

MATH-303201

This question paper consists of 4 printed pages, each of which is identified by the reference MATH 303201

No calculators allowed

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Examination for the Module MATH 3032

(January 2008)

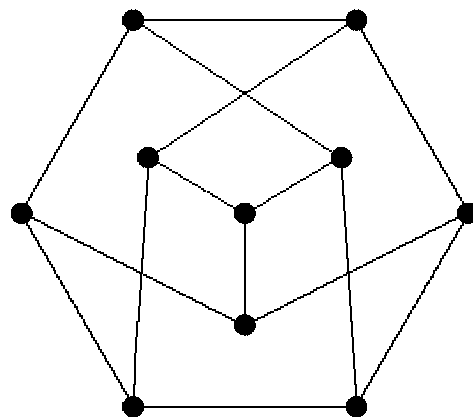
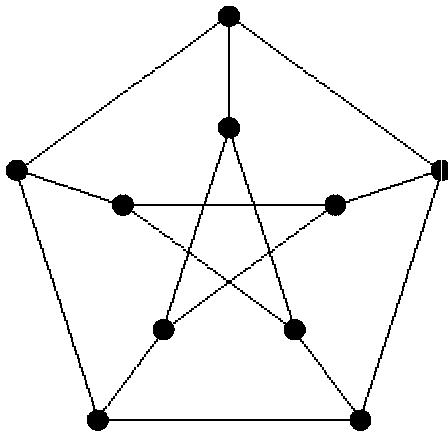
**GRAPH THEORY**

Time allowed : 2 hours

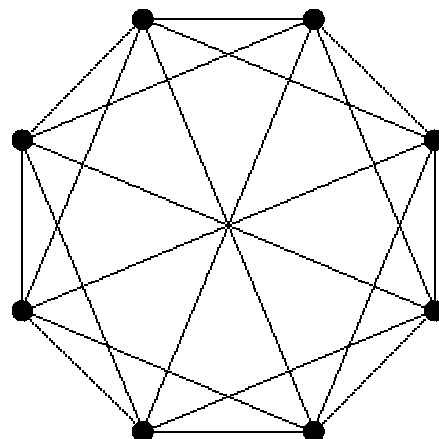
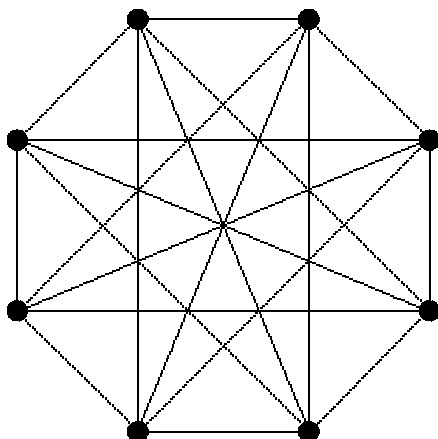
**Do not answer more than *FOUR* questions.**

All questions carry equal marks.

1. (a) Show that the two graphs below are isomorphic:



- (b) Find the complements of the two graphs below. Hence, or otherwise, say, giving reasons, whether the graphs are isomorphic or not.



(c) The *girth* of a graph  $G$  is the length of a shortest circuit in  $G$  (if  $G$  has no circuits we define the girth of  $G$  to be infinite).

Show that a  $k$ -regular graph of girth four has at least  $2k$  vertices.

Give, for each  $k \geq 2$ , an example of a  $k$ -regular graph of girth four with *exactly*  $2k$  vertices.

Show that a  $k$ -regular graph of girth five has at least  $k^2 + 1$  vertices.

2. (a) Define the terms: *Euler tour*, *Eulerian graph*.

Give (without proof) a necessary and sufficient condition for a graph  $G$  to be Eulerian.

