

University of Leeds
Department of Pure Mathematics

MATH 3102 MATHEMATICAL LOGIC 2
MATH 5103 ADVANCED LOGIC

Course Notes

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1 Historical Background

1900: David Hilbert at 1900 ICM in Paris.

Emerging Themes.

(a) The scope of algorithms in Mathematics.

Are there algorithms for solving certain general classes of problems in Mathematics?

Example. (Hilbert's 10th Problem)

If $p(z_1, \dots, z_n, x)$ is a polynomial with integer coefficients, for which m does

$$p(z_1, \dots, z_n, m) = 0$$

have integer solutions m_1, \dots, m_n ?

Example. Is there an algorithm for deciding for a given sentence of predicate calculus whether it is logically valid or not?

More generally, do there exist unsolvable problems in Mathematics.

(b) The role of formalism in Mathematics.

The development of consistent axiomatic theories for important areas of Mathematics in order to

- (1) Put Mathematics on a sure footing, and
- (2) Provide a route to systematically finding solutions to important open problems.

For example Cantor's Continuum Problem of deciding how many real numbers there are.

Hilbert's Programme. (1904-1928) Capture Mathematics in complete consistent theories.

Bertrand Russell's Role.

(a) Russell's Paradox (1901)

Define $S = \{ x \mid x \notin x \}$ (an allowable definition in the set theory of Cantor and Frege). Then $x \in S \Leftrightarrow x \notin x$.

But what happens if one puts $x = S$?

(b) Principia Mathematica (1901-1910 with A.N. Whitehead).

1908-22 Development of axioms of set theory (\mathcal{ZF}) by Zermelo and Fraenkel.

1930, Hilbert: "There is no such thing as an unsolvable problem."

1931, Gödel's Incompleteness Theorem.

- Any theory containing enough basic facts about arithmetic (expressible in first order logic) must be incomplete (i.e. be incapable of proving all the true statements of arithmetic).
- And be incapable of proving its own consistency. (cf. Theme (b) above)

First formalisation of the notion of computable functions

(relevant to Theme (a) above)

- (i) Recursive functions (Gödel, Kleene).
- (ii) λ -computable functions (Church, Kleene).
- (iii) Turing computable functions.

Discovery of unsolvable problems in Mathematics.

- Turing: unsolvability of the Halting Problem for Turing machines.

- Church: Undecidability of logical validity, also of \mathcal{PA} (and hence \mathcal{ZF}).

Can we restrict the scope of these negative results?

Maybe they are based on artificial tricks and do not concern real mathematics? Maybe unprovability and incomputability in Mathematics can be transcended by a real world context.

1939 Invention of the oracle Turing machine (for modelling computability in the real world).

1963 Cohen showed that Cantor's continuum problem is not decidable in \mathcal{ZF} .

1970 Davis, Matiashevich and Robinson showed that Hilbert's 10th problem is unsolvable.

1994 "Nonrigidity" of the Turing universe: reality is not well defined in a universe without external interference (relevant to the lack of definition of phenomena at quantum level—Heisenberg uncertainty etc.)

2 Revision of Predicate Calculus - The Axiomatic Method.

Motivation. A theory is a set of assumptions with some rules for deducing theorems from them.

1. Language \mathcal{L} .

Variables x_1, x_2, \dots

Constant Symbols c_1, c_2, \dots

Predicate Letters $A_1^1, A_2^1, \dots, A_1^2, A_2^2, \dots$ (where A_i^n has n variable places).

Function Letters $f_1^1, f_2^1, \dots, f_1^2, f_2^2, \dots$

Punctuation $(,), \dots$

Logical Connectives $\neg, \rightarrow, (\wedge, \vee, \leftrightarrow)$

Quantifiers $\forall, (\exists)$

In a particular theory, we may cut down the symbols available - e.g. by disposing of certain constants symbols, predicate letters and functions. In general A FIRST ORDER LANGUAGE \mathcal{L}' is \mathcal{L} (above) with a possibly restricted number of constants, predicates, and functions.

