $\Rightarrow A$  recursive.

*Proof.* See Problems 4, Question 6. □

**Basic Fact 5.**  $A \leq_m B$  c.e.  $\Rightarrow A$  c.e.

*Proof.* See Problems 4, Question 6. □

**Definition 8.4.** We define  $T_{\mathcal{T}} = \{ m \mid Th_{\mathcal{T}}(m) \}$ .

**Corollary 8.5.** If  $\mathcal{T}$  is computably axiomatisable, then  $T_{\mathcal{T}}$  is c.e. Hence  $T_{\mathcal{PA}}$  is c.e.

*Proof.* By Corollary 7.12, if  $\mathcal{T}$  is computably axiomatisable then  $Th_{\mathcal{T}}(m)$  is  $\Sigma_1^0$ . So  $T_{\mathcal{T}}$  is  $\Sigma_1^0$  and hence, by Basic Fact 3  $T_{\mathcal{T}}$  is c.e.

*Question 8.6 (CRUCIAL).* What can  $\mathcal{PA}$  say about  $T_{\mathcal{PA}}$ ? Or about c.e. sets in general? □

**Definition 8.7.** (1)  $\mathcal{T}$  is consistent iff for each wf  $\varphi$  of  $\mathcal{L}_{\mathcal{T}}$  either  $\vdash_{\mathcal{T}} \varphi$  or not  $\vdash_{\mathcal{T}} \neg\varphi$ . (Equivalently, “not”  $\vdash_{\mathcal{T}} \varphi \wedge \neg\varphi$ .)

(2)  $\mathcal{T}$  (in the language of  $\mathcal{PA}$ ) is  $\omega$ -consistent iff for each  $\varphi(x_i)$  of  $\mathcal{L}_{\mathcal{PA}}$ , if

$$\vdash_{\mathcal{T}} (\exists x_i) \neg\varphi(x_i) \quad \text{then} \quad \text{“not” } \vdash_{\mathcal{T}} \varphi(\overline{m}), \quad \text{for some } m.$$

*Note 8.8.* If  $\mathcal{T}$  is not consistent then  $\vdash_{\mathcal{T}} \psi$  for all  $\psi$  in  $\mathcal{L}_{\mathcal{T}}$

*Proof.* Problems 4. Question 3. □

*Note 8.9.* If  $\mathcal{T}$  is  $\omega$ -consistent, then  $\mathcal{T}$  is consistent.

*Proof.* Problems 4. Question 4. □

*Note 8.10.* If  $\underline{\underline{\mathbb{N}}}$  is a model of  $\mathcal{PA}$ , then  $\mathcal{PA}$  is  $\omega$ -consistent.

*Proof.* Say that  $\underline{\underline{\mathbb{N}}} \models \mathcal{PA}$  and that  $\vdash_{\mathcal{PA}} \varphi(\bar{m})$  for every  $m \in \mathbb{N}$ . Then  $\underline{\underline{\mathbb{N}}} \models \neg(\exists x_i)\neg\varphi(x_i)$  and so

$$\text{“not” } \vdash_{\mathcal{PA}} (\exists x_i)\neg\varphi(x_i)$$

(since otherwise  $\underline{\underline{\mathbb{N}}} \models \varphi(x_i)$ ). □

**Definition 8.11.**  $S$  is semi-representable in  $\mathcal{PA}$  iff there exists a wf  $\varphi$  of  $\mathcal{L}_{\mathcal{PA}}$  such that

$$m \in S \quad \Leftrightarrow \quad \vdash_{\mathcal{PA}} \varphi(\bar{m})$$

for each  $S \subseteq \mathbb{N}$ .

**Theorem 8.12.** *Assume that  $\mathcal{PA}$  is  $\omega$ -consistent. Then, for any  $S \subseteq \mathbb{N}$*

$$\begin{aligned} \underline{(a)} \ S \text{ c.e.} & \quad \Leftrightarrow \quad \underline{(b)} \ S \text{ semi-representable in } \mathcal{PA} \\ & \quad \Leftrightarrow \quad \underline{(c)} \ S \leq_m T_{\mathcal{PA}}. \end{aligned}$$

*Proof.* (a)  $\Rightarrow$  (b) Suppose that  $S$  is c.e. and so  $\Sigma_1^0$  (Basic Fact 3). So  $m \in S$  iff  $\exists n R(m, n)$ , such that  $R$  is recursive. Thus  $R$  is representable in  $\mathcal{PA}$  by  $\psi(x_0, x_1)$  say. Let  $\varphi(x_0) =_{\text{def}} \exists x_1 \psi(x_0, x_1)$ .

