

10 Undecidability and Creative Sets.

Note 10.1. Gödel's Theorem shows that (computably) axiomatisable theories cannot fully describe incomputable sets.

Question 10.2. But are there any natural examples of such sets?

Definition 10.3. \mathcal{T} is decidable iff $T_{\mathcal{T}}$ (the set of Gödel numbers of theorems of \mathcal{T}) is recursive/computable.

Intuition. \mathcal{T} is decidable iff we can computably decide for each φ if $\vdash_{\mathcal{T}} \varphi$ or not.

Note 10.4. IF \mathcal{T} is decidable then \mathcal{T} is computably axiomatisable.

Proof. Suppose that \mathcal{T} is decidable. So $T_{\mathcal{T}}$ is computable. Define $\Gamma = \{\varphi \mid gn(\varphi) \in T_{\mathcal{T}}\}$. Then Γ is a set of axioms for \mathcal{T} and $Ax_{\mathcal{T}}(m) \Leftrightarrow m \in T_{\mathcal{T}}$, so is computable. \square

Corollary 10.5. \mathcal{PA}^* (the TRUE THEORY of \mathbb{N}) is not computably axiomatisable and so not decidable.

Proof. We have seen in Note 9.9 that \mathcal{PA}^* is a complete extension of \mathcal{PA} . So by Gödel's Theorem \mathcal{PA}^* is not axiomatisable. \square

Corollary 10.6. \mathcal{PA} is not decidable.

Proof. K is c.e. by Corollary 8.17 and so $K \leq_m T_{\mathcal{PA}}$ by Theorem 8.12. However K is not computable by Corollary 8.17.

So $T_{\mathcal{PA}}$ is not computable by Basic Fact 4. \square

Note 10.7. If any given incomputable set S is semi-representable in a given axiomatisable theory \mathcal{T} , then \mathcal{T} is not decidable.

Definition 10.8. $C \subseteq \mathbb{N}$ is creative iff C is c.e. and there exists a computable (recursive) function f such that, for all i ,

$$W_i \subseteq \overline{C} \Rightarrow f(i) \in \overline{C} - W_i.$$

Note 10.9. If C is creative then C is not computable.

Proof. Exercise. \square

Example 10.10 (Example of Creative Set). K is creative...

(a) K is c.e.

(b) Define $f(i) = i$ for all i . Suppose that $W_i \subseteq \overline{K}$.

Then

$$\begin{aligned} f(i) \in W_i &\Leftrightarrow i \in W_i && \text{by definition of } f, \\ &\Leftrightarrow i \in K && \text{by definition of } K, \\ &\Leftrightarrow f(i) \in K && \text{by definition of } f. \end{aligned}$$

So $f(i) \notin W_i$ and $f(i) \notin K$ (since $S_i \cap K = \emptyset$).

Thus $f(i) \in \overline{K} - W_i$.

Theorem 10.11. *Suppose that S is c.e. and C is creative. Then*

$$C \leq_m S \quad \Leftrightarrow \quad S \text{ creative.}$$

Proof. See Computability Theory Page 112. (Not needed for the exam.)

□

Corollary 10.12. $T_{\mathcal{P}A}$ is creative.

Proof. \mathcal{K} is creative, $T_{\mathcal{P}A}$ is c.e., and $\mathcal{K} \leq_m T_{\mathcal{P}A}$.

□

11 Church's Theorem

Idea If one checks the proof that all recursive functions are representable in \mathcal{PA} one sees that (PA9) is not fully used. This means that one can replace (PA9) with a single axiom (PA9)* which is not a scheme like (PA9).

Theorem 11.1 (Raphael Robinson's Theorem). *There is a finitely axiomatisable theory \mathcal{RR} in the language of \mathcal{PA} satisfying both (a) and (b) below.*

(a) *All recursive functions are representable in \mathcal{RR} .*

(b) $\vdash_{\mathcal{RR}} \varphi \Rightarrow \vdash_{\mathcal{PA}} \varphi$ (so that, if \mathcal{PA} is ω -consistent, so is \mathcal{RR}).

Corollary 11.2. *If \mathcal{PA} is ω -consistent, then \mathcal{RR} is incomplete and undecidable.*

Proof. The proof is the same as for \mathcal{PA} . In effect, (i) as in Theorem 8.12 show that S c.e. $\Rightarrow S$ semi-representable in \mathcal{RR} and then (ii) apply Theorem 9.2. \square

We now use \mathcal{RR} to show that there is no computer program to test for logical validity.

We need 3 Lemmas.