2. Terms in \mathcal{L} . (Inductive definition)

(i) $x_1, x_2, \dots, c_1, c_2, \dots$ are terms.

(ii) If t_1, \dots, t_n are terms, then $f_i^n(t_1, \dots, t_n)$ is a term.

3. Formulas of \mathcal{L} . (Inductive definition)

(i) (Atomic Formulas) $A_j^k(t_1, \dots, t_k)$ is a formula if t_1, \dots, t_k are terms.

(ii) If \mathcal{A} and \mathcal{B} are formulas, then so are $(\neg\mathcal{A})$, $(\mathcal{A} \rightarrow \mathcal{B})$ and $(\forall x_i)\mathcal{A}$.

Notation. A formula of \mathcal{L} derived as above is said to be a well-formed formula (wf) of \mathcal{L} . A sentence of \mathcal{L} is a wf with no free variables.

4. The Axioms of $\mathcal{K}_{\mathcal{L}}$. (or just \mathcal{K})

(K1) $(\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{A}))$

(K2) $(\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})) \rightarrow ((\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\mathcal{A} \rightarrow \mathcal{C}))$

(K3) $(\neg\mathcal{A} \rightarrow \neg\mathcal{B}) \rightarrow (\mathcal{B} \rightarrow \mathcal{A})$

- (K4) $((\forall x_i)\mathcal{A} \rightarrow \mathcal{A})$ if x_i does not occur free in \mathcal{A}
(K5) $((\forall x_i)\mathcal{A}(x_i) \rightarrow \mathcal{A}(t))$ if t is free for x_i in $\mathcal{A}(x_i)$
(K6) $(\forall x_i)(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\mathcal{A} \rightarrow (\forall x_i)\mathcal{B})$, if x_i does not occur free in \mathcal{A} .

where \mathcal{A} , \mathcal{B} and \mathcal{C} are formulas of \mathcal{L} .

5. Rules of deduction for \mathcal{K} .

(i) Modus Ponens (MP): $\mathcal{A}, (\mathcal{A} \rightarrow \mathcal{B}) \vdash \mathcal{B}$

That is, from \mathcal{A} and $(\mathcal{A} \rightarrow \mathcal{B})$ we can deduce \mathcal{B} .

(ii) Generalisation (Gen): $\mathcal{A} \vdash (\forall x_i)\mathcal{A}$.

Definitions. We now get the following definitions:

(a) A proof in \mathcal{K} is a sequence of wfs $\mathcal{A}_1, \dots, \mathcal{A}_n$ such that, for each i , either \mathcal{A}_i is an axiom of \mathcal{K} or \mathcal{A}_i is a direct consequence of some of the preceding wfs by virtue of one of the rules of deduction (MP or Gen).

(b) A theorem of \mathcal{K} is a wf φ such that there is a proof in \mathcal{K} , the last wf of which is φ . Such a proof is called a proof of φ in \mathcal{K} (denoted $\vdash_{\mathcal{K}} \varphi$).

(c) If Γ is a (finite) set of wfs $\mathcal{B}_1, \dots, \mathcal{B}_k$ then a deduction of (formula) φ from Γ is a sequence of wfs $\mathcal{A}_1, \dots, \mathcal{A}_n$ such that, for each i , either \mathcal{A}_i is an axiom of \mathcal{K} or is contained in Γ or \mathcal{A}_i is a direct consequence of some of the preceding wfs by virtue of one of the rules of deduction (MP or Gen). In this case we say that φ is a consequence of Γ in \mathcal{K} or that Γ proves φ in \mathcal{K} (denoted by $\Gamma \vdash_{\mathcal{K}} \varphi$).

Notation. A sentence \mathcal{A} is a wf (of \mathcal{K}) with no free variables.