We show that  $m \in S \quad \Leftrightarrow \quad \vdash_{\mathcal{PA}} \varphi(\bar{m})$

(i) Suppose that  $m \in S$ , so that  $\exists n R(m, n)$ .

Thus  $R(m, n)$  holds for some  $n \in \mathbb{N}$ .

So  $\vdash_{\mathcal{PA}} \psi(\bar{m}, \bar{n})$  (since  $\psi$  represents  $R$ ).

But  $\vdash_{\mathcal{PA}} \psi(\bar{m}, \bar{n}) \rightarrow \exists x_1 \psi(\bar{m}, x_1)$  (logically valid wf)

So  $\vdash_{\mathcal{PA}} \exists x_1 \psi(\bar{m}, x_1)$  (MP)

That is  $\varphi(\bar{m})$ .

(ii) Suppose that  $\vdash_{\mathcal{PA}} \varphi(\bar{m})$ ; in other words that  $\vdash_{\mathcal{PA}} \exists x_1 \psi(\bar{m}, x_1)$ .

Hence<sup>1</sup>  $\vdash_{\mathcal{PA}} \exists x_1 \neg (\neg\psi(\bar{m}, x_1))$

So, by  $\omega$ -consistency, “not”  $\vdash_{\mathcal{PA}} \neg\psi(\bar{m}, \bar{n})$ , some  $n$ .

So  $R(m, n)$  holds (since  $\psi$  represents  $R$ ). This means  $\exists n R(m, n)$ , giving  $m \in S$ .

Thus  $S$  is semi-representable in  $\mathcal{PA}$ .

(b)  $\Leftrightarrow$  (c) Assume that  $S$  is semi-representable via some wf  $\varphi$ .

In other words,  $m \in S \Leftrightarrow \vdash_{\mathcal{PA}} \varphi(\bar{m})$ .

So

$$\begin{aligned} m \in S &\Leftrightarrow Th_{\mathcal{PA}}(gn(\varphi(\bar{m}))) \\ &\Leftrightarrow \underbrace{gn(\varphi(\bar{m}))}_{\substack{\text{recursive} \\ \text{function} \\ \text{of } m}} \in T_{\mathcal{PA}}. \end{aligned}$$

Thus  $S \leq_m T_{\mathcal{PA}}$ .

(c)  $\Leftrightarrow$  (a) If  $S \leq_m T_{\mathcal{PA}}$  then, as  $T_{\mathcal{PA}}$  is c.e. (Corollary 8.5), we know that  $S$  is c.e. by Basic Fact 5.  $\square$

*Note 8.13.* By Corollary 8.5, if  $\mathcal{T}$  is computably axiomatisable then  $T_{\mathcal{T}}$  is c.e., so  $T_{\mathcal{T}} \leq_m T_{\mathcal{PA}}$ .

Notice also that Note 8.13 shows that  $T_{\mathcal{PA}}$  subsumes all computably axiomatisable theories  $\mathcal{T}$  in an abstract sense.

**Corollary 8.14.** *If  $\mathcal{PA}$  is  $\omega$ -consistent then, for all  $S \subseteq \mathbb{N}$ ,*

$$S \text{ computable} \quad \Leftrightarrow \quad S \text{ representable in } \mathcal{PA}.$$

*Proof.* ( $\Rightarrow$ ) Theorem 6.1.

( $\Leftarrow$ ) Suppose that  $S$  is representable in  $\mathcal{PA}$  via  $\varphi(x_0)$ . Thus:

$$m \in S \quad \Rightarrow \quad \vdash_{\mathcal{PA}} \varphi(\bar{m})$$

and

$$m \notin S \quad \Rightarrow \quad \vdash_{\mathcal{PA}} \neg\varphi(\bar{m}).$$

But we also have  $\vdash_{\mathcal{PA}} \varphi(\bar{m}) \Rightarrow m \in S$ ,

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<sup>1</sup>By logical validity  $\vdash_{\mathcal{PA}} \exists x_1 \psi(\bar{m}, x_1) \rightarrow \exists x_1 \neg(\neg\psi(\bar{m}, x_1))$  so this follows by one application of MP.