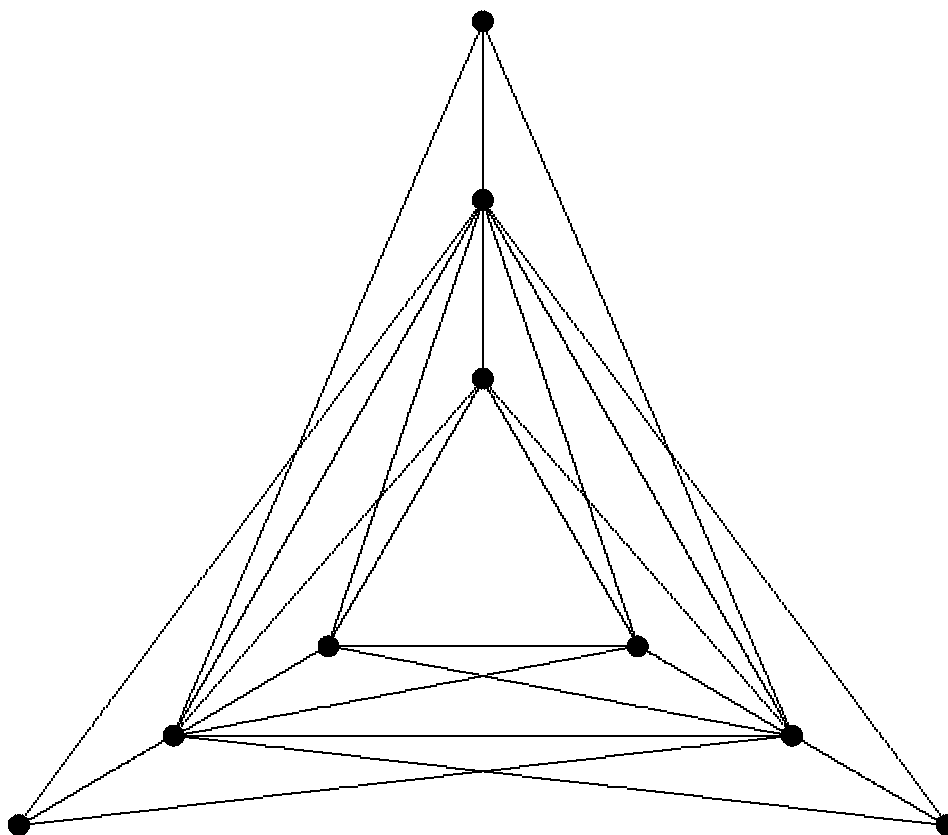
Is it possible for a knight to travel round an  $8 \times 8$  chessboard in such a way that every possible move (where the direction of the move is not relevant) occurs exactly once?

Give reasons for your answer.

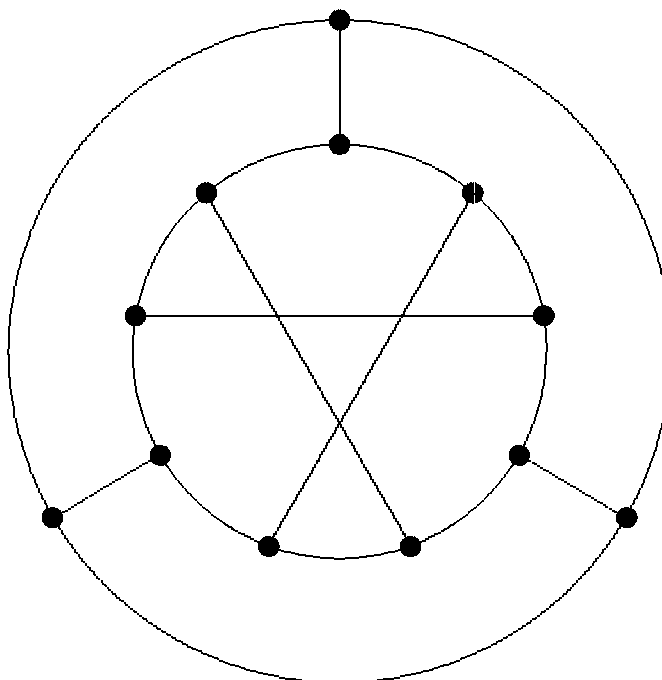
(b) Let  $S$  be a non-empty set of vertices of a Hamiltonian graph  $G$ .

Show that the number  $c(G - S)$  of components of  $G - S$  is not greater than the number  $|S|$  of vertices in  $S$ .

Say, giving reasons, whether or not the graph  $G$  below is Hamiltonian.



(c) Show that the Tietze graph (below) is not Hamiltonian.



3. (a) Prove *Euler's Formula* for a connected plane graph  $G$  with  $\nu$  vertices,  $\varepsilon$  edges and  $\varphi$  faces:

$$\nu - \varepsilon + \varphi = 2.$$

- (b) If  $G$  is a plane graph, define the *dual graph*  $G^*$  of  $G$ .

Deduce that  $\nu(G^*) = \varphi(G)$ ,  $\varepsilon(G^*) = \varepsilon(G)$  and  $d_{G^*}(f^*) = d_G(f)$  for each face  $f$  of  $G$  (where  $f^*$  is the vertex of  $G^*$  corresponding to  $f$ ).