Theorem 2.1 (Deduction Theorem). *If $\Gamma, \mathcal{A} \vdash_{\mathcal{K}} \mathcal{B}$ then $\Gamma \vdash_{\mathcal{K}} \mathcal{A} \rightarrow \mathcal{B}$ (with \mathcal{A} a sentence).*

Theorem 2.2 (Gödel's Completeness Theorem for \mathcal{K}).

$$\begin{aligned} \vdash_{\mathcal{K}} \mathcal{A} &\Leftrightarrow \mathcal{A} \text{ is } \underline{\text{logically valid}} \text{ (written } \models \mathcal{A}) \\ &\Leftrightarrow_{\text{defn}} \mathcal{A} \text{ is true in every interpretation/model } \mathfrak{M} \text{ of } \mathcal{K} \\ &\text{(written } \mathfrak{M} \models \mathcal{A}) \text{ for every } \mathfrak{M}. \end{aligned}$$

Definition 2.3. A First Order Theory consists of \mathcal{K} with a first order language \mathcal{L}' and certain additional axioms Γ . In other words, an extension of \mathcal{K} .

Definition 2.4. The Theory \mathcal{S} is consistent iff it is not the case that $\vdash_{\mathcal{S}} \varphi \wedge \neg\varphi$ for any φ . In other words \mathcal{S} cannot prove a contradiction.

Theorem 2.5 (Gödel). *Any consistent first order theory has a model.*

Theorem 2.6 (Löwenheim Skolem Theorem). *If \mathcal{S} is a first order theory then if \mathcal{S} has a model (i.e. is consistent), then \mathcal{S} has a countable model.*

Theorem 2.7 (Compactness Theorem). *Let \mathcal{S} be a first order theory with set of axioms Γ say. Then \mathcal{S} has a model iff every finite subset of Γ has a model.*

Proof. (You do not need to know this proof).

(\Rightarrow) If $\mathfrak{M} \models \mathcal{S}$ then $\mathfrak{M} \models \Gamma$ and so $\mathfrak{M} \models \Gamma'$ for every finite $\Gamma' \subseteq \Gamma$.

(\Leftarrow) If \mathcal{S} has no model then \mathcal{S} is inconsistent (by Theorem 2.5). So $\vdash_{\mathcal{S}} \varphi \wedge \neg\varphi$ (some wf φ of $\mathcal{L}_{\mathcal{S}}$). I.e. $\Gamma \vdash \varphi \wedge \neg\varphi$ which means that there is some (finite) deduction of $\varphi \wedge \neg\varphi$ from Γ . Hence $\Gamma' \vdash \varphi \wedge \neg\varphi$, for some finite $\Gamma' \subseteq \Gamma$ (by finiteness of the deduction/proof) and so Γ' has no model. \square

3 The first order theory for arithmetic (or first order Peano Arithmetic).

Definition 3.1. We define $\underline{\mathbb{N}}$ to be the mathematical structure consisting of:

- (1) The domain $\mathbb{N} = \{0, 1, 2, \dots\}$
- (2) The usual functions of addition $+$, multiplication \times and successor $'$ (where $x' = x + 1$).
- (3) The distinguished element 0 .
- (4) The usual relation $=$ of equality.

We write $\underline{\mathbb{N}} = \langle \mathbb{N}, 0, +, \times, ', = \rangle$.

Note 3.2. $\underline{\mathbb{N}}$ is important as it forms a basic component of all everyday mathematics.

We want to develop a first order theory \mathcal{PA} such that:

- (a) The language of $\mathcal{L}_{\mathcal{PA}}$ can be interpreted in $\underline{\mathbb{N}}$.
- (b) The axioms of \mathcal{PA} can be interpreted in $\underline{\mathbb{N}}$.
- (c) All the usual basic first order statements about $\underline{\mathbb{N}}$ are derivable as theorems of \mathcal{PA} .

1. The language $\mathcal{L}_{\mathcal{PA}}$.

Variables: x_1, x_2, \dots

Constant Symbols: Just a_1 written $\bar{0}$. (Intended interpretation 0 .)