(since otherwise  $m \notin S \Rightarrow \vdash_{\mathcal{PA}} \neg\varphi(\bar{m})$  contradicting consistency of  $\mathcal{PA}$ ).

Similarly  $\vdash_{\mathcal{PA}} \neg\varphi(\bar{m}) \Rightarrow m \in \bar{S}$ .

Thus  $S$  and  $\bar{S}$  are semi-represented in  $\mathcal{PA}$  by  $\varphi(x_0)$  and  $\neg\varphi(x_0)$  respectively.

Hence both  $S$  and  $\bar{S}$  are c.e. by Theorem 8.12.

Thus  $S$  is computable by Basic Fact 2.  $\square$

*Note 8.15.* By Theorem 8.12 the c.e. sets are exactly those semi-represented by wfs of the form  $\varphi(x_0)$ .

As a result we can deduce the following Theorem.

**Theorem 8.16** (Enumeration Theorem for c.e. sets). *There is a listing  $W_0, W_1, \dots$  of all c.e. sets, and a computable relation  $T_1(i, p, m)$  (the Kleene  $T$ -predicate) such that, for all  $m, i$ ,*

$$m \in W_i \quad \Leftrightarrow \quad \exists p T_1(i, p, m).$$

*Proof.* Define

$$W_i = \begin{cases} \text{The set semi-represented by } gn^{-1}(i) \\ \text{if } \underbrace{gn^{-1}(i) \text{ is of the form } \varphi(x_0)} \\ \quad \text{Call this} \\ \quad \text{computable relation} \\ \quad \text{Form}^*(i) \\ \emptyset \text{ otherwise.} \end{cases}$$

Then

$$m \in W_i \quad \Leftrightarrow \quad \text{Form}^*(i) \ \& \ \vdash_{\mathcal{PA}} \varphi_i(\bar{m})$$

$$\Leftrightarrow \quad \underbrace{\exists p \left( \text{Form}^*(i) \ \& \ \text{Proof}_{\mathcal{PA}}(p) \ \& \ (p)_{lh(p)} = gn \left( \overbrace{gn^{-1}(i)[\bar{m}/x_0]} \right) \right)}_{\text{computable relation of } p, i, m}$$

If  $\text{Form}^*(i)$  holds then this is of the form  $\varphi(\bar{m})$

$$\text{So } m \in W_i \quad \Leftrightarrow \quad \exists p T_1(i, p, m) \text{ as required.} \quad \square$$

**Corollary 8.17.** *There exist a c.e. set that is not computable.*

*Proof.* Define  $K = \{ m \mid m \in W_m \}$ .

(i) We have that

$$\begin{aligned} m \in K &\Leftrightarrow m \in W_m \\ &\Leftrightarrow \underbrace{\exists p T_1(m, p, m)}_{\Sigma_1^0 \text{ relation}}. \end{aligned}$$

So  $K$  is c.e. by Basic Fact 3.

(ii) Suppose that  $K$  is computable. Then  $\overline{K}$  is computable and so c.e. (Basic Fact1).

So, for some  $i$ ,  $\overline{K} = W_i$ .

Thus  $m \in \overline{K} \Leftrightarrow m \in W_i$ .

Putting  $m = i$ , we have that

$$\begin{aligned} i \in \overline{K} &\Leftrightarrow i \in W_i \\ &\Leftrightarrow i \in K \quad (\text{by definition of } K). \end{aligned}$$

A contradiction! □

## 9 Gödel's Incompleteness Theorem

**Definition 9.1.** Theory  $\mathcal{T}$  is complete iff for every sentence  $\varphi$  of  $\mathcal{L}_{\mathcal{T}}$ , either  $\vdash_{\mathcal{PA}} \varphi$  or  $\vdash_{\mathcal{PA}} \neg\varphi$ .

**Theorem 9.2.** (1) Let  $\mathcal{T}$  be a (consistent) computably axiomatisable theory in which  $K$  is semi-representable. Then  $\mathcal{T}$  is not complete.  
(2) In particular,  $\mathcal{PA}$  is not complete (assuming  $\omega$ -consistency).

*Proof.* (1) Suppose that  $m \in K \Leftrightarrow \vdash_{\mathcal{PA}} \varphi(\overline{m})$ , for all  $m \in \mathbb{N}$ .