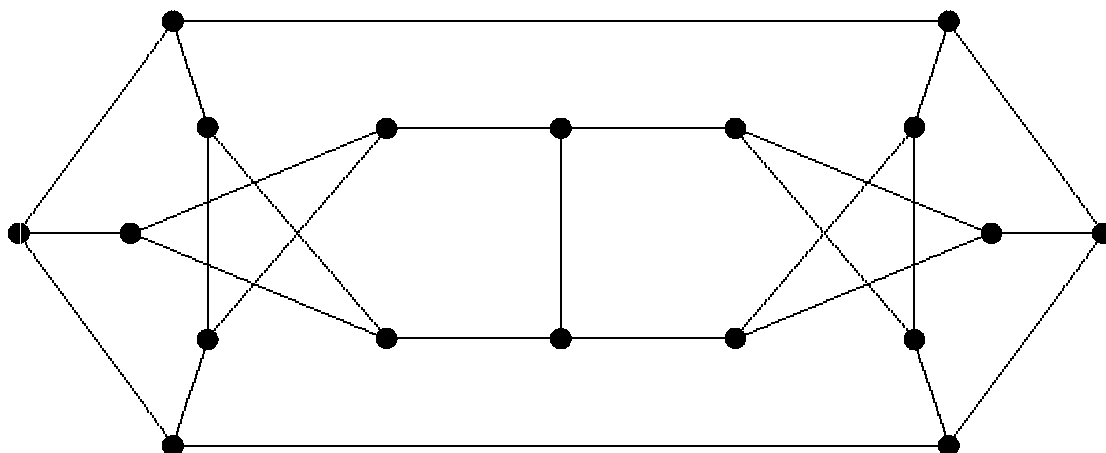
A plane graph is said to be *self-dual* if and only if it is isomorphic to its dual.

- (i) Show that if  $G$  is self-dual, then  $\varepsilon = 2\nu - 2$ .

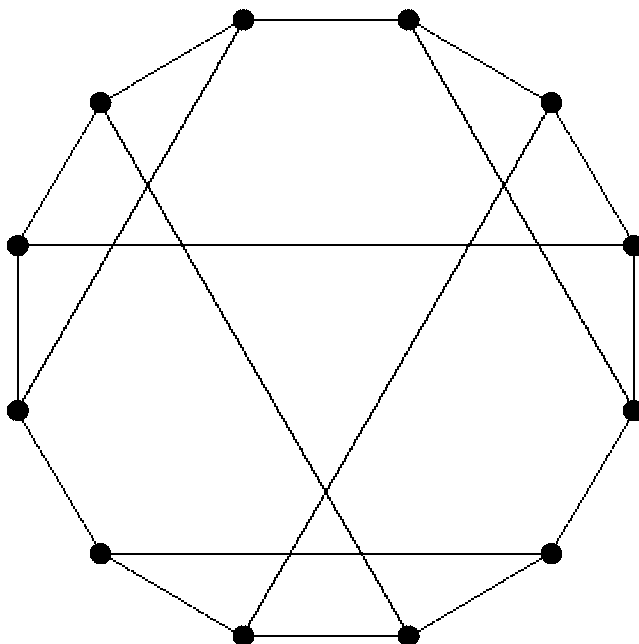
- (ii) Find a self-dual plane graph on 6 vertices.

- (c) *Wagner's Theorem* says that a graph  $G$  is planar if, and only if, it contains no subgraph contractible to  $K_5$  or  $K_{3,3}$ .

Use *Wagner's Theorem* to show that the *Second Blanusa Snark* (below) is non planar:



4. (a) Define:  $D$  is a *tournament*.  
 Show that any tournament contains at most one source and at most one sink.
- (b) Define:  $D$  is *semi-Hamiltonian* for  $D$  a digraph.  
 Prove that every tournament  $D$  is semi-Hamiltonian.
- (c) By considering a vertex of maximum outdegree, or otherwise, prove that:  
 Any tournament contains a vertex from which every other vertex is reachable by a directed path of length at most two.
5. (a) Show that any Eulerian map  $G$  is 2-face-colourable.
- (b) Let  $G$  be a plane graph in which all the faces are bounded by triangles.  
 By using part (a), or otherwise, show that if  $G$  is Eulerian, then  $G$  is 3-colourable.
- (c) Find an embedding of the Franklin graph (below) on the Klein bottle.



Deduce that the chromatic number  $\chi(K)$  of a Klein bottle  $K$  is at least 6.  
 Use Heawood's inequality

$$\chi(S) \leq \frac{1}{2}(7 + \sqrt{49 - 24n})$$

for a surface  $S$  of Euler characteristic  $n < 2$  to show that  $6 \leq \chi(K) \leq 7$  for the Klein bottle  $K$ .

**END**