Predicate Letters: Just A_1^2 written $=$. (Intended interpretation usual $=$ in $\underline{\mathbb{N}}$.)

Function Symbols: f_1^1, f_1^2, f_2^2 , where we write $f_1^1(x_i) = x_i'$, $f_1^2(x_i, x_j) = x_i + x_j$ and $f_2^2(x_i, x_j) = x_i \times x_j$. (Intended interpretations are the usual $'$, $+$, and \times respectively.)

+ Usual Punctuation and Logical Symbols.

2. Terms of $\mathcal{L}_{\mathcal{PA}}$. Defined in the usual way.

Notation. Write $\bar{m} = \bar{0} \overbrace{m \dots}$. Thus $\overline{m+1} = \bar{m}'$. (The intended interpretation: interpret \bar{m} as $m \in \mathbb{N}$). $\bar{0}, \bar{1}, \bar{2}, \dots$ are called the numerals of

\mathcal{PA} .

3. Formulas of $\mathcal{L}_{\mathcal{PA}}$. Defined in the usual way.

4. Axioms of \mathcal{PA} .

(a) Logical axioms: (K1) - (K6) in $\mathcal{L}_{\mathcal{PA}}$.

(b) The special axioms of \mathcal{PA} .

$$(PA1) \quad x_1 = x_2 \rightarrow (x_1 = x_3 \rightarrow x_2 = x_3)$$

$$(PA2) \quad x_1 = x_2 \rightarrow x'_1 = x'_2$$

$$(PA3) \quad \bar{0} \neq x'_1$$

$$(PA4) \quad x'_1 = x'_2 \rightarrow x_1 = x_2$$

$$(PA5) \quad x_1 + \bar{0} = x_1$$

$$(PA6) \quad x_1 + x'_2 = (x_1 + x_2)'$$

$$(PA7) \quad x_1 \times \bar{0} = \bar{0}$$

$$(PA8) \quad x_1 \times x'_2 = x_1 \times x_2 + x_1$$

$$(PA9) \quad \text{If } \varphi(x_i) \text{ is a wf of } \mathcal{L}_{\mathcal{PA}}: \quad (\varphi(\bar{0}) \rightarrow ((\forall x_i)(\varphi(x_i) \rightarrow \varphi(x'_i)) \rightarrow (\forall x_i)\varphi(x_i)))$$

Note that (PA9) is a scheme for induction.

5. Rules of deduction for \mathcal{PA} . The usual rules MP and Gen.

Example 3.3. Proof of $t = x_2 \rightarrow (t = x_3 \rightarrow x_2 = x_3)$ where t is some term of $\mathcal{L}_{\mathcal{PA}}$.

$$1. \quad \vdash x_1 = x_2 \rightarrow (x_1 = x_3 \rightarrow x_2 = x_3) \quad (PA1)$$

$$2. \quad \vdash \forall x_1(x_1 = x_2 \rightarrow (x_1 = x_3 \rightarrow x_2 = x_3)) \quad (1, \text{Gen})$$

$$3. \quad \vdash \forall x_1(x_1 = x_2 \rightarrow (x_1 = x_3 \rightarrow x_2 = x_3)) \rightarrow (t = x_2 \rightarrow (t = x_3 \rightarrow x_2 = x_3)) \quad (K5)$$

$$4. \quad \vdash (t = x_2 \rightarrow (t = x_3 \rightarrow x_2 = x_3)) \quad (2,3,MP)$$

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Note 3.4. By repeating the above process twice more we can obtain (PA1)' : $\vdash t_1 = t_2 \rightarrow (t_1 = t_3 \rightarrow t_2 = t_3)$ where t_1, t_2, t_3 are terms of $\mathcal{L}_{\mathcal{PA}}$. We can derive similar transformations of the other special axioms of \mathcal{PA} .