But, for some  $m$ ,  $m \in K \not\Rightarrow \vdash_{\mathcal{PA}} \neg\varphi(\overline{m})$ ,

since, by Problems 5, Question 7,  $K$  is not representable in  $\mathcal{T}$ .

So, for some  $m_0$  say we have that  $m_0 \in \overline{K}$  but “not”  $\vdash_{\mathcal{PA}} \neg\varphi(\overline{m_0})$ .

However we also have “not”  $\vdash_{\mathcal{PA}} \varphi(\overline{m_0})$  (since  $m_0 \notin K$ ).

Thus neither  $\varphi(\overline{m_0})$  nor  $\neg\varphi(\overline{m_0})$  are provable in  $\mathcal{T}$ .

(2) Since  $K$  is c.e.,  $K$  is semi-representable in  $\mathcal{PA}$  by Theorem 8.12. □

**Corollary 9.3.** Let  $\mathcal{T}$  be an  $\omega$ -consistent (computably) axiomatisable theory in which every computable relation is representable. Then  $\mathcal{T}$  is incomplete.

*Proof.* By Problems 5 Question 6(b) (i.e. the extension of Theorem 8.12 from  $\mathcal{PA}$  to  $\mathcal{T}$ ), if  $\mathcal{T}$  is as stated then every c.e. set is semi-representable in  $\mathcal{T}$ —and in particular  $K$  is. So by Theorem 9.2,  $\mathcal{T}$  is not complete. □

**Definition 9.4.** (1)  $\mathcal{T}'$  is an extension of  $\mathcal{T}$  if  $\mathcal{T}' = \mathcal{T} \cup \Sigma$  for some set  $\Sigma$  of wfs of  $\mathcal{L}_{\mathcal{T}}$ .

(2) Further,  $\mathcal{T}'$  is a finite extension of  $\mathcal{T}$  if  $\Sigma$  is finite.

**Theorem 9.5** (Gödel's (First) Incompleteness Theorem). If  $\mathcal{T} = \mathcal{PA} \cup \Sigma$  is an  $\omega$ -consistent (computably) axiomatisable extension of  $\mathcal{PA}$ , then  $\mathcal{T}$  is not complete.

*Proof.* If  $\varphi$  is a wf of  $\mathcal{L}_{\mathcal{PA}}$ , then

$$\vdash_{\mathcal{PA}} \varphi \quad \Rightarrow \quad \Sigma \vdash_{\mathcal{PA}} \varphi \quad \Rightarrow \quad \vdash_{\mathcal{T}} \varphi.$$

In particular, if  $\varphi$  represents a relation  $R$  in  $\mathcal{PA}$  then  $\varphi$  represents  $R$  in  $\mathcal{T}$ . Hence every computable relation is representable in  $\mathcal{T}$  (as well as in  $\mathcal{PA}$ ). The result now follows by Corollary 9.3.  $\square$

*Note 9.6* (Rosser). All the previous results, including Gödel's Incompleteness Theorem, can be proved using just consistency of  $\mathcal{PA}$ .

*Note 9.7*. We can formulate “ $\mathcal{PA}$  is consistent” as a number theoretic statement  $Con_{\mathcal{PA}}$  (say), and get

**Theorem 9.8** (Gödel's Second Incompleteness Theorem). “Not”  $\vdash_{\mathcal{PA}} Con_{\mathcal{PA}}$  (i.e.  $\mathcal{PA}$  is unable to prove its own consistency).

*Note 9.9*.  $\mathcal{PA}$  does have a complete extension (assuming that  $\underline{\mathbb{N}} \models \mathcal{PA}$ ).

*Proof*. Define

$$\Sigma = \{ \varphi \mid \varphi \text{ is a sentence of } \mathcal{L}_{\mathcal{PA}} \text{ and } \underline{\mathbb{N}} \models \varphi \}.$$

Define  $\mathcal{PA}^* = \mathcal{PA} \cup \Sigma$ . Then

- (1)  $\mathcal{PA}^*$  is consistent (since  $\underline{\mathbb{N}} \models \mathcal{PA}$  and  $\underline{\mathbb{N}} \models \Sigma$  and so  $\underline{\mathbb{N}} \models \mathcal{PA} \cup \Sigma$ ) and
- (2)  $\mathcal{PA}^*$  is complete (since for each sentence  $\varphi$  either  $\underline{\mathbb{N}} \models \varphi$  or  $\underline{\mathbb{N}} \models \neg\varphi$ ).  $\square$