Lemma 11.3. *If $\mathcal{T}' = \mathcal{T} \cup \Sigma$ is a finite extension of \mathcal{T} , then $T_{\mathcal{T}' \leq_m} T_{\mathcal{T}}$.*

Proof. Let $\Sigma = \{\varphi_1, \dots, \varphi_n\}$, with each φ_i a sentence of $\mathcal{L}_{\mathcal{T}}$. Then for all $m \in \mathbb{N}$

$$\begin{aligned}
 m \in T_{\mathcal{T}'} &\Leftrightarrow \vdash_{\mathcal{T}'} gn^{-1}(m) \\
 &\Leftrightarrow \varphi_1, \dots, \varphi_n \vdash_{\mathcal{T}} gn^{-1}(m) \\
 &\Leftrightarrow \varphi_1 \wedge \dots \wedge \varphi_n \vdash_{\mathcal{T}} gn^{-1}(m) \\
 &\Leftrightarrow \vdash_{\mathcal{T}} \varphi_1 \wedge \dots \wedge \varphi_n \rightarrow gn^{-1}(m) \quad (\text{Deduction Theorem}) \\
 &\Leftrightarrow \underbrace{\vdash_{\mathcal{T}} gn\left(\varphi_1 \wedge \dots \wedge \varphi_n \rightarrow gn^{-1}(m)\right)}_{\text{computable function of } m} \in T_{\mathcal{T}}.
 \end{aligned}$$

Thus $T_{\mathcal{T}' \leq_m} T_{\mathcal{T}}$ as required. \square

Lemma 11.4. *Let $\mathcal{PC}_{\mathcal{L}_{\mathcal{P}A}}$ be predicate calculus with language restricted to $\mathcal{L}_{\mathcal{P}A}$. Then $\mathcal{PC}_{\mathcal{L}_{\mathcal{P}A}}$ is undecidable.*

Proof. \mathcal{RR} is a finite extension of $\mathcal{PC}_{\mathcal{L}_{\mathcal{P}A}}$. So $T_{\mathcal{RR}} \leq_m T_{\mathcal{PC}_{\mathcal{L}_{\mathcal{P}A}}}$ by Lemma 11.3. But \mathcal{RR} is undecidable and so $T_{\mathcal{PC}_{\mathcal{L}_{\mathcal{P}A}}}$ is not computable (Basic Fact 4). Thus $\mathcal{PC}_{\mathcal{L}_{\mathcal{P}A}}$ is undecidable. \square

Lemma 11.5. (1) *if φ is a wf of $\mathcal{L}_{\mathcal{P}A}$, then*

$$\vdash_{\mathcal{PC}_{\mathcal{L}_{\mathcal{P}A}}} \varphi \quad \Leftrightarrow \quad \vdash_{\mathcal{PC}} \varphi.$$

(2) *So $T_{\mathcal{PC}_{\mathcal{L}_{\mathcal{P}A}}} \leq_m T_{\mathcal{PC}}$.*

Proof. Assume that φ is a wf of $\mathcal{L}_{\mathcal{P}A}$. Then

$$\begin{aligned} \vdash_{\mathcal{PC}_{\mathcal{L}_{\mathcal{P}A}}} \varphi &\Leftrightarrow \varphi \text{ logically valid} && \text{(by Gödel's Adequacy Theorem)} \\ &\Leftrightarrow \vdash_{\mathcal{PC}} \varphi. \end{aligned}$$

Thus $T_{\mathcal{PC}_{\mathcal{L}_{\mathcal{P}A}}} \Leftrightarrow \text{Form}_{\mathcal{P}A}(m) \wedge m \in T_{\mathcal{PC}}$.

Define

$$f(m) = \begin{cases} m & \text{if } \text{Form}_{\mathcal{P}A}(m) \\ 0 & \text{otherwise.} \end{cases}$$

Then f is computable (since $\text{Form}_{\mathcal{P}A}(m)$ is) and

$$m \in T_{\mathcal{PC}_{\mathcal{L}_{\mathcal{P}A}}} \Leftrightarrow f(m) \in T_{\mathcal{PC}}.$$

\square

Theorem 11.6 (Church's Theorem). *\mathcal{PC} is not decidable. Hence there is no computer program for deciding for a given wf whether or not it is logically valid.*

Proof. $T_{\mathcal{PC}_{\mathcal{L}_{\mathcal{P}A}}} \leq_m T_{\mathcal{PC}}$ by Lemma 11.5. Hence, by Lemma 11.4 and Basic Fact 4, $T_{\mathcal{PC}}$ is not computable. \square

In fact,

Corollary 11.7. *$T_{\mathcal{PC}}$ is creative.*

Proof. Let $K = \{ m \mid m \in W_m \}$. And so K is creative. Then K is semi-representable in \mathcal{RR} , so

$$\begin{aligned}
 K &\leq_m T_{\mathcal{RR}} && \text{using Theorem 11.1} \\
 &\leq_m T_{\mathcal{PC}_{\mathcal{LPA}}} && \text{by Lemma 11.3} \\
 &\leq_m T_{\mathcal{PC}} && \text{by Lemma 11.5.}
 \end{aligned}$$

Since K is creative, so is $T_{\mathcal{RR}}$ (by Theorem 10.11) and likewise so is $T_{\mathcal{PC}}$. □

12 Some Examples of Decidable Theories

Example 12.1. Propositional Calculus (\mathcal{PR})—i.e. the theory with no function symbols or quantifiers and only 0-place predicate symbols—is decidable.