Example 3.5. Proof of $x_1 = x_1$ (Hamilton's axiom E1 [?]).

$$1. \quad \vdash x_1 + \bar{0} = x_1 \quad (PA5)$$

2. $\vdash x_1 + \bar{0} = x_1 \rightarrow (x_1 + \bar{0} = x_1 \rightarrow x_1 = x_1)$ (PA1)'
(with $t_1 = x_1 + \bar{0}$, $t_2 = x_1$, $t_3 = x_1$).
3. $\vdash x_1 + \bar{0} = x_1 \rightarrow x_1 = x_1$ (1,2,MP)
4. $\vdash x_1 = x_1$ (1,3, MP).

Note 3.6. By Gödel's Completeness Theorem for $\mathcal{K}_{\mathcal{L}_{\mathcal{PA}}}$, (see Theorem 2.2) every logically valid formula of $\mathcal{L}_{\mathcal{PA}}$ is provable in \mathcal{PA} (because all the axioms of $\mathcal{K}_{\mathcal{L}_{\mathcal{PA}}}$ are axioms of \mathcal{PA}). In particular every instance of a tautology is provable in \mathcal{PA} .

Example 3.7. Proof of $x_1 = x_2 \rightarrow x_2 = x_1$.

1. $\vdash x_1 = x_2 \rightarrow (x_1 = x_1 \rightarrow x_2 = x_1)$ (N1)'
2. $\vdash (x_1 = x_2 \rightarrow (x_1 = x_1 \rightarrow x_2 = x_1)) \rightarrow (x_1 = x_1 \rightarrow (x_1 = x_2 \rightarrow x_2 = x_1))$
(instance of a tautology).

3. $\vdash (x_1 = x_1 \rightarrow (x_1 = x_2 \rightarrow x_2 = x_1))$ (1,2,MP)
4. $\vdash x_1 = x_1$ (Example 3.5)
5. $\vdash x_1 = x_2 \rightarrow x_2 = x_1$ (3,4,MP)

Notice that, arguing as in Example 3.3 (and Note 3.4) we can further obtain

$\vdash_{\mathcal{PA}} t_1 = t_2 \rightarrow t_2 = t_1$ for any terms t_1, t_2 of $\mathcal{L}_{\mathcal{PA}}$.

Note 3.8. (Note on models of \mathcal{PA}).

- (1) We always interpret $=$ as the usual identity relation. (That is, all models of \mathcal{PA} are normal).
- (2) We have designed \mathcal{PA} so that $\underline{\mathbb{N}} = \langle \mathbb{N}, 0, ', +, \times, = \rangle$ is a model of \mathcal{PA} (i.e. all axioms of \mathcal{PA} are true in $\underline{\mathbb{N}}$). However, $\underline{\mathbb{N}}$ is not the only model of \mathcal{PA} . This is because we cannot say in $\mathcal{L}_{\mathcal{PA}}$ “every x in a model of \mathcal{PA} is a number” (i.e. an interpretation of a numeral \bar{n}).

Definition 3.9. Let \mathfrak{M} be an interpretation of $\mathcal{L}_{\mathcal{PA}}$ in which every axiom of \mathcal{PA} is true. That is, \mathfrak{M} is a model of \mathcal{PA} . Then if $\mathfrak{M} \not\cong \underline{\mathbb{N}}$ (\mathfrak{M} is not isomorphic to $\underline{\mathbb{N}}$) we say that \mathfrak{M} is a non standard model of \mathcal{PA} . (Note that $\mathfrak{M} \cong \mathfrak{M}'$ iff $\exists f : \mathbb{M} \xrightarrow[\text{onto}]{1-1} \mathbb{M}'$ such that for every wf φ , $\mathfrak{M} \models_{a_1, \dots, a_n} \varphi(x_1, \dots, x_n) \Leftrightarrow \mathfrak{M}' \models_{f(a_1), \dots, f(a_n)} \varphi(x_1, \dots, x_n)$).

