1) Given a map, we can get a graph by replacing countries by vertices, and joining vertices by edges whenever the corresponding countries have a border in common. Find a graph corresponding to the map below (not forgetting the country round the outside).

Find a colouring of the vertices of the graph such that no two adjacent vertices have the same colour, and such that only 4 colours are used.

2) Show that if $G$ is a simple graph with $\varepsilon$ edges and $\nu$ vertices, then $\varepsilon \leq \binom{\nu}{2}$. When do we get equality?

3) Show that, up to isomorphism, there are exactly 4 simple graphs on 3 vertices. How many simple graphs (up to isomorphism) are there on 4 vertices?

4) Show that the following graphs are isomorphic:
5) Show that the following graphs are not isomorphic:

6) The line graph \( L(G) \) of a simple graph \( G(V, E) \) is the graph whose vertices are in 1-1 correspondence with the edges of \( G \), 2 vertices of \( L(G) \) being adjacent if and only if the corresponding edges of \( G \) are adjacent.

Show that the line graphs of the 2 graphs below are isomorphic:

Show that the line graph of \( G \) has \( \varepsilon \) vertices and \( \sum_{v \in V} \binom{d(v)}{2} \) edges (where \( d(v) \) is the degree of \( v \) in \( G \)).

7) Show that: (a) \( K_{m,n} \) has \( mn \) edges;
   (b) if \( G \) is simple and bipartite, then \( \varepsilon \leq \frac{\varepsilon^2}{4} \).
Can you find a bipartite, simple graph with 10 edges and 6 vertices?

8) Show that if a \( k \)-regular bipartite graph (with \( k > 0 \)) has bipartition \( (V_1, V_2) \), then \( |V_1| = |V_2| \).

9) Let \( M \) be the incidence matrix and \( A \) the adjacency matrix of a graph \( G \).
   (a) Show that every column sum of \( M \) is 2.
   (b) What are the column sums of \( A \)?
10) Let \( G \) be bipartite. Show that the vertices of \( G \) can be listed in such a way that the adjacency matrix of \( G \) has the form: 
\[
\begin{pmatrix}
0 & \mathbf{A}_{12} \\
\ldots & \ldots \\
\mathbf{A}_{21} & 0
\end{pmatrix}
\] 
where \( \mathbf{A}_{21} \) is the transpose of \( \mathbf{A}_{12} \).

11) Show that if \( G \) is simple, the entries on the diagonals of both \( \mathbf{M}' \) and \( \mathbf{A}^2 \) are the degrees of the vertices of \( G \) (where \( \mathbf{M}' = \) the transpose of \( \mathbf{M} \)).

12) Show that, in any group of two or more people, there are always two with exactly the same number of friends inside the group.

13) Show that in a group of six people, either there are three people who know each other, or there are three people none of whom knows either of the other two.

14) If \( G \) has vertices \( v_1, v_2, \ldots, v_n \), the sequence \( (d(v_1), d(v_2), \ldots, d(v_n)) \) is called a degree sequence of \( G \). Show that a sequence \( (d_1, d_2, \ldots, d_n) \) of non-negative integers is a degree sequence of some graph if and only if \( \sum_{i=1}^{n} d_i \) is even.

15) A sequence \( \mathbf{d} = (d_1, d_2, \ldots, d_n) \) is graphic if there is a simple graph with degree sequence \( \mathbf{d} \). Show that the sequences \((7, 6, 5, 4, 3, 3, 2)\) and \((6, 6, 5, 4, 3, 3, 1)\) are not graphic.

16) Give examples (where they exist) of

(a) a bipartite graph which is regular,

(b) a restriction \( G[V'] \) of a complete graph \( G \) which is bipartite,

(c) a Platonic graph which is bipartite,

(d) a cubic graph on nine vertices,

(e) a simple graph which is isomorphic to its line graph,

(f) a Platonic graph which is the line graph of another Platonic graph.
17) Show that the graph below is planar:

Show that it can be drawn in the plane in such a way that every edge is a straight line.

Hand in solutions to **FOUR questions**