Proof. Note that:

$$\vdash_{\mathcal{PR}} \varphi \iff \varphi \text{ is a tautology.}$$

And so the truth table method gives us a way of deciding whether or not φ is a tautology. \square

Example 12.2. Let $\mathcal{T} = \mathcal{PC} \cup \{\varphi\}$ where $\varphi = \forall x \forall y (x = y)$. (I.e. φ says that “there is only one element in any model of \mathcal{T} ”.) Then \mathcal{T} is decidable.

Proof. Let ψ be a sentence of \mathcal{PC} . Then

$$\begin{aligned} \vdash_{\mathcal{T}} \psi & \quad \text{iff} \quad \varphi \vdash_{\mathcal{PC}} \psi \\ & \quad \text{iff} \quad \vdash_{\mathcal{PC}} \varphi \rightarrow \psi \quad \text{by the deduction theorem,} \end{aligned}$$

and so, by Gödel’s completeness/adequacy theorem,

$$\text{iff} \quad \varphi \rightarrow \psi \quad \text{is logically valid.}$$

and, since $\varphi \rightarrow \psi$ is true in every interpretation in which φ is false,

$$\text{iff}$$

ψ is true in every interpretation of ψ having just one element. And we show that this is the case

$$\text{iff}$$

ψ^* is a tautology, where ψ^* is obtained from ψ by removing all quantifiers, variables and associated brackets. For example, if ψ is the formula $\forall x \exists y (A_0(x, y) \rightarrow A_1(y))$, then ψ^* is $A_0 \rightarrow A_1$. Accordingly we now prove this last equivalence.

(\Rightarrow) Suppose that ψ is true in every one element interpretation \mathfrak{M} ,

but that ψ^* is false for some assignment of t (true) and f (false) to the A_i 's in ψ^* . Choose \mathfrak{M} so that

$$\mathfrak{M} = \langle \{a\}, S_1, \dots, S_k \rangle$$

as follows. If $A_i(x_1, \dots, x_r)$ occurs in ψ then $S_i(x_1, \dots, x_r)$ is its interpretation in \mathfrak{M} . However, since \mathfrak{M} only has one element $S_i(x_1, \dots, x_r)$ can be thought of as predicating set (i.e. as a one place relation). Now the point here is—thinking of the S_i as sets—to define $S_i = \{a\}$ (i.e. $S_i(a, \dots, a) = 1$) if the proposition A_i is assigned to t in in the assignment under which ψ^* is false, and to define $S_i = \emptyset$ (i.e. $S_i(a, \dots, a) = 0$) if the proposition A_i is assigned to f . It is now easy to check that $\mathfrak{M} \not\models \psi$. A contradiction.

(\Leftarrow) Now suppose that there is some interpretation

$$\mathfrak{M} = \langle \{a\}, R_1, \dots, R_k \rangle$$

such that $\mathfrak{M} \not\models \psi$. Assign t to A_i if $R_i(a, \dots, a) = 1$ in \mathfrak{M} and f to A_i if $R_i(a, \dots, a) = 0$. It can now be checked that ψ^* is mapped to f under this assignment.

The result now follows from Example 12.1 since it suffices to write an algorithm that converts any sentence ψ of \mathcal{PC} into a proposition ψ^* of \mathcal{PR} and that then uses an algorithm for Example 12.1 to decide whether ψ^* is a tautology or not. \square

Example 12.3. A sentence of \mathcal{PC} is existential if it is in prenex normal form and all the quantifiers in its prefix are existential. Then the set of of existential sentences of \mathcal{PC} is decidable. Also the set of quantifier free formulas, the set of universal formulas and the set of $\forall - \exists$ formulas are all decidable etc.

Example 12.4. If \mathcal{T} is complete and recursively axiomatisable, then \mathcal{T} is decidable.

Proof. As \mathcal{T} is computably/recursively axiomatisable, $T_{\mathcal{T}}$ is c.e. Also $X_{\mathcal{T}} =_{\text{def}} \{ m \mid \vdash_{\mathcal{T}} \neg gn^{-1}(m) \}$ is c.e. since many-one reducible to $T_{\mathcal{T}}$. Also, $Y_{\mathcal{T}} =_{\text{def}} \{ m \mid \neg \text{Sent}_{\mathcal{T}}(m) \}$ is recursive and so c.e. However this means that $\overline{T_{\mathcal{T}}}$ is the union of two c.e. sets ($\overline{T_{\mathcal{T}}} = X_{\mathcal{T}} \cup Y_{\mathcal{T}}$)

and so c.e.

Thus $T_{\mathcal{T}}$ is computable/recursive (as both $T_{\mathcal{T}}$ and $\overline{T_{\mathcal{T}}}$ are c.e.).

In other words, \mathcal{T} is decidable. □