Theorem 3.10. *If $\mathfrak{M} \models \mathcal{PA}$ (that is \mathfrak{M} is a model of \mathcal{PA}), then the domain (\mathbb{M}) of \mathfrak{M} is infinite.*

Proof. Notice that since $\vdash_{\mathcal{PA}} t = t$ (see Example 3.5) for any term t , we have that if $m = n$ then $\vdash_{\mathcal{PA}} \bar{m} = \bar{n}$. In fact, by Lemma 3.11 below we have the converse: if $m \neq n$ then $\vdash_{\mathcal{PA}} \neg(\bar{m} = \bar{n})$.

So supposing that $\mathfrak{M} \models \mathcal{PA}$ we know that $\mathfrak{M} \models \neg(\bar{m} = \bar{n})$ for each $m, n \in \mathbb{N}$ with $m \neq n$. Now say, that for each n we interpret \bar{n} as $a_n \in \mathbb{M}$ (the domain of \mathfrak{M}). Then $a_n \neq a_m$ for each $m \neq n$. So $\{a_0, a_1, \dots\}$ is an infinite set of distinct members of \mathbb{M} . \square

Notation. Supposing t_1, t_2 to be any terms in $\mathcal{L}_{\mathcal{PA}}$ we use $t_1 \neq t_2$ as shorthand for the wf $\neg(t_1 = t_2)$.

Lemma 3.11. *If $m \neq n$ then $\vdash_{\mathcal{PA}} \bar{m} \neq \bar{n}$.*

Proof. W.l.o.g suppose that $n > m$, i.e. $n = m + k$ for some $k > 0$. Note that

$$\overline{0 + k} = \bar{0} \overbrace{''\dots''}^k$$

and that more generally

$$\overline{m + k} = \bar{0} \overbrace{''\dots''}^{m+k}$$

We can split the proof into two cases (for simplicity).

Case $m = 0$ (And so $\bar{m} = \bar{0}$ and $\overline{m + k} = \bar{k}$)

$$(1) \vdash_{\mathcal{PA}} \neg(\bar{0} = \bar{k}) \quad (\text{PA3})'$$

In other words, $\vdash_{\mathcal{PA}} \neg(\bar{m} = \bar{n})$ (since $\bar{m} = \bar{0}$ and $\bar{n} = \overline{m + k} = \bar{k}$).

Case $m > 0$

$$(1) \bar{m} = \overline{m + k} \vdash_{\mathcal{PA}} \bar{m} = \overline{m + k},$$

$$(2) \vdash_{\mathcal{PA}} \bar{m} = \overline{m + k} \rightarrow \overline{m - 1} = \overline{m + k - 1} \quad (\text{instance of (PA4)}'),$$

$$(3) \bar{m} = \overline{m + k} \vdash_{\mathcal{PA}} \overline{m - 1} = \overline{m + k - 1} \quad (1,2,\text{MP}),$$

(4) If $m = 1$ we have

$$\bar{m} = \overline{m + k} \vdash_{\mathcal{PA}} \bar{0} = \bar{k}. \quad (3.1)$$

If $m > 1$ then simply continue the proof by repeating these steps.

(For example:

(4*) $\vdash_{\mathcal{PA}} \overline{m-1} = \overline{m+k-1} \rightarrow \overline{m-2} = \overline{m+k-2}$ (instance of (PA4)'),

(5*) $\overline{m} = \overline{m+k} \vdash_{\mathcal{PA}} \overline{m-2} = \overline{m+k-2}$ (3,4*,MP),

etc.)

Therefore, when $m > 1$ we also obtain (in m steps)

(4') $\overline{m} = \overline{m+k} \vdash_{\mathcal{PA}} \overline{0} = \overline{k}$,

and so—after (4) or (4')—we proceed...

(5) $\vdash_{\mathcal{PA}} \overline{m} = \overline{m+k} \rightarrow \overline{0} = \overline{k}$, (deduction theorem)

(6) $\vdash_{\mathcal{PA}} (\overline{m} = \overline{m+k} \rightarrow \overline{0} = \overline{k}) \rightarrow (\neg(\overline{0} = \overline{k}) \rightarrow \neg(\overline{m} = \overline{m+k}))$
(instance of taut.),

(7) $\vdash_{\mathcal{PA}} \neg(\overline{0} = \overline{k}) \rightarrow \neg(\overline{m} = \overline{m+k})$ (5,6,MP)

(8) $\vdash_{\mathcal{PA}} \neg(\overline{0} = \overline{k})$ (instance of (PA3)'),

(9) $\vdash_{\mathcal{PA}} \neg(\overline{m} = \overline{m+k})$ (7,8,MP).

However, $m+k = n$. So we have obtained (for $m > 0$)

$$\vdash_{\mathcal{PA}} \neg(\overline{m} = \overline{n}). \quad (3.2)$$

(Therefore Lemma 3.11 holds for both the case $m = 0$ and the case $m > 0$.) \square

Theorem 3.12. *There is a non-standard model \mathfrak{M} of \mathcal{PA} .*

Proof. Add a new individual constant c to the language $\mathcal{L}_{\mathcal{PA}}$. We define an extension \mathcal{PA}^* of \mathcal{PA} by adding to \mathcal{PA} the new special axioms:

$$c \neq \overline{0}, \quad c \neq \overline{1}, \quad c \neq \overline{2}, \dots, \quad c \neq \overline{n}, \dots$$

Claim. \mathcal{PA}^* has a model \mathfrak{M} .

Proof of Claim. Use the Compactness Theorem as follows.

Given any finite subset Γ of the set of axioms of \mathcal{PA}^* we can find a numeral \overline{m} which is not contained in any formula of Γ . So we can interpret c as (the number) m . Then, for every φ in Γ either

(a) φ is an axiom of \mathcal{PA} and so $\underline{\mathbb{N}} \models \varphi$, or

(b) φ is of the form $c \neq \bar{n}$ where $n \neq m$. Thus $c \neq \bar{n}$ is true in $\underline{\mathbb{N}}$ since c is interpreted as m .

So $\underline{\mathbb{N}} \models \Gamma$. Hence every finite subset of the axioms of \mathcal{PA}^* has a model.

This means that, by the Compactness Theorem (see Theorem 2.7), \mathcal{PA}^* has a model \mathfrak{M} , say. That is $\mathfrak{M} \models \mathcal{PA}^*$. \square

In particular $\mathfrak{M} \models \mathcal{PA}$. However, $\mathfrak{M} \not\cong \underline{\mathbb{N}}$. Why? Well, let a_n be the interpretation of \bar{n} in \mathfrak{M} for each $n \in \mathbb{N}$. Then, in any putative isomorphism f between $\underline{\mathbb{N}}$ and \mathfrak{M} we must put n in correspondence with a_n , for each $n \in \mathbb{N}$. But this leaves out the interpretation k of c in \mathfrak{M} since $k \neq a_n$ for any n (so f is not onto). \square

Note. We can use the Lowenheim Skolem Theorem (Theorem 2.6) to make \mathfrak{M} countable.

What do non standard models look like?

Let $\omega = (0, 1, 2, \dots)$ and $\omega^* = (\dots, 2, 1, 0)$.

Now “add” these together to get $\omega + \omega^* = (0, 1, 2, \dots, \dots, 2, 1, 0)$.

Then any countable non-standard model has the ordering got by replacing every element of the rational numbers $\mathbb{Q} \geq 0$ by $(\omega + \omega^*)$.

But there are many different ways of defining the arithmetical operations $+$ and \times on the domain of this model. And no computable way of doing this:

Theorem 3.13 (Tennenbaum, 1959). *There is no countable non-standard model of \mathcal{PA} in which you can compute $+$ or \times .*